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Don
10th Standard
Mathematics
First Edition : 2019

PREFACE

Don Mathematics for 10th standard is released with much pride. This guide is prepared based on the Tamilnadu Government's latest new syllabus.

In this book, 'KEY POINTS' and 'IMPORTANT FORMULAE' are given for headings. All the textual questions are solved. Also enormous additional creative questions are solved. Latest Govt. model question paper is given at the end of this book.

All the Textual MCQs and additional creative MCQs are given with full solution. We firmly believe that this book will be of immense help to the students to score centum in their exam.

Wishing you all the best!

S.A. Suresh Kumar MCA, MBA
for Don Publications (P) Ltd

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Published by:

Don Publications (P) Ltd.

3/100/1, Uthamapandiankulam, Tuticorin Main Road, **TIRUNELVELI** - 627351.
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MATHEMATICAL SYMBOLS

SYMBOL	MEANING
$+$	Addition
$-$	Subtraction
$\times$	Multiplication
$\div$	Division
$\pm$	Plus or Minus
$=$	Equal to
$\neq$	Not equal to
$\approx$	Equivalent to
$\simeq$	Approximately equal to
$\equiv$ (or) $\cong$	Congruent
$\equiv$	Identically equal to
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\vert \quad \vert$	Absolute value
$\propto$	Proportional to
∞	Infinity
$\cup$	Union
$\cap$	Intersection
$\mathbb{U}$	Universal set
$\in$	Belongs to
$\notin$	Does not belong to
$\subset$	Proper subset of
$\subseteq$	Subset of or is contained in
$\not\subset$	Not a proper subset of
$\not\subseteq$	Not a subset of or is not contained in
$\Longrightarrow$	Implies
$\Leftrightarrow$	Implies and is implied by
$:$	Ratio
$.$	Decimal
$\%$	Percent [out of 100]

SYMBOL	MEANING
$\sqrt{}$	Square root
$\sqrt[3]{}$	Cubic root
$\parallel$	Parallel to
$\perp$	Perpendicular to
$\angle$	Angle
$\triangle$	Triangle
$\bigcirc$	Circle
$\square$	Square
$\boxed{}$	Rectangle
$\therefore$	Therefore
$\because$	Since (or) because
π	Pi
$\sum$	Summation
A' (or) A^c	Complement of A
$\varnothing$ (or) $\{\}$	Empty set or null set or void set
$n(A)$	Number of elements in the set A
$P(A)$	Power set of A
$\vert\vert\vert^{ly}$	Similarly
Δ	Symmetric difference
$\mathbb{N}$	Set of Natural numbers
$\mathbb{W}$	Set of Whole numbers
$\mathbb{Z}$	Set of all Integers
$\mathbb{R}$	Set of Real numbers
$\mid$ (or) $:$	Such that
x''	x seconds
x'	x minutes
x°	x degrees
$\overline{AB}$	Segment AB
$\overrightarrow{AB}$	Ray AB
$\overleftrightarrow{AB}$	Line AB
x^n	x to the power n

UNIT 1

RELATIONS AND FUNCTIONS

MIND MAP

- SETS
- CARTESIAN PRODUCT
- RELATIONS
- FUNCTIONS
  - One-One function
  - Onto function
  - Many-One function
  - Into function
  - Identity function
  - Reciprocal function
  - Constant function
  - Linear function
  - Quadratic function
  - Cubic function
  - Real Valued function

CARTESIAN PRODUCT

Key Points

- The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the Cartesian product of non-empty sets A and B and is denoted by $A \times B$.
- For three non-empty sets A, B and C, $A \times B \times C$ is the set of all ordered triplets having first element from A, second element from B and third element from C.
- If A and B are two finite sets, then the number of elements in $A \times B$, i.e., $n(A \times B) = n(A) \times n(B)$.
- If either A or B is an infinite set, then $A \times B$ is an infinite set.
- If A, B and C are finite sets, then $n(A \times B \times C) = n(A) \times n(B) \times n(C)$.
- Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$, $b = d$.
- $A \times B \neq B \times A$ unless $A = B$.

- If one of A and B or both A and B are null sets then $A \times B = \phi$.
- $A \times B$ and $B \times A$ are equivalent sets.
- For any three sets A, B and C
  (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
  (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- If $A \subseteq B$, then $(A \times B) \cap (B \times A) = A^2$

Worked Examples

1.1 If A = {1, 3, 5} and B = {2, 3} then

(i) find A × B and B × A.

(ii) Is $A \times B \neq B \times A$? If not why?

(iii) Show that $n(A \times B) = n(B \times A) = n(A) \times n(B)$

Sol : Given that A = {1, 3, 5} and B = {2, 3}

$A \times B = \{1, 3, 5\} \times \{2, 3\}$
$= \{(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)\}$ ... (1)

$B \times A = \{2, 3\} \times \{1, 3, 5\}$
$= \{(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)\}$ ... (2)

From (1) and (2) we conclude that $A \times B \neq B \times A$ as $(1, 2) \neq (2, 1)$ and $(1, 3) \neq (3, 1)$, etc.

$n(A) = 3$; $n(B) = 2$

From (1) and (2) we observe that,

$n(A \times B) = n(B \times A) = 6$

we see that, $n(A) \times n(B) = 3 \times 2 = 6$ and

$n(B) \times n(A) = 2 \times 3 = 6$

Hence,

$n(A \times B) = n(B \times A) = n(A) \times n(B) = 6$

Thus, $n(A \times B) = n(B \times A) = n(A) \times n(B)$.

1.2 If A × B = {(3, 2), (3, 4), (5, 2), (5, 4)} then find A and B.

Sol : $A \times B = \{(3, 2) (3, 4) (5, 2), (5, 4)\}$

We have A = {set of all first co-ordinates of elements of A × B}.

Therefore, A = {3, 5}

B = {set of all second co-ordinates of elements of A × B}.

Therefore, B = {2, 4}

Thus A = {3, 5} and B = {2, 4}.

1.3 Let $A = \{x \in \mathbb{N} \mid 1 < x < 4\}$, $B = \{x \in \mathbb{W} \mid 0 \leq x < 2\}$ and $C = \{x \in \mathbb{N} \mid x < 3\}$. Then verify that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Sol : $A = \{x \in \mathbb{N} \mid 1 < x < 4\} = \{2, 3\}$

$B = \{x \in \mathbb{W} \mid 0 \leq x < 2\} = \{0, 1\}$

$C = \{x \in \mathbb{N} \mid x < 3\} = \{1, 2\}$

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$B \cup C = \{0, 1\} \cup \{1, 2\}$
$= \{0, 1, 2\}$

$A \times (B \cup C) = \{2, 3\} \times \{0, 1, 2\}$
$= \{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$ ... (1)

$A \times B = \{2, 3\} \times \{0, 1\}$
$= \{(2, 0), (2, 1), (3, 0), (3, 1)\}$

$A \times C = \{2, 3\} \times \{1, 2\}$
$= \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$(A \times B) \cup (A \times C) = \{(2, 0), (2, 1), (3, 0), (3, 1)\} \cup \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$(A \times B) \cup (A \times C) = \{(2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)\}$ ... (2)

From (1) and (2),

$A \times (B \cup C) = (A \times B) \cup (A \times C)$ is verified.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$B \cap C = \{0, 1\} \cap \{1, 2\} = \{1\}$

$A \times (B \cap C) = \{2, 3\} \times \{1\} = \{(2, 1), (3, 1)\}$ ... (1)

$A \times B = \{2, 3\} \times \{0, 1\} = \{(2, 0), (2, 1), (3, 0), (3, 1)\}$

$A \times C = \{2 \times 3\} \times \{1, 2\}$
$= \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$(A \times B) \cap (A \times C) = \{(2, 0), (2, 1), (3, 0), (3, 1)\} \cap \{(2, 1), (2, 2), (3, 1), (3, 2)\}$

$(A \times B) \cap (A \times C) = \{(2, 1), (3, 1)\}$ ... (2)

From (1) and (2),

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ is verified.

Progress Check

1. For any two non-empty sets A and B, $A \times B$ is called as _____

Ans : Cartesian Product

2. If $n(A \times B) = 20$ and $n(A) = 5$ then $n(B)$ is _____

Ans : $n(B) = 4$

3. If $A = \{-1, 1\}$ and $B = \{-1, 1\}$ then geometrically describe the set of points of $A \times B$

Ans : $A \times B = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$

4. If A, B are the line segments given by the intervals $(-4, 3)$ and $(-2, 3)$ respectively, represent the cartesian product of A and B.

Ans : $A \times B = \{(-4, -2), (-4, 3), (3, -2), (3, 3)\}$

Thinking Corner

1. When will $A \times B$ be equal to $B \times A$?

Ans : $A \times B = B \times A$, When $A = B$.

Exercise 1.1

1. Find $A \times B$, $A \times A$ and $B \times A$ if

(i) $A = \{2, -2, 3\}$ and $B = \{1, -4\}$

(ii) $A = B = \{p, q\}$

(iii) $A = \{m, n\}$; $B = \phi$

Sol :

(i) $A=\{2, -2, 3\}$ and $B = \{1, -4\}$

$A \times B = \{2, -2, 3\} \times \{1, -4\}$

$= \{(2, 1), (2, -4), (-2, 1), (-2, -4), (3, 1), (3, -4)\}$

$A \times A = \{2, -2, 3\} \times \{2, -2, 3\}$

$= \{(2, 2), (2, -2), (2, 3), (-2, 2), (-2, -2), (-2, 3), (3, 2), (3, -2), (3, 3)\}$

$B \times A = \{1, -4\} \times \{2, -2, 3\}$

$= \{(1, 2), (1, -2), (1, 3), (-4, 2), (-4, -2), (-4, 3)\}$

(ii) $A = B = \{p, q\}$

$A \times B = \{p, q\} \times \{p, q\}$

$= \{(p, p), (p, q), (q, p), (q, q)\}$

$A \times A = \{p, q\} \times \{p, q\}$

$= \{(p, p), (p, q), (q, p), (q, q)\}$

$B \times A = \{p, q\} \times \{p, q\}$

$= \{(p, p), (p, q), (q, p), (q, q)\}$

$\therefore A \times B = A \times A = B \times A$

Since $A = B$

(iii) $A = \{m, n\}, B = \phi$

If either A or B are null sets, then $A \times B$ will also be an empty set.

i.e., $A = \phi$ (or) $B = \phi$

then $A \times B = \phi$, $B \times A = \phi$

and $A \times A = \{m, n\} \times \{m, n\}$

$= \{(m, m), (m, n), (n, m), (n, n)\}$

2. Let $A = \{1, 2, 3\}$ and $B = \{x \mid x$ is a prime number less than $10\}$. Find $A \times B$ and $B \times A$.

Sol : $A = \{1, 2, 3\}$; $B = \{2, 3, 5, 7\}$

$A \times B = \{(1, 2), (1, 3), (1, 5), (1, 7), (2, 2), (2, 3), (2, 5), (2, 7), (3, 2), (3, 3), (3, 5), (3, 7)\}$

$B \times A = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (5, 1), (5, 2), (5, 3), (7, 1), (7, 2), (7, 3)\}$

3. If $B \times A = \{(-2, 3), (-2, 4), (0, 3), (0, 4), (3, 3), (3, 4)\}$ find A and B.

Sol : From $B \times A$, All the first entries belong to the set B and all the second entries belong to A.

$\therefore A = \{3, 4\}$

$B = \{-2, 0, 3\}$

4. If $A = \{5, 6\}$, $B = \{4, 5, 6\}$, $C = \{5, 6, 7\}$, Show that $A \times A = (B \times B) \cap (C \times C)$.

Sol : $A = \{5, 6\}$, $B = \{4, 5, 6\}$, $C = \{5, 6, 7\}$

LHS: $A \times A = \{5, 6\} \times \{5, 6\}$

$= \{(5, 5), (5, 6), (6, 5), (6, 6)\}$

$B \times B = \{4, 5, 6\} \times \{4, 5, 6\}$

$= \{(4, 4), (4, 5), (4, 6), (5, 4), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)\}$

RHS:

$C \times C = \{5, 6, 7\} \times \{5, 6, 7\}$

$= \{(5, 5), (5, 6), (5, 7), (6, 5), (6, 6), (6, 7), (7, 5), (7, 6), (7, 7)\}$

$(B \times B) \cap (C \times C) = \{(5, 5), (5, 6), (6, 5), (6, 6)\}$

$\therefore$ LHS = RHS

$A \times A = (B \times B) \cap (C \times C)$

Hence proved.

5. Given A = {1, 2, 3}, B = {2, 3, 5}, C = {3, 4} and D = {1, 3, 5}, check if $(A\cap C)\times(B\cap D) = (A\times B)\cap(C\times D)$ is true?

Sol : A = {1, 2, 3}, B = {2, 3, 5}, C = {3, 4}, D = {1, 3, 5}

$A\cap C = \{3\}$

$B\cap D = \{3, 5\}$

$(A\cap C)\times(B\cap D) = \{3\}\times\{3, 5\}$

$= \{(3, 3), (3, 5)\}$ .....**(1)**

$A\times B = \{(1,2), (1,3), (1,5), (2,2), (2,3), (2,5), (3,2), (3,3), (3,5)\}$

$C\times D = \{(3,1), (3,3), (3,5), (4,1), (4,3), (4,5)\}$

$(A\times B)\cap(C\times D) = \{(3, 3), (3, 5)\}$ .....**(2)**

From (1) and (2), it is clear that

$\therefore \quad (A\cap C)\times(B\cap D) = (A\times B)\cap(C\times D)$

Hence it is true.

6. Let A = $\{x\in \mathbb{W} \mid x<2\}$, B = $\{x\in\mathbb{N} \mid 1<x\le 4\}$ and C = {3, 5}. Verify that

(i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$

(ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$

(iii) $(A\cup B)\times C = (A\times C)\cup(B\times C)$

Sol : Given $A = \{x\in\mathbb{W} \mid x<2\}\Rightarrow A = \{0, 1\}$

$B = \{x\in\mathbb{N} / 1<x\le 4\}\Rightarrow B = \{2, 3, 4\}$

$C = \{3, 5\}$

(i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$

$B\cup C = \{2, 3, 4, 5\}$

$A\times(B\cup C) = \{0, 1\}\times\{2, 3, 4, 5\}$

$=\{(0,2), (0,3), (0,4), (0,5), (1,2), (1,3), (1,4), (1,5)\}$ ... (1)

$A\times B = \{0, 1\}\times\{2, 3, 4\}$

$= \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$

$A\times C = \{0, 1\}\times\{3, 5\}$

$= \{(0, 3), (0, 5), (1, 3), (1, 5)\}$

$(A\times B)\cup(A\times C) = \{(0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5)\}$....(2)

From (1) × (2), it is clear that

$A\times(B\cup C) = (A\times B)\cup(A\times C)$

Hence verified.

(ii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$

$A = \{0, 1\}, B = \{2, 3, 4\}, C = \{3, 5\}$

$B\cap C = \{3\}$

$A\times(B\cap C) = \{0, 1\}\times\{3\}$

$= \{(0, 3), (1, 3)\}$ ... (1)

$A\times B = \{(0,2), (0,3), (0,4), (1,2), (1,3), (1,4)\}$

$A\times C = \{(0, 3), (0, 5), (1, 3), (1, 5)\}$

$(A\times B)\cap(A\times C) = \{(0, 3), (1, 3)\}$ ... (2)

From (1) and (2), it is clear that

$A\times(B\cap C) = (A\times B)\cap(A\times C)$

Hence verified.

(iii) $(A\cup B)\times C = (A\times C)\cup(B\times C)$

$A = \{0, 1\}, \quad B = \{2, 3, 4\}, \quad C = \{3, 5\}$

$(A\cup B) = \{0, 1, 2, 3, 4\}$

$(A\cup B)\times C = \{0, 1, 2, 3, 4\}\times\{3, 5\}$

$= \{(0, 3), (0, 5), (1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5)\}$ ... (1)

$A\times C = \{0, 1\}\times\{3, 5\}$

$= \{(0, 3), (0, 5), (1, 3), (1, 5)\}$

$B\times C = \{2, 3, 4\}\times\{3, 5\}$

$= \{(2, 3), (2, 5), (3, 3), (3, 5), (4, 3), (4, 5)\}$

$(A\times C)\cup(B\times C) = \{(0,3), (0,5), (1,3), (1,5), (2,3), (2,5), (3,3), (3,5), (4,3), (4,5)\}$ ... (2)

From (1) and (2), it is clear that

$(A\cup B)\times C = (A\times C)\cup(B\times C)$

Hence verified.

7. Let A is the set of all natural numbers less than 8, B is the set of all prime numbers less than 8, C is the set of even prime number. Verify that

(i) $(A\cap B)\times C = (A\times C)\cap(B\times C)$

(ii) $A\times(B-C) = (A\times B)-(A\times C)$

Sol : Given: 'A' is the set of all natural numbers less than 8.

$A = \{1, 2, 3, 4, 5, 6, 7\}$

'B' is the set of all prime numbers less than 8

$B = \{2, 3, 5, 7\}$

'C' is the set of all even prime number

$C = \{2\}$

(i) Verify

$(A\cap B)\times C = (A\times C)\cap(B\times C)$

$A\cap B = \{2, 3, 5, 7\}$

$(A\cap B)\times C = \{2, 3, 5, 7\}\times\{2\}$

$=\{(2, 2), (3, 2), (5, 2), (7, 2)\}$ ... (1)

$A\times C = \{1, 2, 3, 4, 5, 6, 7\}\times\{2\}$

$= \{(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (7, 2)\}$

$B\times C = \{2, 3, 5, 7\}\times\{2\}$

$= \{(2, 2), (3, 2), (5, 2), (7, 2)\}$

$(A\times C)\cap(B\times C) = \{(2, 2), (3, 2), (5, 2), (7, 2)\}$ ... (2)

From (1) and (2) it is clear that

$(A\cap B)\times C = (A\times C)\cap(B\times C)$

Hence verified.

(ii) Verify

$A\times(B-C) = (A\times B)-(A\times C)$

$B-C = \{2, 3, 5, 7\}-\{2\} = \{3, 5, 7\}$

$$A \times (B - C) = \{1, 2, 3, 4, 5, 6, 7\} \times \{3, 5, 7\}$$
$$= \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7), (5,3), (5,5), (5,7), (6,3), (6,5), (6,7), (7,3), (7,5), (7,7)\} \quad ...(1)$$

$$A \times B = \{1, 2, 3, 4, 5, 6, 7\} \times \{2, 3, 5, 7\}$$
$$= \{(1,2), (1,3), (1,5), (1,7), (2,2), (2,3), (2,5), (2,7), (3,2), (3,3), (3,5), (3,7), (4,2), (4,3), (4,5), (4,7), (5,2), (5,3), (5,5), (5,7), (6,2), (6,3), (6,5), (6,7), (7,2), (7,3), (7,5), (7,7)\}$$

$$A \times C = \{1, 2, 3, 4, 5, 6, 7\} \times \{2\}$$
$$= \{(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(7,2)\}$$

$$(A \times B) - (A \times C) = \{(1,3), (1,5), (1,7), (2,3), (2,5), (2,7), (3,3), (3,5), (3,7), (4,3), (4,5), (4,7), (5,3), (5,5), (5,7), (6,3), (6,5), (6,7), (7,3), (7,5), (7,7)\} \quad ...(2)$$

From (1) and (2), it is clear that

$$A \times (B - C) = (A \times B) - (A \times C)$$

Hence verified.

RELATIONS

Key Points

- Let A and B be two sets. Then a relation R from A to B is a subset of A × B.
  Thus $R \subseteq A \times B$
- A and B are two non-empty sets having 'm' and 'n' number of elements respectively. Then A × B will have 'mn' elements. Total number of relations is 2^{mn}.
- A relation from set A to a set B can be represented in any one of the following forms.
  (i) Roster form (ii) Set-Builder form
  (iii) An Arrow diagram (iv) A set of ordered pairs
- If R is a relation from set A to set B, then the set of all first components or co-ordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or co-ordinates of the ordered pairs in R is called the range of R.
- In a non-empty set A, a relation from A to itself i.e., a subset of A × A is called a relation on set A.

Worked Examples

1.4 Let A = {3, 4, 7, 8} and B = {1, 7, 10}. Which of the following sets are relations from A to B?

(i) $R_1 = \{(3, 7), (4, 7), (7, 10), (8, 1)\}$

(ii) $R_2 = \{(3, 1), (4, 12)\}$

(iii) $R_3 = \{(3, 7), (4, 10), (7, 7), (7, 8), (8, 11), (8, 7), (8, 10)\}$

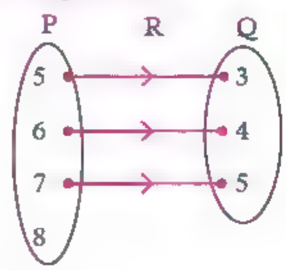
Sol : $A \times B = \{(3,1), (3,7), (3,10), (4,1), (4,7), (4,10), (7,1), (7,7), (7,10), (8,1), (8,7), (8,10)\}$

(i) We note that, $R_1 \subseteq A \times B$. Thus, R_1 is a relation from A to B.

(ii) Here, $(4,12) \in R_2$, but $(4,12) \notin A \times B$. So, R_2 is not a relation from A to B.

(iii) Here, $(7,8) \in R_3$, but $(7,8) \notin A \times B$. So, R_3 is not a relation from A to B.

1.5 The arrow diagram shows a relationship between the sets P and Q. Write the relation in (i) Set builder form (ii) Roster form (iii) What is the domain and range of R ?



Sol :

(i) Set builder form of
$R = \{(x, y) \mid y = x - 2, x \in P, y \in Q\}$

(ii) Roster form $R = \{(5, 3), (6, 4), (7, 5)\}$

(iii) Domain of $R = \{5, 6, 7\}$;
Range of $R = \{3, 4, 5\}$

Progress Check

1. Which of the following are relations from A to B?
   (i) {(1, b), (1, c), (3, a), (4, b)}
   (ii) {(1, a), (b, 4), (c, 3)}
   (iii) {(1, a), (a, 1), (2, b), (b, 2)}

Ans: Let A = {1, 2, 3, 4} and B = {a, b, c}
Relations from A to B
(i) {(1, b), (1, c), (3, a), (4, b)}

2. Which of the following are relations from B to A?
   (i) {(c, a), (c, b), (c, 1)}
   (ii) {(c, 1), (c, 2), (c, 3), (c, 4)}
   (iii) {(a, 4), (b, 3), (c, 2)}

Ans: Let A = {1, 2, 3, 4} and B = {a, b, c}
Relations from B to A
(iii) {(a, 4), (b, 3), (c, 2)}

Exercise 1. 2

1. Let A = {1, 2, 3, 7} and B = {3, 0, – 1, 7}, which of the following are relation from A to B?
   (i) R_1 = {(2, 1), (7, 1)}
   (ii) R_2 = {(– 1, 1)}
   (iii) R_3 = {(2, – 1), (7, 7), (1, 3)}
   (iv) R_4 = {(7, – 1), (0, 3), (3, 3), (0, 7)}

Sol : A = {1, 2, 3, 7}, B = {3, 0, –1, 7}

A × B = {(1, 3), (1, 0), (1, –1), (1, 7), (2, 3), (2, 0), (2, –1), (2, 7), (3, 3), (3, 0), (3, –1), (3, 7), (7, 3), (7, 0), (7, –1), (7, 7)}

(i) R_1 = {(2, 1), (7, 1)}
Since (2, 1) and (7, 1) are not the elements of A × B, R_1 is not a relation from A to B.
Moreover $1 \notin B$.

(ii) R_2 = {(– 1, 1)}, $(-1, 1) \notin A \times B$,
∴ R_2 is not a relation from A to B.
But $(-1, 1) \in (B \times A)$ as $-1 \in B$ and $1 \in A$.

(iii) R_3 = {(2, – 1), (7, 7), (1, 3)}
It is clear that $R_3 \subseteq A \times B$
∴ R_3 is a relation from A to B.

(iv) R_4 = {(7, – 1), (0, 3), (3, 3), (0, 7)}
In this (0, 3) and (0, 7) ∈ R_4
But (0, 3) and (0, 7) are not the elements of A × B. Hence R_4 is not a relation from A to B.

2. Let A = {1, 2, 3, 4,..., 45} and R be the relation defined as "is square of" on A. Write R as a subset of A × A. Also, find the domain and range of R .

Sol : A= {1, 2, 3, 4, ...,45}
Relation is "is square of" and $A \to A$ on A
A × A={(1, 1), (1, 2), (1, 3), (1, 4).....(45, 45)}
The square of '1' is $1 \in A$ and $(1, 1) \in A \times A$
The square of 2 is $4 \in A$ and $(4, 2) \in A \times A$
The square of 3 is $9 \in A$ and $(9, 3) \in A \times A$
The square of 4 is $16 \in A$ and $(16, 4) \in A \times A$
The square of 5 is $25 \in A$ and $(25, 5) \in A \times A$
The square of 6 is $36 \in A$ and $(36, 6) \in A \times A$
The square of 7 is $49 \notin A$.
R ={(1, 1), (4, 2), (9, 3), (16, 4), (25, 5), (36, 6)}
Domain of R = {1, 4, 9, 16, 25, 36}
Range of R = {1, 2, 3, 4, 5, 6}

3. A Relation R is given by the set {(x, y) / y = x + 3, $x \in$ {0, 1, 2, 3, 4, 5}}. Determine its domain and range.

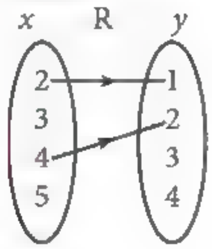
Sol :
Given Set = {(x, y) / y = x + 3, $x \in$ {0, 1, 2, 3, 4, 5}}
When x = 0, y = 0 + 3 = 3
When x = 1, y = 1 + 3 = 4
When x = 2, y = 2 + 3 = 5
When x = 3, y = 3 + 3 = 6
When x = 4, y = 4 + 3 = 7
When x = 5, y = 5 + 3 = 8
∴ Relation R = {(0, 3), (1, 4), (2, 5), (3, 6), (4, 7), (5, 8)}
Domain = {0, 1, 2, 3, 4, 5}
Range = {3, 4, 5, 6, 7, 8}

4. Represent each of the given relations by (a) an arrow diagram, (b) a graph and (c) a set in roster, wherever possible.
   (i) {(x, y) | x = 2y, $x \in$ {2, 3, 4, 5}, $y \in$ {1, 2, 3, 4}}
   (ii) {(x, y) | y = x + 3, x, y are natural numbers < 10}

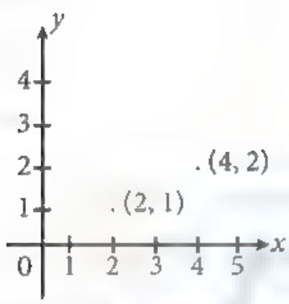
Sol :
(i) Given Set-Builder form
{(x, y) / x = 2y, $x \in$ {2, 3, 4, 5}, $y \in$ {1, 2, 3, 4}}
x = 2y
When y = 1, x = 2 (1) = 2 ∈ x

When $y = 2$, $x = 2(2) = 4 \in x$
When $y = 3$, $x = 2(3) = 6 \notin x$
When $y = 4$, $x = 2(4) = 8 \notin x$
$\therefore$ Relation $R = \{(2, 1), (4, 2)\}$

(a) Arrow diagram



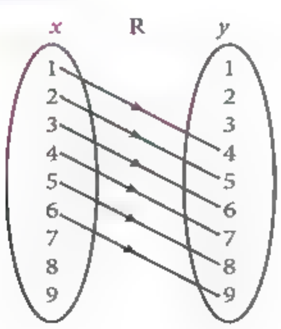
(b) Graph



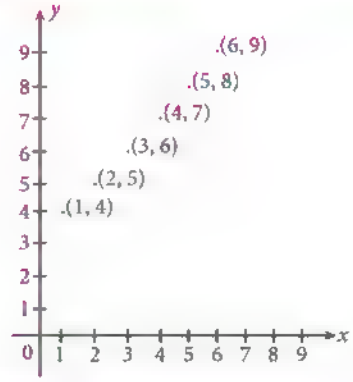
(c) Roster form $R = \{(2,1), (4,2)\}$

(ii) Given set
$\{(x, y) / y = x + 3, x, y$ are natural numbers $< 10\}$
When $x = 1$, $y = 1 + 3 = 4$
When $x = 2$, $y = 2 + 3 = 5$
When $x = 3$, $y = 3 + 3 = 6$
When $x = 4$, $y = 4 + 3 = 7$
When $x = 5$, $y = 5 + 3 = 8$
When $x = 6$, $y = 6 + 3 = 9$
When $x = 7$, $y = 7 + 3 = 10$ is not possible
Since x and y are less than 10.
$\therefore$ Relation $R = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

(a) Arrow diagram



(b) Graph



(c) Roster form

$R = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$

5. A company has four categories of employees given by Assistants (A), Clerks (C), Managers (M) and an Executive Officer (E). The company provide ₹ 10,000, ₹ 25,000, ₹ 50,000 and ₹ 1,00,000 as salaries to the people who work in the categories A, C, M and E respectively. If A_1, A_2, A_3, A_4 and A_5 were Assistants; C_1, C_2, C_3, C_4 were Clerks; M_1, M_2, M_3 were managers and E_1, E_2 were Executive officers and if the relation R is defined by xRy, where x is the salary given to person y, express the relation R through an ordered pair and an arrow diagram.

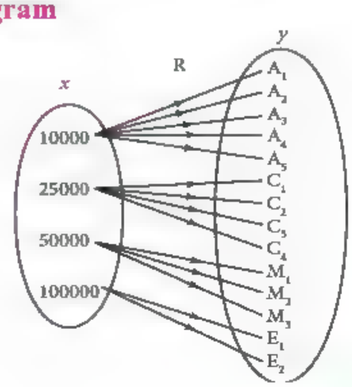
Sol :

Ordered Pair : The Domain of the relation is about the salaries given to person.

Relation is $R = \{(10000, A_1), (10000, A_2), (10000, A_3), (10000, A_4), (10000, A_5), (25000, C_1), (25000, C_2), (25000, C_3), (25000\ C_4), (50000, M_1), (50000, M_2), (50000, M_3), (100000, E_1), (100000, E_2)\}$

Relation R defined by x R y
'x' is the salary given to person 'y'.

Arrow diagram



Don

FUNCTIONS

Key Points

- Let A and B be two non-empty sets, then a relation from A to B i.e., a subset of $A \times B$ is called a function from A to B, if
  - (i) for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$
  - (ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$
- A function 'f' from a set 'A' to set 'B' associates each element of set A to a unique element of set B.
- Let $f : A \to B$. Then, the set A is domain of 'f' and B is co-domain of 'f'. The set of all images of elements of A is known as the range of 'f' or image set of A under f and is denoted by f(A).

$$\text{i.e., } f(A) = \{f(x) : x \in A\} = \text{Range of } f$$

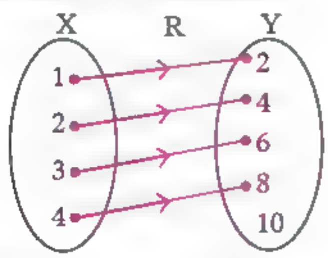
$$\therefore f(A) \subseteq B$$

- Not every curve in the cartesian plane is the graph of a function.
- Vertical line test: A set of points in the cartesian plane is the graph of a function if and only if no vertical straight line intersects the curve more than once.

Worked Examples

1.6 Let $X = \{1, 2, 3, 4\}$ and $Y = \{2, 4, 6, 8, 10\}$ and $\mathbb{R} = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$. Show that $\mathbb{R}$ is a function and find its domain, co-domain and range?

Sol : Pictorial representation of $\mathbb{R}$ is given in the figure. From the diagram, we see that for each $x \in X$, there exists only one $y \in Y$. Thus all elements in X have only image in Y. Therefore $\mathbb{R}$ is a function.



Domain X = {1, 2, 3, 4};
Co-domain Y = {2, 4, 6, 8, 10};
Range of f = {2, 4, 6, 8}.

1.7 A relation 'f' is defined by $f(x) = x^2 - 2$ where, $x \in \{-2, -1, 0, 3\}$

(i) List the elements of f (ii) Is f a function?

Sol : $f(x) = x^2 - 2$ where $x \in \{-2, -1, 0, 3\}$

(i)
$$f(-2) = (-2)^2 - 2 = 2$$
$$f(-1) = (-1)^2 - 2 = -1$$
$$f(0) = (0)^2 - 2 = -2$$
$$f(3) = (3)^2 - 2 = 7$$

Therefore, $f = \{(-2, 2), (-1, -1), (0, -2), (3, 7)\}$

(ii) We note that each element in the domain of f has a unique image. Therefore f is a function.

1.8 If $X = \{-5, 1, 3, 4\}$ and $Y = \{a, b, c\}$, then which of the following relations are functions from X to Y?

(i) $R_1 = \{(-5, a), (1, a), (3, b)\}$

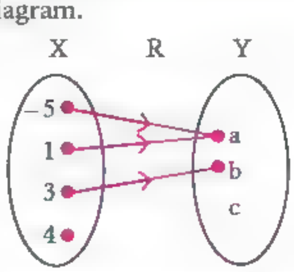
(ii) $R_2 = \{(-5, b), (1, b), (3, a), (4, c)\}$

(iii) $R_3 = \{(-5, a), (1, a), (3, b), (4, c), (1, b)\}$

Sol :

(i) $R_1 = \{(-5, a), (1, a), (3, b)\}$

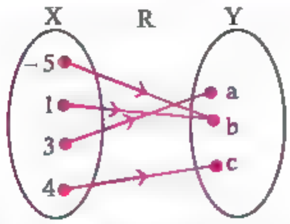
We may represent the relation $\mathbb{R}_1$ in an arrow diagram.



$\mathbb{R}_1$ is not a function as $4 \in X$ does not have an image in Y.

(ii) $\mathbb{R}_2 = \{(-5, b), (1, b), (3, a), (4, c)\}$

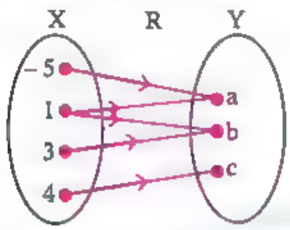
Arrow diagram of $\mathbb{R}_2$ is shown in Figure.



$\mathbb{R}_2$ is a function as each element of X has an unique image in Y.

(iii) $\mathbb{R}_3 = \{(-5, a), (1, a), (3, b), (4, c), (1, b)\}$

Representing $\mathbb{R}_3$ in an arrow diagram.



$\mathbb{R}_3$ is not a function as $1 \in X$ has two images $a \in Y$ and $b \in Y$.

Note that the image of an element should always be unique.

1.9 Given $f(x) = 2x - x^2$, find

(i) $f(1)$ (ii) $f(x+1)$ (iii) $f(x) + f(1)$

Sol :

(i) Replacing x with 1, we get

$$f(1) = 2(1) - (1)^2 = 2 - 1 = 1$$

(ii) Replacing x with x + 1, we get

$$\begin{aligned} f(x+1) &= 2(x+1) - (x+1)^2 \\ &= 2x + 2 - (x^2 + 2x + 1) \\ &= -x^2 + 1 \end{aligned}$$

(iii) $f(x) + f(1) = (2x - x^2) + 1 = -x^2 + 2x + 1$

[Note that $f(x) + f(1) \neq f(x+1)$.

In general $f(a+b)$ is not equal to $f(a) + f(b)$]

Progress Check

1. **Relations are subsets of ____; Functions are subsets of ____.**

Ans : Cartesian product; Relations.

2. **True or False: All the elements of a relation should have images.**

Ans : False

3. **True or False: All the elements of a function should have images.**

Ans : True

4. **True or False: If $\mathbb{R} : A \to B$ is a relation then the domain of $\mathbb{R} = A$**

Ans : True

5. **If f: $\mathbb{N} \to \mathbb{N}$ is defined as $f(x) = x^2$ the pre-image(s) of 1 and 2 are ____ and ____.**

Ans : 1 and None

6. **The difference between relation and function is ____.**

Ans : Every function is a Relation, but the relation need not be a function.

7. **Let A and B be two non-empty finite sets. Then which one among the following two collection is large?**
   (i) The number of relations between A and B.
   (ii) The number of functions between A and B.

Ans :

(i) The number of relations between A and B is large.

(ii) Number of relation is always greater than number of functions.

Thinking Corner

1. **Is the relation representing the association between planets and their respective moons a function?**

Ans : Yes.

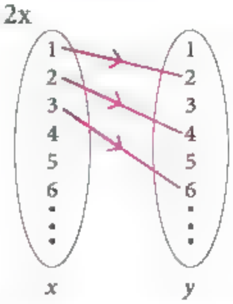
Exercise 1.3

1. **Let $f = \{(x, y) | x, y \in \mathbb{N}$ and $y = 2x\}$ be a relation on N. Find the domain, co-domain and range. Is this relation a function?**

Sol :

$f = \{(x, y)/x, y \in \mathbb{N}$ and $y = 2x\}$

Given that $y = 2x$



'x' is always a natural number. Domain is the set of all first entries.

So, Domain-Set of natural numbers = N and y is always an even number as y = 2x

Range = Set of even natural numbers

Co-domain = Set of natural numbers = N

Here, the first elements (x) are having unique images. So, this relation is a function.

2. Let X = {3, 4, 6, 8}. Determine whether the relation $\mathbb{R} = \{(x, f(x)) \mid x \in X, f(x) = x^2 + 1\}$ is a function from X to $\mathbb{N}$?

Sol : Given X = {3, 4, 6, 8}

Relation $R = \{(x, f(x)) / x \in X, f(x) = x^2 + 1\}$

When x = 3, $f(3) = (3)^2 + 1 = 9 + 1 = 10 \in N$

When x = 4, $f(4) = (4)^2 + 1 = 16 + 1 = 17 \in N$

When x = 6, $f(6) = (6)^2 + 1 = 36 + 1 = 37 \in N$

When x = 8, $f(8) = (8)^2 + 1 = 64 + 1 = 65 \in N$

R = {(3, 10), (4, 17), (6, 37), (8, 65)}

Since, all the elements of X are having natural numbers as images, it is a function from X to N.

3. Given the function f : $x \to x^2 - 5x + 6$, evaluate

(i) f (– 1) **(ii) f (2a)**

(iii) f (2) **(iv) f (x – 1)**

Sol : $f(x) = x^2 - 5x + 6$

(i) $f(-1) = (-1)^2 - 5(-1) + 6 = 1 + 5 + 6 = 12$

(ii) $f(2a) = (2a)^2 - 5(2a) + 6 = 4a^2 - 10a + 6$

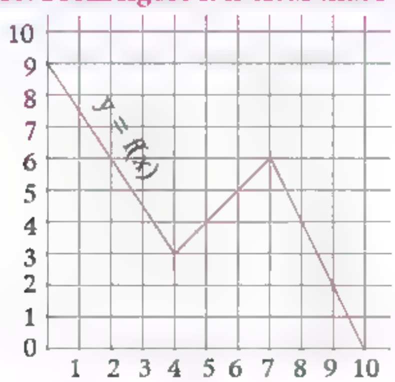
(iii) $f(2) = (2)^2 - 5(2) + 6 = 4 - 10 + 6 = 0$

(iv) $f(x-1) = (x-1)^2 - 5(x-1) + 6$

$= x^2 - 2x + 1 - 5x + 5 + 6$

$= x^2 - 7x + 12$

4. A graph representing the function f (x) is given below figure. From figure it is clear that f (9) = 2.



(i) Find the following values of the function

(a) f (0) **(b) f (7)**

(c) f (2) **(d) f (10)**

(ii) For what value of x is f (x) = 1?

(iii) Describe the following (a) Domain (b) Range.

(iv) What is the image of 6 under f?

Sol :

(i) (a) f (0) = 9

(b) f (7) = 6

(c) f (2) = 6

(d) f (10) = 0

(ii) For what value of x is f (x) = 1

From the graph, it is known that when x = 9.5, f (9.5) = 1

(iii) (a) Domain = $\{x / 0 \le x \le 10, x \in R\}$

(b) Range = $\{x / 0 \le x \le 9, x \in R\}$

(iv) Image of '6' under f is '5'.

5. Let f (x) = 2x + 5. If $x \neq 0$ then find $\frac{f(x+2) - f(2)}{x}$.

Sol :

$f(x) = 2x + 5, \quad x \neq 0$

$$\frac{f(x+2) - f(2)}{x} = \frac{[2(x+2)+5] - [2(2)+5]}{x}$$

$$= \frac{2x+4+5-9}{x} = \frac{2x+9-9}{x}$$

$$= \frac{2x}{x} = 2$$

6. A function f is defined by f (x) = 2x – 3

(i) find $\frac{f(0) + f(1)}{2}$

(ii) find x such that f (x) = 0

(iii) find x such that f (x) = x.

(iv) find x such that f (x) = f (1 – x).

Sol : f (x) = 2x – 3

(i) $$\frac{f(0)+f(1)}{2} = \frac{[2(0)-3]+[2(1)-3]}{2}$$

$$= \frac{0-3+2-3}{2} = \frac{2-6}{2}$$

$$= \frac{-4}{2} = -2$$

(ii) Given f (x) = 0

$\therefore$ 2x – 3 = 0

2x = 3 $\Rightarrow$ x = 3/2

(iii) Given f (x) = x

2x – 3 = x

2x – x = 3 $\Rightarrow$ x = 3

(iv) Given f (x) = f (1 – x)

2x – 3 = 2 (1 – x) – 3

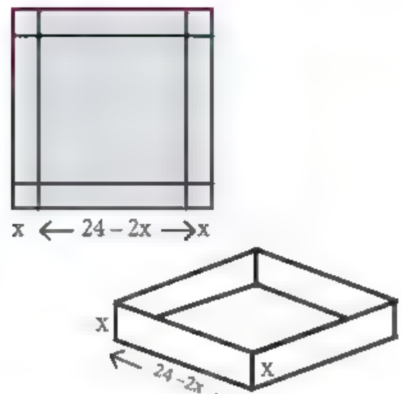
2x – 3 = 2 – 2x – 3

2x + 2x = 2 – 3 + 3

4x = 2

x = 2/4 = 1/2

7. An open box is to be made from a square piece of material, 24 cm on a side, by cutting equal squares from the corners and turning up the sides as shown figure. Express the volume V of the box as a function of x.



Sol : From the diagram, the solid is a cuboid.

Volume of cuboid = length × breadth × height

where $l = 24 - 2x$, $b = 24 - 2x$, $h = x$

$\therefore$ Volume $V(x) = (24 - 2x)(24 - 2x)x$

$V(x) = x(24 - 2x)^2, x > 0$

$= 4x^3 - 96x^2 + 576x, x > 0$

So, the domain is $0 < x < 12$

8. A function f is defined by $f(x) = 3 - 2x$. Find x such that $f(x^2) = (f(x))^2$.

Sol : $f(x) = 3 - 2x$

Given $f(x^2) = [f(x)]^2$

$3 - 2x^2 = (3 - 2x)^2$

$3 - 2x^2 = 9 - 12x + 4x^2$

$6x^2 - 12x + 6 = 0$

$6(x^2 - 2x + 1) = 0$

$(x - 1)^2 = 0 \Rightarrow x = 1.$

9. A plane is flying at a speed of 500 km per hour. Express the distance d travelled by the plane as function of time t in hours.

Sol : Let the distance be 'd'

Speed = 500 km/hr,

Time = 't' hours

$\therefore$ Distance = Speed × time

$d(t) = 500\,t$

10. The data in the adjacent table depicts the length of a woman's forehand and her corresponding height. Based on this data, a student finds a relationship between the height (y) and the forehand length (x) as $y = ax + b$, where a, b are constants.

Length x of forehand (in cm)	Height y (in inch)
35	56
45	65
50	69.5
55	74

(i) Check if this relation is a function.

(ii) Find a and b.

(iii) Find the height of a woman whose forehand length is 40 cm.

(iv) Find the length of forehand of a woman if her height is 53.3 inches.

Sol : $y = ax + b$; x = forehand length; y = height

x	y
35	56
45	65
50	69.5
55	74

For all the x-values, there is an image which is 'y'. Moreover, the difference between two consecutive 'y' values is constant.

$\therefore$ In $y = ax + b$,

(i) the Relation

$R = \{(35, 56), (45, 65), (50, 69.5), (55, 74)\}$ is a function.

(ii) In $y = ax + b$

when $x = 35, y = 56$

$\Rightarrow 56 = 35a + b$ ....(1)

when $x = 45, y = 65$

$\Rightarrow 65 = 45a + b$ ....(2)

Solving (1) and (2), we get $a = 0.90$ and $b = 24.5$

(iii) Given, forehand length is 40 cm

i.e., when $x = 40, y = ax + b$

So, $y = (0.90)(40) + 24.5 = 60.5$

$\therefore$ Height of woman is 60.5 inches

(iv) Given height is 53.3 inches

ie. when $y = 53.3, x = ?$

$\Rightarrow 53.3 = 0.9x + 24.5$

$53.3 - 24.5 = 0.9x$

$\Rightarrow x = \dfrac{28.8}{0.9} = 32$

length of forehand is 32 cm.

REPRESENTATION OF FUNCTIONS

Worked Examples

1.10 Using vertical line test, determine which of the following curves (figure a, b, c, d) represent a function?

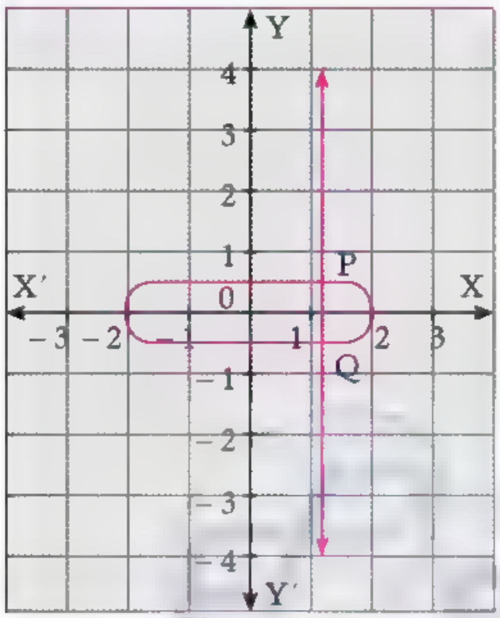


Figure (a)

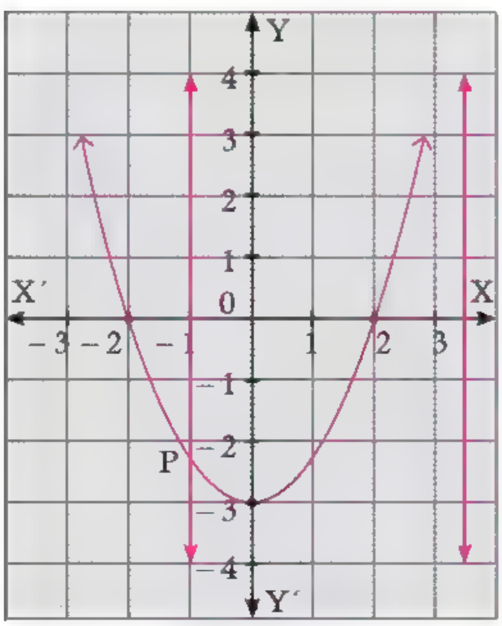


Figure (b)

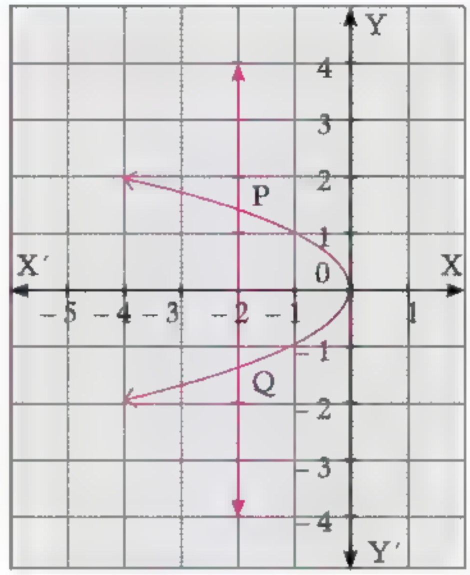


Figure (c)

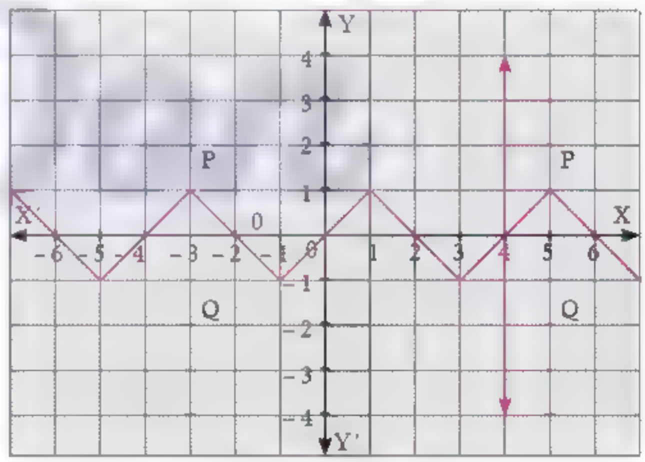


Figure (d)

Sol :

The curves in figure (a) and (c) do not represent a function as the vertical lines meet the curves in two points P and Q.

The curves in figure (b) and (d) represent a function as the vertical lines meet the curve in at most one point.

1.11 Let A = {1, 2, 3, 4} and B = {2, 5, 8, 11, 14} be two sets. Let f : $A \to B$ be a function given by f (x) = 3x – 1. Represent this function as

(i) by arrow diagram

(ii) in a table form

(iii) as a set of ordered pairs

(iv) in a graphical form

Sol :

A = {1, 2, 3, 4}; B = {2, 5, 8, 11, 14};

f (x) = 3x – 1

f (1) = 3 (1) – 1 = 3 – 1 = 2

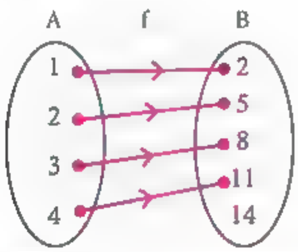
f (2) = 3 (2) – 1 = 6 – 1 = 5

f (3) = 3 (3) – 1 = 9 – 1 = 8

f (4) = 4 (3) – 1 = 12 – 1 = 11

(i) Arrow diagram

Let us represent the function $f : A \rightarrow B$ by an arrow diagram



(ii) Table form

The given function f can be represented in a tabular form as given below

x	1	2	3	4
f (x)	2	5	8	11

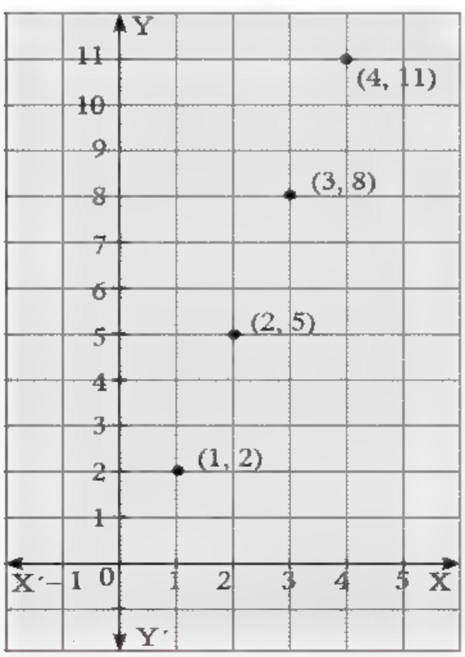
(iii) Set of ordered pairs

The function f can be represented as a set of ordered pairs as

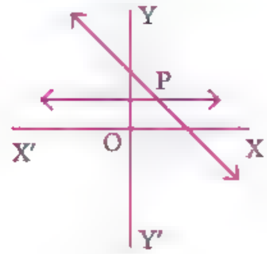
f = {(1, 2), (2, 5), (3, 8), (4, 11)}

(iv) Graphical form

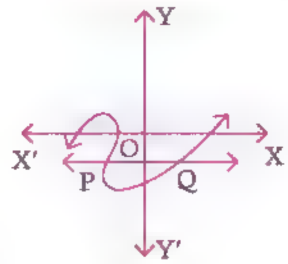
In the adjacent xy-plane the points (1, 2), (2, 5), (3, 8), (4, 11) are plotted.



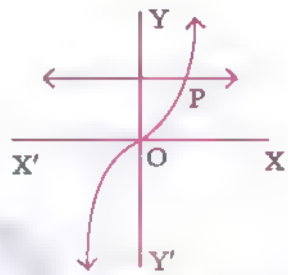
1. 12. Using horizontal line test fig (a), fig (b), fig (c), determine which of the following functions are one-one.



(a)



(b)



(c)

Sol : The curves in fig (a) and fig (c), represent a one-one function as the horizontal lines meet the curves in only one point P.

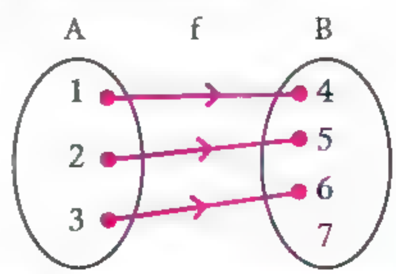
The curve in fig (b) does not represent a one-one function, since, the horizontal line meet the curve in two points P and Q.

1. 13. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one but not onto function.

Sol : A = {1, 2, 3}, B = {4, 5, 6, 7};

f = {(1, 4), (2, 5), (3, 6)}

Then f is a function from A to B and for different elements in A, there are different images in B. Hence f is one-one function. Note that the element 7 in the co-domain does not have any pre-image in the domain. Hence f is not onto.



Therefore f is one-one but not an onto function.

1. 14. **If $A = \{-2, -1, 0, 1, 2\}$ and $f: A \to B$ is an onto function defined by $f(x) = x^2 + x + 1$ then find B.**

Sol : Given $A = \{-2, -1, 0, 1, 2\}$ and

$$f(x) = x^2 + x + 1.$$
$$f(-2) = (-2)^2 + (-2) + 1 = 3$$
$$f(-1) = (-1)^2 + (-1) + 1 = 1$$
$$f(0) = 0^2 + 0 + 1 = 1$$
$$f(1) = 1^2 + 1 + 1 = 3$$
$$f(2) = 2^2 + 2 + 1 = 7$$

Since, f is an onto function, range of f = B Co-domain.

Therefore, $B = \{1, 3, 7\}$.

1. 15 **Let f be a function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 3x + 2,\ x \in \mathbb{N}$**

(i) Find the images of 1, 2, 3

(ii) Find the pre-images of 29, 53

(iii) Identify the type of function

Sol : The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = 3x + 2$

(i) If $x = 1$, $f(1) = 3(1) + 2 = 5$

If $x = 2$, $f(2) = 3(2) + 2 = 8$

If $x = 3$, $f(3) = 3(3) + 2 = 11$

The images of 1, 2, 3 are 5, 8, 11 respectively.

(ii) If x is the pre-image of 29, then

$$f(x) = 29.$$

Hence $3x + 2 = 29 \Rightarrow 3x = 27 \Rightarrow x = 9$.

Similarly, if x is the pre-image of 53, then $f(x) = 53$. Hence $3x + 2 = 53$

$3x = 51 \Rightarrow x = 17$.

Thus the pre-images of 29 and 53 are 9 and 17 respectively.

(iii) Since different elements of $\mathbb{N}$ have different images in the co-domain, the function f is one-one function.

The co-domain of f is $\mathbb{N}$.

But the range of $f = \{5, 8, 11, 14, 17, ...\}$ is a subset of $\mathbb{N}$.

Therefore f is not an onto function. That is, f is an into function.

Thus f is one-one and into function.

1. 16 **Forensic scientists can determine the height (in cms) of a person based on the length of their thigh bone. They usually do so using the function $h(b) = 2.47b + 54.10$ where b is the length of the thigh bone.**

(i) Check if the function h is one-one

(ii) Also find the height of a person if the length of his thigh bone is 50 cms.

(iii) Find the length of the thigh bone if the height of a person is 147.96 cms.

Sol :

(i) To check if h is one - one, we assume that $h(b_1) = h(b_2)$.

Then we get,

$$2.47 b_1 + 54.10 = 2.47 b_2 + 54.10$$
$$2.47 b_1 = 2.47 b_2 \Rightarrow b_1 = b_2$$

Thus, $h(b_1) = h(b_2) \Rightarrow b_1 = b_2$.

So, the function h is one - one.

(ii) If the length of the thigh bone $b = 50$, then the height is

$h(50) = (2.47 \times 50) + 54.10 = 177.6$ cms.

(iii) If the height of a person is 147. 96 cms, then $h(b) = 147.96$ and so the length of the thigh bone is given by $2.47b + 54.10 = 147.96$.

$$b = \frac{93.86}{2.47} = 38$$

Therefore, the length of the thigh bone is 38 cms.

1. 17. **Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 3x - 5$. Find the values of a and b given that (a, 4) and (1, b) belong to f.**

Sol : $f(x) = 3x - 5$ can be written as

$f = \{(x, 3x - 5) \mid x \in \mathbb{R}\}$

(a, 4) means the image of a is 4.

That is, $f(a) = 4$

$$3a - 5 = 4$$
$$\Rightarrow a = 3$$

(1, b) means the image of 1 is b.

That is, $f(1) = b$

$$3(1) - 5 = b \Rightarrow b = -2$$

1. 18. **The distance S (in kms) travelled by a particle in time 't' hours is given by $S(t) = \frac{t^2 + t}{2}$. Find the distance travelled by the particle after**

(i) three and half hours

(ii) eight hours and fifteen minutes.

Sol : The distance travelled by the particle is given by

$$S(t) = \frac{t^2 + t}{2}$$

(i) $t = 3.5$ hours. Therefore,

$$S(3.5) = \frac{(3.5)^2 + 3.5}{2}$$
$$= \frac{15.75}{2} = 7.875$$

The distance travelled in 3. 5 hours is 7. 875 kms.

(ii) t = 8.25 hours. Therefore,

$$S(8.25) = \frac{(8.25)^2 + 8.25}{2}$$
$$= \frac{76.3125}{3}$$
$$= 38.15625$$

The distance travelled in 8.25 hours is 38.16 kms, approximately.

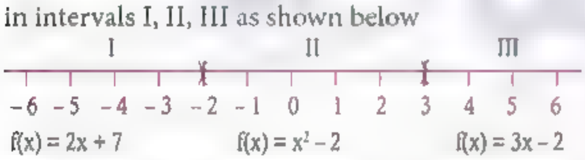
1. 19. **If the function f: $R \rightarrow R$ defined by**

$$f(x) = \begin{cases} 2x+7, & x < -2 \\ x^2 - 2, & -2 \le x < 3 \\ 3x - 2, & x \ge 3 \end{cases}$$

(i) f (4) **(ii) f (– 2)**

(iii) f (4) + 2 f (1) **(iv) $\frac{f(1) - 3f(4)}{f(-3)}$**

Sol : The function f is defined by three values in intervals I, II, III as shown below



For a given value of x = a, find out the interval at which the point a is located, there after find f (a) using the particular value defined in that interval.

(i) First, we see that, x = 4 lie in the third interval. Therefore,

$$f(x) = 3x - 2; f(4) = 3(4) - 2 = 10$$

(ii) x = – 2 lies in the second interval. Therefore,

$f(x) = x^2 - 2; f(-2) = (-2)^2 - 2 = 2$

(iii) From **(i)**, f (4) = 10.

To find f (1), first we see that x = 1 lies in the second interval.

Therefore,

$$f(x) = x^2 - 2 \Rightarrow f(1) = 1^2 - 2 = -1$$

Therefore, $f(4) + 2f(1) = 10 + 2(-1) = 8$

(iv) We know that f (1) = – 1 and f (4) = 10. For finding f (– 3), we see that x = – 3, lies in the first interval.

Therefore, $f(x) = 2x + 7$; thus,

$$f(-3) = 2(-3) + 7 = 1$$

Hence, $\frac{f(1) - 3f(4)}{f(-3)} = \frac{-1 - 3(10)}{1} = -31$

Progress Check

State True (or) False

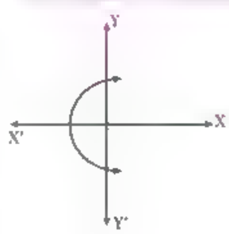
1. **All one - one functions are onto functions.**
   Ans : False
2. **There will be no one - one function from A to B when n (A) = 4, n (B) = 3.**
   Ans : True
3. **All onto functions are one - one functions.**
   Ans : False
4. **There will be no onto function from A to B when n (A) = 4, n (B) = 5.**
   Ans : True
5. **If f is a bijection from A to B, then n (A) = n (B).**
   Ans : True
6. **If n (A) = n (B), then f is a bijection from A to B.**
   Ans : False
7. **All constant functions are bijections.**
   Ans : False

Thinking Corner

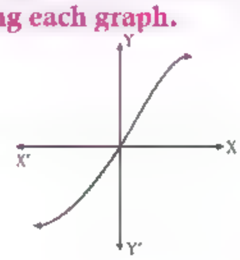
1. **Can there be a one to many function?**
   Ans : There cannot be a one to many function as the elements in Co-domain should have only one pre-image in the domain.
2. **Is an identity function one-one function?**
   Ans : Yes. It is one-to-one function.

Exercise 1. 4

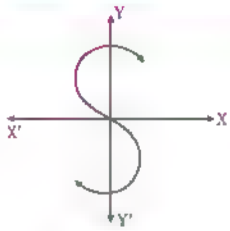
1. **Determine whether the graph given below represent functions. Give reason for your answers concerning each graph.**



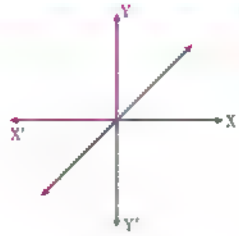
(i)



(ii)



(iii)



(iv)

Sol :

(i) It is not a function. Since, a vertical line intersects the curve in two points.

(ii) It is a function. Any vertical line drawn, will intersect the curve at only one point.

(iii) It is not a function. Vertical line intersecting the curve at two points.

(iv) It is a function. Vertical line intersects the curve at only one point.

2. Let f : $A \to B$ be a function defined by f (x) = $\frac{x}{2}-1$, where A = {2, 4, 6, 10, 12}, B = {0, 1, 2, 4, 5, 9}. Represent f by

(i) set of ordered pairs; (ii) a table;

(iii) an arrow diagram; (iv) a graph

Sol : $f: A \to B$, $f(x) = \frac{x}{2}-1$

A = {2, 4, 6, 10, 12}, B = {0, 1, 2, 4, 5, 9}

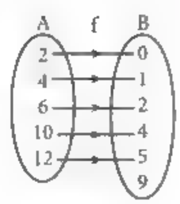
(i) Set of ordered pairs

={(2, 0), (4, 1), (6, 2), (10, 4), (12, 5)}

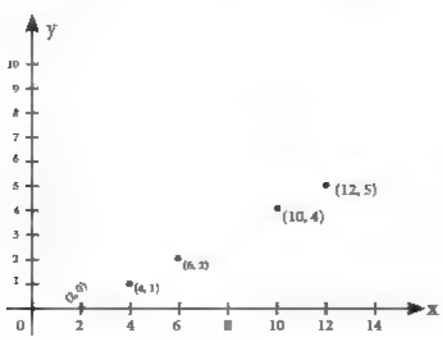
(ii) Table

x	2	4	6	10	12
f (x)	0	1	2	4	5

(iii) Arrow diagram



(iv) Graph



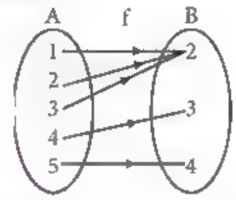
3. Represent the function f = {(1, 2), (2, 2), (3, 2), (4, 3), (5, 4)} through

(i) an arrow diagram

(ii) a table form (iii) a graph

Sol : Given function f = {(1, 2), (2, 2), (3, 2), (4, 3), (5, 4)}

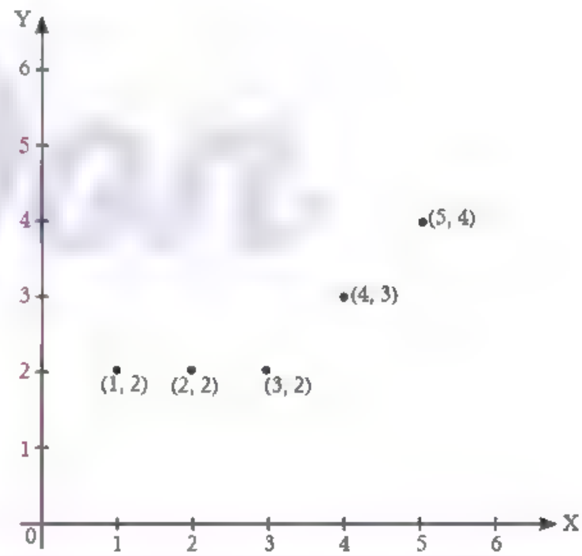
(i) Arrow diagram



(ii) Table form

x	1	2	3	4	5
f (x)	2	2	2	3	4

(iii) Graph



4. Show that the function f : $\mathbb{N} \to \mathbb{N}$ be defined by f (x) = 2x – 1 is one - one but not onto.

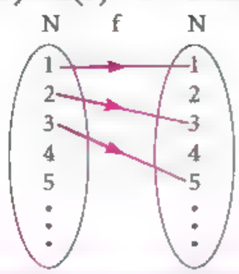
Sol : Given function f : $\mathbb{N} \to \mathbb{N}$

f (x) = 2x – 1

This function maps every element from the domain to element that is twice minus one the original. 2x – 1 is always an odd number when $x \in N$.

Clearly, each element from the domain is mapped to different element in the co-domain. So, the function is one-to-one. On the other hand, there are no elements in the domain that would map to even numbers. So, the function is not onto.

when $x = 1$, $f(1) = 2(1) - 1 = 1$
when $x = 2$, $f(2) = 2(2) - 1 = 3$
when $x = 3$, $f(3) = 2(3) - 1 = 5$ and so on.



5. Show that the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(m) = m^2 + m + 3$ is one-one function.

Sol : $f : \mathbb{N} \to \mathbb{N}$

$f(m) = m^2 + m + 3$

when $m = 1$, $f(1) = (1)^2 + 1 + 3 = 5$
when $m = 2$, $f(2) = (2)^2 + 2 + 3 = 9$
when $m = 3$, $f(3) = (3)^2 + 3 + 3 = 15$ and so on.

Clearly, A function for which every element of the range of the function corresponds to exactly one element of the domain.

$\therefore$ So, it is one-to-one function.

6. Let $A = \{1, 2, 3, 4\}$ and $B = \mathbb{N}$. Let $f : A \to B$ be defined by $f(x) = x^3$

(i) Find the range of f

(ii) Identify the type of function

Sol : $A = \{1, 2, 3, 4\}$ and $B = N$

(i) $f : A \to B$

$$f(x) = x^3$$
$$f(1) = (1)^3 = 1 \in N$$
$$f(2) = (2)^3 = 8 \in N$$
$$f(3) = (3)^3 = 27 \in N$$
$$f(4) = (4)^3 = 64 \in N$$
$$\text{Range of } f = \{1, 8, 27, 64\}$$

(ii) 'f' is a function from 'A' to 'B' and all the elements in 'A' having different images in 'B'.

$\therefore$ $f : A \to B$ is a function. It is one-to-one and into function and also called a cubic function.

7. In each of the following cases state whether the function is bijective or not. Justify your answer.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$

(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3 - 4x^2$

Sol : (i) $f : R \to R$ and $f(x) = 2x + 1$

when $x = -1$, $f(-1) = 2(-1) + 1 = -1 \in R$
when $x = 0$, $f(0) = 2(0) + 1 = 1 \in R$
when $x = 1$, $f(1) = 2(1) + 1 = 3 \in R$ and so on.

For every value of $x \in R$, $f(x)$ also $\in R$.

$\therefore$ The function is well defined and it is one-to-one function (Injective)

For $f(x) : R \to R$, the domain and range are also well defined. So it is an onto function. (Surjective)

Thus, the function is one -to -one onto i.e. Bijective function.

(ii) $f : R \to R$, $f(x) = 3 - 4x^2$

when $x = 0$, $f(0) = 3 - 4(0) = 3 \in R$
when $x = 1$, $f(1) = 3 - 4(1)^2 = -1 \in R$
when $x = 2$, $f(2) = 3 - 4(2)^2 = -13 \in R$
when $x = -1$, $f(-1) = 3 - 4(-1)^2 = -1 \in R$
when $x = -2$, $f(-2) = 3 - 4(-2)^2 = -13 \in R$

From this, it is clear that two or more elements having same image in the co-domain. So, it is not one-to-one and it is many-to-one function. Hence it is not Bijective.

8. Let $A = \{-1, 1\}$ and $B = \{0, 2\}$. If the function $f : A \to B$ defined by $f(x) = ax + b$. Is an onto function? Find a and b.

Sol : $A = \{-1, 1\}$, $B = \{0, 2\}$

$f : A \to B$, $f(x) = ax + b$

when $x = -1$, $f(-1) = 0$
when $x = 1$, $f(1) = 2$

Since, there is a constant difference between x and f (x), it is an onto function.

Substituting the values, we get

$$-a + b = 0$$
$$a + b = 2$$

Solving these equations, we get $a = 1$, $b = 1$

9. If the function f is defined by

$$f(x) = \begin{cases} x+2 & \text{if } x > 1 \\ 2 & \text{if } -1 \le x \le 1 \\ x-1 & \text{if } -3 < x < -1 \end{cases}$$ **; Find the values of**

(i) f (3) **(ii) f (0)**

(iii) f (– 1.5) **(iv) f (2) + f (– 2)**

Sol :

$$f(x) = \begin{cases} x+2 & \text{if } x > 1 \\ 2 & \text{if } -1 \le x \le 1 \\ x-1 & \text{if } -3 < x < -1 \end{cases}$$

(i) $f(3) = 3 + 2 = 5$ $[\because 3 > 1]$

(ii) $f(0) = 2$ $[\because -1 \le 0 \le 1]$

(iii) $f(-1.5) = -1.5 - 1 = -2.5$ $[\because -3 < -1.5 < -1]$

(iv) $f(2) + f(-2) = (2 + 2) + (-2 - 1)$
$= 4 - 3 = 1$ $[\because 2 > 1 \text{ and } -3 < -2 < -1]$

10. A function $f : [-5, 9] \to \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} 6x+1 & if\ -5 \le x < 2 \\ 5x^2 - 1 & if\ \ 2 \le x < 6 \\ 3x - 4 & if\ \ 6 \le x \le 9 \end{cases}$$

Find (i) $f(-3) + f(2)$ (ii) $f(7) - f(1)$

(iii) $2f(4) + f(8)$ (iv) $\frac{2f(-2) - f(6)}{f(4) + f(-2)}$

Sol : $f : [-5, 9] \to R$

$$f(x) = \begin{cases} 6x+1 & if\ -5 \le x < 2 \\ 5x^2 - 1 & if\ \ 2 \le x < 6 \\ 3x - 4 & if\ \ 6 \le x \le 9 \end{cases}$$

(i) $f(-3) + f(2) = [6(-3) + 1] + [5(2)^2 - 1]$

$= (-18 + 1) + (20 - 1)$

$= -17 + 19 = 2$ $[\because -5 \le -3 < 2$, $2 \le 2 < 6]$

(ii) $f(7) - f(1) = [3(7) - 4] - [6(1) + 1]$

$= (21 - 4) - (6 + 1)$

$= 17 - 7 = 10$ $[\because 6 \le 7 \le 9$, $-5 \le 1 < 2]$

(iii) $2f(4) + f(8) = 2[5(4)^2 - 1] + [3(8) - 4]$

$= 2[80 - 1] + [24 - 4]$

$= 158 + 20 = 178$ $[\because 2 \le 4 < 6$, $6 \le 8 \le 9]$

(iv) $\frac{2f(-2) - f(6)}{f(4) + f(-2)}$

$f(-2) = 6(-2) + 1 = -12 + 1 = -11$ $[\because -5 \le -2 < 2]$

$f(6) = 3(6) - 4 = 18 - 4 = 14$ $[\because 6 \le 6 < 9]$

$f(4) = 5(4)^2 - 1 = 80 - 1 = 79$ $[\because 2 \le 4 < 6]$

$f(-2) = -11$

$$\therefore \frac{2f(-2) - f(6)}{f(4) + f(-2)} = \frac{2(-11) - 14}{79 - 11}$$

$$= \frac{-22 - 14}{68} = -\frac{36}{68} = -\frac{9}{17}$$

11. The distance S an object travels under the influence of gravity in time t seconds is given by $S(t) = \frac{1}{2}gt^2 + at + b$, where, (g is the acceleration due to gravity), a, b are constants. Check if the function S (t) is one-one.

Sol : Distance travelled by an object is given to be $S(t) = \frac{1}{2}gt^2 + at + b$

'g'- acceleration due to gravity is a constant.

'g' is the acceleration due to gravity 'a' and 'b' are constants.

't' is a variable. (t - time)

At different values of 't' , S(t) is having different values. (images in codomain) clearly $f : t \to S(t)$ is one-to-one function.

Given $S(t) = \frac{1}{2}gt^2 + at + b$ (a, b are constants)

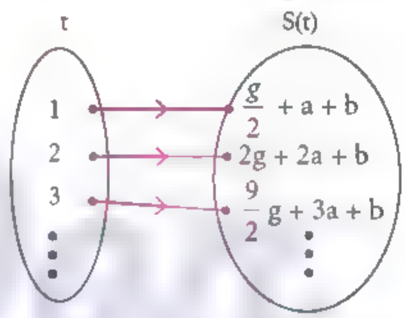
t = time in seconds

Let us take t = 1, 2, 3 ..... Seconds

When $t = 1 \Rightarrow S(1) = \frac{g}{2} + a + b$

When $t = 2 \Rightarrow S(2) = 2g + 2a + b$

When $t = 3 \Rightarrow S(3) = \frac{9}{2}g + 3a + b$ and so on.



All the elements of t, having different images in S(t). Hence it is an One-to-one function.

12. The function 't' which maps temperature in Celsius (C) into temperature in Fahrenheit (F) is defined by t (C) = F where $F = \frac{9C}{5} + 32$. Find,

(i) t (0) **(ii) t (28)**

(iii) t (– 10)

(iv) the value of C when t (C) = 212

(v) the temperature when the Celsius value is equal to the Fahrenheit value.

Sol :

Given t (C) = F where $F = \frac{9C}{5} + 32$.

C - Celsius, F - Fahrenheit

$$\therefore t(C) = \frac{9C}{5} + 32$$

(i) $t(0) = \frac{9(0)}{5} + 32 = 0 + 32 = 32\,°F$

(ii) $t(28) = \frac{9(28)}{5} + 32 = \frac{252}{5} + 32$

$= 50.4 + 32 = 82.4°F$

(iii) $t(-10) = \frac{9(-10)}{5} + 32 = -18 + 32 = 14°F$

(iv) Given t (C) = 212

$$\therefore \frac{9C}{5}+32 = 212 \Rightarrow \frac{9C}{5} = 212-32$$

$$C = 180 \times \frac{5}{9} = 100\,^{\circ}C$$

(v) The temperature when the Celsius value is equal to the Fahrenheit value.

$$\therefore F = C$$

$$\frac{9C}{5}+32 = C$$

$$\frac{9C}{5} - C = -32 \Rightarrow \frac{9C-5C}{5} = -32$$

$$4C = -32 \times 5 \Rightarrow C = -\frac{160}{4}$$

$$^{\circ}C = -40$$

COMPOSITION OF FUNCTIONS

Key Points

- Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$, then their sum $f + g$ i.e., $(f + g) : X \to R$ is the function defined by $(f + g)(x) = f(x) + g(x)$, for all $x \in X$.
- Their difference i.e., $(f - g) : X \to R$ is the function defined by $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.
- The multiplication (αf) is a function from $X \to R$ defined by a scalar by $(\alpha f)(x) = \alpha\, f(x)$, $x \in X$.
- The product $fg : X \to R$ is defined by $(fg)(x) = f(x).g(x)$, for all $x \in X$.
- The quotient $\frac{f}{g}$ is a function defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in X$.
- The composition of two functions f and g is denoted by $f \circ g$ and $(f \circ g)(x) = f[g(x)]$.
- Composition of functions is not always commutative i.e., $f \circ g \neq g \circ f$.
- Composition of functions is always associative.

Worked Examples

1. 20. **Find $f \circ g$ and $g \circ f$ when $f(x) = 2x + 1$ and $g(x) = x^2 - 2$.**

Sol :

$$f(x) = 2x + 1,\ g(x) = x^2 - 2$$

$$(f \circ g)(x) = f(g(x)) = f(x^2 - 2)$$
$$= 2(x^2 - 2) + 1 = 2x^2 - 3$$

$$(g \circ f)(x) = g(f(x)) = g(2x + 1)$$
$$= (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

Thus $f \circ g = 2x^2 - 3$, $g \circ f = 4x^2 + 4x - 1$.

From the above, we see that $f \circ g \neq g \circ f$.

1. 21. **Represent the function $f(x) = \sqrt{2x^2 - 5x + 3}$ as a composition of two functions.**

Sol :

We set $f_2(x) = 2x^2 - 5x + 3$ and $f_1(x) = \sqrt{x}$

Then, $f(x) = \sqrt{2x^2 - 5x + 3}$

$$= \sqrt{f_2(x)}$$
$$= f_1[f_2(x)] = f_1 f_2(x)$$

1.22. **If $f(x) = 3x - 2$, $g(x) = 2x + k$ and $f \circ g = g \circ f$, then find the value of k.**

Sol :

$$f(x) = 3x - 2,\ g(x) = 2x + k$$

$$(f \circ g)(x) = f(g(x)) = f(2x + k)$$
$$= 3(2x + k) - 2 = 6x + 3k - 2$$

Thus, $(f \circ g)(x) = 6x + 3k - 2.$

$$(g \circ f)(x) = g(3x - 2) = 2(3x - 2) + k$$

Thus, $(g \circ f)(x) = 6x - 4 + k.$

Given that $f \circ g = g \circ f$

Therefore, $6x + 3k - 2 = 6x - 4 + k$

$$6x - 6x + 3k - k = -4 + 2 \Rightarrow k = -1$$

1. 23. **Find k if $f \circ f(k) = 5$ where $f(k) = 2k - 1$.**

Sol :

$$f \circ f(k) = f(f(k)) = f(2k - 1)$$

$= 2(2k-1)-1 = 4k-3$

Thus, $f \circ f(k) = 4k-3$

But, it is given that $f \circ f(k) = 5$

Therefore $4k-3 = 5 \Rightarrow k = 2.$

1.24. **If f (x) = 2x + 3, g (x) = 1 – 2x and h (x) = 3x. Prove that $f \circ (g \circ h) = (f \circ g) \circ h$.**

Sol :

$f(x) = 2x+3,\ g(x) = 1-2x,\ h(x) = 3x$

Now, $(f \circ g)(x) = f(g(x)) = f(1-2x)$
$= 2(1-2x)+3 = 5-4x$

Since, $(f \circ g) \circ h(x) = (f \circ g)(h(x))$
$= (f \circ g)(3x)$
$= 5-4(3x) = 5-12x$ ... (1)

$(g \circ h)(x) = g(h(x)) = g(3x)$
$= 1-2(3x) = 1-6x$

Since, $f \circ (g \circ h)(x) = f(1-6x) = 2(1-6x)+3$
$= 5-12x$ ... (2)

From (1) and (2), we get

$(f \circ g) \circ h = f \circ (g \circ h)$

1.25. **Find x if gff (x) = fgg (x), given f (x) = 3x + 1 and g (x) = x + 3.**

Sol : $gff(x) = g[f\{f(x)\}]$

(This means "g of f of f of x")

$= g[f(3x+1)] = g[3(3x+1)+1]$
$= g(9x+4) = [(9x+4)+3] = 9x+7$

$fgg(x) = f[g\{g(x)\}]$

(This means "f of g of g of x")

$= f[g(x+3)] = f[(x+3)+3]$

$f(x+6) = [3(x+6)+1] = 3x+19$

These two quantities being equal, we get 9x + 7 = 3x + 19. Solving this equation we obtain x = 2.

Progress Check

State your answer for the following questions by selecting the correct option.

1. **Composition of functions is commutative**
   (a) Always true (b) Never true
   (c) Sometimes true

   Ans : (c) Sometimes true

2. **Composition of functions is associative**
   (a) Always true (b) Never true
   (c) Sometimes true

   Ans : (a) Always true

3. **Is a constant function a linear function?**

   Ans : Yes

4. **Is quadratic function a one-one function?**

   Ans : No

5. **Is cubic function a one -one function?**

   Ans : Yes

6. **Is the reciprocal function a bijection?**

   Ans : Yes

7. **Is f : $A \to B$ is a constant function, then the range of f will have _____ elements.**

   Ans : One element

Thinking Corner

1. **If f (x) = x^m and g (x) = x^n does $f \circ g = g \circ f$?**

   Ans : $f(x) = x^m,\ g(x) = x^n$

   $f \circ g = f[g(x)] = f(x^n) = (x^n)^m = x^{nm}$

   $g \circ f = g[f(x)] = g[x^m] = (x^m)^n = x^{mn}$

   $\therefore f \circ g = g \circ f$

Exercise 1.5

1. Using the functions f and g given below, find $f \circ g$ and $g \circ f$. Check whether $f \circ g = g \circ f$.

(i) $f(x) = x-6,\ g(x) = x^2$

(ii) $f(x) = \frac{2}{x},\ g(x) = 2x^2-1$

(iii) $f(x) = \frac{x+6}{3},\ g(x) = 3-x$

(iv) $f(x) = 3+x,\ g(x) = x-4$

(v) $f(x) = 4x^2-1,\ g(x) = 1+x$

Sol :

(i) $f(x) = x-6,\ g(x) = x^2$

$f \circ g = f[g(x)] = f[x^2] = x^2-6$

$g \circ f = g[f(x)] = g[x-6] = (x-6)^2$
$= x^2-12x+36$

$\therefore f \circ g \neq g \circ f$

(ii) $f(x) = \frac{2}{x},\ g(x) = 2x^2-1$

$f \circ g = f[g(x)] = f[2x^2-1] = \frac{2}{2x^2-1}$

$g \circ f = g[f(x)] = g\left[\frac{2}{x}\right] = 2\left(\frac{2}{x}\right)^2 - 1$

$= 2\left(\frac{4}{x^2}\right)-1 = \frac{8-x^2}{x^2}$

$\therefore f \circ g \neq g \circ f$

(iii) $f(x) = \frac{x+6}{3}, g(x) = 3 - x$

$f \circ g = f[g(x)] = f(3-x) = \frac{3-x+6}{3} = \frac{9-x}{3}$

$g \circ f = g[f(x)] = g\left(\frac{x+6}{3}\right) = 3 - \left(\frac{x+6}{3}\right)$

$= \frac{9-x-6}{3} = \frac{3-x}{3} \therefore f \circ g \neq g \circ f$

(iv) $f(x) = 3 + x, g(x) = x - 4$

$f \circ g = f[g(x)] = f[x-4] = 3+x-4 = x-1$

$g \circ f = g[f(x)] = g[3+x] = 3+x-4 = x-1$

$\therefore f \circ g = g \circ f$

(v) $f(x) = 4x^2 - 1, g(x) = 1 + x$

$f \circ g = f[g(x)] = f(1+x) = 4(1+x)^2 - 1$

$= 4(1 + 2x + x^2) - 1$

$= 4 + 8x + 4x^2 - 1$

$= 4x^2 + 8x + 3$

$g \circ f = g[f(x)] = 1 + 4x^2 - 1$

$= 4x^2$

$\therefore f \circ g \neq g \circ f$

2. Find the value of k, such that $f \circ g = g \circ f$.

(i) $f(x) = 3x + 2, g(x) = 6x - k$

(ii) $f(x) = 2x - k, g(x) = 4x + 5$

Sol :

(i) $f(x) = 3x + 2, g(x) = 6x - k$

$f \circ g = f[g(x)] = f(6x - k)$

$= 3(6x - k) + 2$

$= 18x - 3k + 2$

$g \circ f = g[f(x)] = 6(3x + 2) - k$

$= 18x + 12 - k$

Given $f \circ g = g \circ f$

$\therefore 18x - 3k + 2 = 18x + 12 - k$

$3k - k = -12 + 2$

$2k = -10$

$k = -5$

(ii) $f(x) = 2x - k, g(x) = 4x + 5$

$f \circ g = f[g(x)] = f[4x + 5]$

$= 2(4x + 5) - k$

$= 8x + 10 - k$

$g \circ f = g[f(x)] = g(2x - k) = 4(2x - k) + 5$

$= 8x - 4k + 5$

Given $f \circ g = g \circ f$

$\therefore 8x + 10 - k = 8x - 4k + 5$

$4k - k = 5 - 10$

$3k = -5$

$k = -5/3$

3. If $f(x) = 2x - 1, g(x) = \frac{x+1}{2}$, show that $f \circ g = g \circ f = x$.

Sol : $f(x) = 2x - 1, g(x) = \frac{x+1}{2}$

$f \circ g = f[g(x)] = f\left(\frac{x+1}{2}\right) = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x$

$g \circ f = g[f(x)] = g(2x - 1) = \frac{2x-1+1}{2}$

$= \frac{2x}{2} = x$

$\therefore f \circ g = g \circ f = x$ Hence proved.

4. (i) If $f(x) = x^2 - 1, g(x) = x - 2$ find a, if $g \circ f(a) = 1$.

(ii) Find k, if $f(k) = 2k - 1$ and $f \circ f(k) = 5$

Sol :

(i) $f(x) = x^2 - 1, g(x) = x - 2$

$f(a) = a^2 - 1$

Given, $(g \circ f)(a) = 1$

$g[f(a)] = 1$

$g[a^2 - 1] = 1$

$a^2 - 1 - 2 = 1$

$a^2 - 3 = 1$

$a^2 = 1 + 3 = 4$

$a = \pm 2$

(ii) $f(k) = 2k - 1$

Given $(f \circ f)(k) = 5$

$f[f(k)] = 1$

$f[2k - 1] = 1$

$\therefore 2(2k - 1) - 1 = 5$

$4k - 2 - 1 = 5$

$4k - 3 = 5$

$4k = 5 + 3 = 8$

$k = 8/4 = 2$

5. Let $A, B, C \subseteq \mathbb{N}$ and a function $f : A \to B$ be defined by $f(x) = 2x + 1$ and $g : B \to C$ be defined by $g(x) = x^2$. Find the range of $f \circ g$ and $g \circ f$.

Sol : $A, B, C \subseteq \mathbb{N}$

$f : A \to B$ defined by $f(x) = 2x + 1$

$g : B \to C$ defined by $g(x) = x^2$

$f \circ g = f[g(x)] = f(x^2) = 2x^2 + 1$

$g \circ f = g[f(x)] = g(2x - 1) = (2x + 1)^2$

$= 4x^2 + 4x + 1$

'x' can take any real value and can produce any real value. Thus, the domain and range of $f \circ g$ and $g \circ f$ is R (Set of real numbers).

6. If $f(x) = x^2 - 1$. Find (a) $f \circ f$ (b) $f \circ f \circ f$

Sol : $f(x) = x^2 - 1$

(a) $f \circ f = f[f(x)] = f(x^2 - 1)$
$= (x^2 - 1)^2 - 1 = x^4 - 2x^2 + 1 - 1 = x^4 - 2x^2$

(b) $f \circ f \circ f = f[f[f(x)]] = f[f(x^2 - 1)] = f[x^4 - 2x^2]$
$= (x^4 - 2x^2)^2 - 1 = x^8 - 4x^6 + 4x^4 - 1$

7. If $f : R \to R$ and $g : R \to R$ are defined by $f(x) = x^5$ and $g(x) = x^4$ then check if f, g are one - one and $f \circ g$ is one - one.

Sol :

$f(x) = x^5, g(x) = x^4$

$f(x) = x^5$

For any value of 'x', f (x) gives us a different value (image) in co domain.

$\therefore$ f (x) is one-one function

$g(1) = 1; \quad g(-1) = 1$

Hence g(x) is not one-one function

$f \circ g = f[g(x)] = f(x^4) = (x^4)^5 = x^{20}$

$f \circ g$ is also One - One function as x is mapped with different value of $f \circ g$.

8. Consider the function f (x), g (x), h (x) as given below. Show that $(f \circ g) \circ h = f \circ (g \circ h)$ in each case.

(i) $f(x) = x - 1$, $g(x) = 3x + 1$ and $h(x) = x^2$

(ii) $f(x) = x^2$, $g(x) = 2x$ and $h(x) = x + 4$

(iii) $f(x) = x - 4$, $g(x) = x^2$ and $h(x) = 3x - 5$

Sol :

(i) $f(x) = x - 1, g(x) = 3x + 1, h(x) = x^2$

$f \circ g = f[g(x)] = f[3x + 1]$
$= (3x + 1) - 1 = 3x$

$(f \circ g) \circ h = (f \circ g)[h(x)] = (f \circ g)[x^2] = 3x^2$

$g \circ h = g[h(x)] = g[x^2] = 3x^2 + 1$

$f \circ (g \circ h) = f[g(h(x))] = f[g(x^2)] = f[3x^2 + 1]$
$= (3x^2 + 1) - 1 = 3x^2$

$\therefore (f \circ g) \circ h = f \circ (g \circ h)$

Hence proved.

(ii) $f(x) = x^2, g(x) = 2x, h(x) = x + 4$

$f \circ g = f[g(x)] = f[2x]$
$= f(2x) = (2x)^2 = 4x^2$

$(f \circ g) \circ h = (f \circ g)[h(x)] = (f \circ g)[x + 4]$
$= 4(x + 4)^2$
$= 4(x^2 + 8x + 16) = 4x^2 + 32x + 64$

$g \circ h = g[h(x)] = g[x + 4] = 2(x + 4)$

$f \circ (g \circ h) = f[g(h(x))] = f[g(x + 4)]$
$= f[2(x + 4)] = [2(x + 4)]^2 = 4(x + 4)^2$
$= 4(x^2 + 8x + 16) = 4x^2 + 32x + 64$

$\therefore (f \circ g) \circ h = f \circ (g \circ h)$

(iii) $f(x) = x - 4, g(x) = x^2, h(x) = 3x - 5$

$f \circ g = f[g(x)] = f(x^2) = x^2 - 4$

$(f \circ g) \circ h = (f \circ g)[h(x)] = (f \circ g)[3x - 5]$
$= (3x - 5)^2 - 4 = 9x^2 - 30x + 25 - 4$
$= 9x^2 - 30x + 21$

$g \circ h = g[h(x)]$
$= g(3x - 5) = (3x - 5)^2 = 9x^2 - 30x + 25$

$f \circ (g \circ h) = f[g(h(x))] = f[g(3x - 5)] = f[(3x - 5)^2]$
$= 9x^2 - 30x + 25 - 4 = 9x^2 - 30x + 21$

$\therefore (f \circ g) \circ h = f \circ (g \circ h)$

9. Let $f = \{(-1, 3), (0, -1), (2, -9)\}$ be a linear function from $\mathbb{Z}$ into $\mathbb{Z}$. Find f (x).

Sol : $f = \{(-1, 3), (0, -1), (2, -9)\}$, $f : \mathbb{Z} \to \mathbb{Z}$

Since 'f' is a linear function

$f(x) = ax + b$

in $(0, -1)$, when $x = 0, f(0) = -1$

$\therefore \quad a(0) + b = -1 \Rightarrow b = -1$

in $(-1, 3)$, when $x = -1, f(-1) = 3$

$\therefore \quad a(-1) + b = 3 \Rightarrow -a - 1 = 3$

$-a = 4 \Rightarrow a = -4$

$\therefore \quad f(x) = -4x - 1$

10. In electrical circuit theory, a circuit C(t) is called a linear circuit if it satisfies the superposition principle given by $C(at_1 + bt_2) = aC(t_1) + bC(t_2)$, where a, b are constants. Show that the circuit $C(t) = 3t$ is linear.

Sol : Given superposition principle is

$C(at_1 + bt_2) = aC(t_1) + bC(t_2)$ a, b are constants

If all the independent sources except for $C(t_1)$ have known fixed values, then

$C(t) = aC(t_1) + d$

where $d = bC(t_2)$

$\therefore$ C (t) is linear.

Aliter:

C(t) is linear and $t = t_1 + t_2$

Let $C(t_1) = t$ and $C(t_2) = 2t$

By Given data,

$C(at_1 + bt_2) = aC(t_1) + bC(t_2)$ ... (1)

Now $C(t) = C(t_1 + t_2)$ $[\because t = t_1 + t_2]$
$= C(t_1) + C(t_2)$ from (1)
$= t + 2t = 3t$

Hence the function C(t) is linear.

Exercise 1. 6

Multiple Choice Questions:

1. If n (A × B) = 6 and A = {1, 3} then n (B) is

(1) 1 (2) 2
(3) 3 (4) 6

[Ans: (3)]

Sol :

$$n(A \times B) = 6$$
$$A = \{1, 3\} \Rightarrow n(A) = 2$$
$$n(B) = \frac{n(A \times B)}{n(A)} = \frac{6}{2} = 3$$

2. A = {a, b, p}, B = {2, 3}, C = {p, q, r, s} then $n[(A \cup C) \times B]$ is

(1) 8 (2) 20
(3) 12 (4) 16

[Ans: (3)]

Sol:

A = {a, b, p}, B = {2, 3}, C = {p, q, r, s}

$$A \cup C = \{a, b, p, q, r, s\}$$
$$n(A \cup C) = 6$$
$$n(B) = 2$$
$$\therefore \; n[(A \cup C) \times B] = 6 \times 2 = 12$$

3. If A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8} then state which of the following statement is true.

(1) $(A \times C) \subset (B \times D)$ (2) $(B \times D) \subset (A \times C)$
(3) $(A \times B) \subset (A \times D)$ (4) $(D \times A) \subset (B \times A)$

[Ans: (1)]

Sol : A = {1, 2}, B = {1, 2, 3, 4},
C = {5, 6} and D = {5, 6, 7, 8}

$A \times C = \{(1, 5), (1, 6), (2, 5), (2, 6)\}$

$B \times D = \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$

Hence $(A \times C) \subset (B \times D)$

4. If there are 1024 relations from a set A = {1, 2, 3, 4, 5} to a set B, then the number of elements in B is

(1) 3 (2) 2
(3) 4 (4) 8

[Ans: (2)]

Sol : A = {1, 2, 3, 4, 5}, n(A) = 5, n(B) = ?

Number of relations from A to B is 1024.

i.e., $2^{mn} = 1024$ where 'mn' is number of elements in A × B.

$$2^{mn} = 2^{10} \Rightarrow mn = 10$$
$$n(B) = \frac{n(A \times B)}{n(A)} = \frac{10}{5} = 2$$

5. The range of the relation R = {(x, x^2) | x is a prime number less than 13} is

(1) {2, 3, 5, 7}
(2) {2, 3, 5, 7, 11}
(3) {4, 9, 25, 49, 121}
(4) {1, 4, 9, 25, 49, 121}

[Ans: (3)]

Sol : Set of prime numbers less than 13 is {2, 3, 5, 7, 11}

Relation R = {(x, x^2)/ x is a prime number less than 13}

R = {(2, 4), (3, 9), (5, 25), (7, 49), (11, 121)}

∴ Range = {(4, 9, 25, 49, 121}

6. If the ordered pairs (a + 2, 4) and (5, 2a + b) are equal then (a, b) is

(1) (2, – 2) (2) (5, 1)
(3) (2, 3) (4) (3, – 2)

[Ans: (4)]

Sol :

Given (a + 2, 4) = (5, 2a + b)

a + 2 = 5	2a + b = 4
a = 5 – 2	2(3) + b = 4
= 3	b = 4 – 6 = – 2

7. Let n (A) = m and n (B) = n then the total number of non-empty relations that can be defined from A to B is

(1) m^n (2) n^m
(3) $2^{mn} - 1$ (4) 2^{mn}

[Ans: (4)]

8. If {(a, 8), (6, b)} represents an identity function, then the value of a and b are respectively

(1) (8, 6) (2) (8, 8)
(3) (6, 8) (4) (6, 6)

[Ans: (1)]

Sol : Given {(a, 8), (6, b)} is an Identity function

i.e., {(8, 8), (6, 6)}

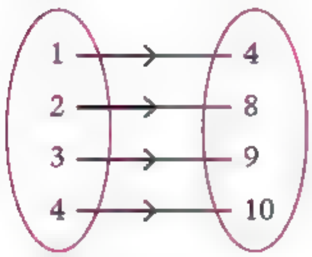
∴ a = 8, b = 6

9. Let A = {1, 2, 3, 4} and B = {4, 8, 9, 10}. A function f : $A \to B$ given by f = {(1, 4), (2, 8), (3, 9), (4, 10)} is a

(1) Many-one function
(2) Identity function
(3) One-to-one function
(4) Into function **[Ans: (3)]**

Sol : A = {1, 2, 3, 4}, B = {4, 8, 9, 10}

A R B



f : A → B

and f = {(1, 4), (2, 8), (3, 9), (4, 10)}

One-to-one function

10. If f (x) = 2x² and g (x) = $\frac{1}{3x}$, Then $f \circ g$ is

(1) $\frac{3}{2x^2}$ (2) $\frac{2}{3x^2}$
(3) $\frac{2}{9x^2}$ (4) $\frac{1}{6x^2}$ **[Ans: (3)]**

Sol :

$$f(x) = 2x^2, \; g(x) = 1/3x$$

$$f \circ g = f[g(x)] = f\left[\frac{1}{3x}\right]$$

$$= 2\left(\frac{1}{3x}\right)^2 = 2\left(\frac{1}{9x^2}\right)$$

$$= \left(\frac{2}{9x^2}\right)$$

11. If f : $A \to B$ is a bijective function and if n (B) = 7, then n (A) is equal to

(1) 7 (2) 49
(3) 1 (4) 14

[Ans: (1)]

Sol :

f : A → B is a bijective function and n(B) = 7.
f is a bijective function i.e., one-to-one onto function.
∴ A and B should have equal number of elements.

n(A) = 7

12. Let f and g be two functions given by f = {(0, 1), (2, 0), (3, – 4), (4, 2), (5, 7)} g = {(0, 2), (1, 0), (2, 4), (– 4, 2), (7, 0)} then the range of $f \circ g$ is

(1) {0, 2, 3, 4, 5} (2) {– 4, 1, 0, 2, 7}
(3) {1, 2, 3, 4, 5} (4) {0, 1, 2} **[Ans: (4)]**

Sol : f = {(0, 1), (2, 0), (3, – 4), (4, 2), (5, 7)}
g = {(0, 2), (1, 0), (2, 4), (– 4, 2), (7, 0)}

Now, f o g(0) = f[2] = 0
f o g(1) = f[0] = 1
f o g(2) = f[4] = 2
f o g(– 4) = f[2] = 0
f o g(7) = f[0] = 1

Range of 'f o g' = {0, 1, 2}

13. Let f (x) = $\sqrt{1+x^2}$ then

(1) f (xy) = f (x). f (y)
(2) $f(xy) \geq f(x).f(y)$
(3) $f(xy) \leq f(x).f(y)$
(4) None of these **[Ans: (3)]**

Sol :

$$f(x) = \sqrt{1+x^2}$$

then $f(y) = \sqrt{1+y^2}$

and $f(xy) = \sqrt{1+x^2y^2}$

Hence, it is clear that f(xy) ≤ f(x) f(y)

14. If g = {(1, 1), (2, 3), (3, 5), (4, 7)} is a function given by g (x) = $\alpha x + \beta$ then the values of α *and* β are

(1) (– 1, 2) (2) (2, – 1)
(3) (– 1, – 2) (4) (1, 2) **[Ans: (2)]**

Sol :

g = {(1, 1), (2, 3), (3, 5), (4, 7)}

$$g(x) = \alpha x + \beta$$

When α = 2 and β = – 1, g(x) = 2x – 1
Which is satisfying 'g'.

15. f (x) = (x + 1)³ – (x – 1)³ represents

(1) a linear function
(2) a cubic function
(3) a reciprocal function
(4) a quadratic function **[Ans: (4)]**

Sol :

$$f(x) = (x+1)^3 - (x-1)^3$$

$$= (x^3 + 3x^2 + 3x + 1) - (x^3 - 3x^2 + 3x - 1)$$

$$= 6x^2 + 2$$

UNIT EXERCISE - 1

1. If the ordered pairs $(x^2 - 3x, y^2 + 4y)$ and $(-2, 5)$ are equal, then find x and y.

Sol :

Given $(x^2 - 3x, y^2 + 4y) = (-2, 5)$

$\therefore\ x^2 - 3x = -2$ | $y^2 + 4y = 5$

$x^2 - 3x + 2 = 0$ | $y^2 + 4y - 5 = 0$

$(x-1)(x-2) = 0$ | $(y+5)(y-1) = 0$

$x = 1, 2$ | $y = -5, 1$

2. The Cartesian product A × A has 9 elements among which (– 1, 0) and (0, 1) are found. Find the set A and the remaining elements of A × A.

Sol : Since A × A has 9 elements ,

A would have 3 elements $(\because 3 \times 3 = 9)$

A × A contains (– 1, 0) and (0, 1)

$\therefore\ -1, 0 \in A$ ... (1)

Similarly (0, 1) is in A × A

So, $0, 1 \in A$ ... (2)

From (1) and (2) $\Rightarrow\ -1, 0, 1 \in A$

$\therefore\quad A = \{-1, 0, 1\}$

$\therefore\ A \times A = \{-1, 0, 1\} \times \{-1, 0, 1\}$

$= \{(-1,-1), (-1,0), (-1,1), (0,-1), (0,0), (0,1), (1,-1), (1,0), (1,1)\}$

$\therefore$ The remaining elements of A × A are

$\{(-1,-1), (-1,1), (0,-1), (0,0), (1,-1), (1,0), (1,1)\}$

3. Given that $f(x) = \begin{Bmatrix} \sqrt{x-1} & x \geq 1 \\ 4 & x < 1 \end{Bmatrix}$. Find

(i) f (0) **(ii) f (3)**

(iii) f (a + 1) in terms of a. (Given that $a \geq 0$)

Sol : $f(x) = \begin{Bmatrix} \sqrt{x-1} & x \geq 1 \\ 4 & x < 1 \end{Bmatrix}$

(i) $f(0) = 4$ $[\because 0 < 1$

(ii) $f(3) = \sqrt{3-1} = \sqrt{2}$ $[\because 3 \geq 1$

(iii) $f(a+1) = \sqrt{a+1-1} = \sqrt{a}$ $[\because \textit{Given } a \geq 0$

$[\because a + 1 \geq 1$

4. Let A = {9, 10, 11, 12, 13, 14, 15, 16, 17} and let $f : A \to N$ be defined by f (n) = the highest prime factor of $n \in A$. Write f as a set of ordered pairs and find the range of f.

Sol : A = {9, 10, 11, 12, 13, 14, 15, 16, 17}

$f : A \to N$, f (n)= the highest prime factor of $n \in A$

$\therefore\ f(9) = 3,\quad f(10) = 5,\quad f(11) = 11$

$f(12) = 3,\quad f(13) = 13,\quad f(14) = 7$

$f(15) = 5,\quad f(16) = 2,\quad f(17) = 17$

Set of ordered pairs

f ={(9, 3), (10, 5), (11, 11), (12, 3), (13, 13), (14, 7), (15, 5), (16, 2), (17, 17)

Range ={2, 3, 5, 7, 11, 13, 17}

5. Find the domain of the function

$f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$

Sol : $f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$

$f(x) = \sqrt{1-t}$

where $t = \sqrt{1-\sqrt{1-x^2}}$

$1 - t \geq 0$

$t \leq 1$

$\sqrt{1-\sqrt{1-x^2}} \leq 1$

Squaring $1 - \sqrt{1-x^2} \leq 1$

$-\sqrt{1-x^2} \leq 0$

$\sqrt{1-x^2} \geq 0$

$1 - x^2 \geq 0$

$x^2 \leq 1$

$\Rightarrow\ x \in [-1, 1]$ i.e., {– 1, 0, 1}

6. If f (x) = x^2, g (x) = 3x and h (x) = x – 2. Prove that $(f \circ g) \circ h = f \circ (g \circ h)$.

Sol : $f(x) = x^2, g(x) = 3x, h(x) = x - 2$

$f \circ g = f[g(x)] = f(3x)$

$= (3x)^2 = 9x^2$

$(f \circ g) \circ h = (f \circ g)[h(x)] = (f \circ g)[x-2]$

$= 9(x-2)^2$

$g \circ h = g[h(x)] = g[x-2] = 3(x-2)$

$f \circ (g \circ h) = f[g(h(x))]$

$= f[3(x-2)] = [3(x-2)]^2 = 9(x-2)^2$

$\therefore\quad (f \circ g) \circ h = f \circ (g \circ h)$

Hence proved.

7. A = {1, 2} and B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}. Verify that whether A × C is a subset of B × D?

Sol :

A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6}, D = {5, 6, 7, 8}

$A \times C = \{1, 2\} \times \{5, 6\}$

$= \{(1,5), (1,6), (2,5), (2,6)\}$

$B \times D = \{1, 2, 3, 4\} \times \{5, 6, 7, 8\}$
$= \{(1, 5), (1, 6), (1, 7), (1, 8), (2, 5), (2, 6), (2, 7), (2, 8), (3, 5), (3, 6), (3, 7), (3, 8), (4, 5), (4, 6), (4, 7), (4, 8)\}$

$\therefore$ $(A \times C)$ is a subset of $(B \times D)$

8. If $f(x) = \frac{x-1}{x+1}, x \neq 1$ show that $f(f(x)) = -\frac{1}{x}$, provided $x \neq 0$.

Sol : $f(x) = \frac{x-1}{x+1}, x \neq -1$

$$f[f(x)] = f\left[\frac{x-1}{x+1}\right]$$

$$= \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{\frac{x-1-(x+1)}{x+1}}{\frac{x-1+(x+1)}{x+1}}$$

$$= \frac{x-1-x-1}{x-1+x+1} = -\frac{2}{2x} = -\frac{1}{x}$$

Hence proved.

9. The function f and g are defined by $f(x) = 6x + 8$; $g(x) = \frac{x-2}{3}$

(i) Calculate the value of $gg\left(\frac{1}{2}\right)$

(ii) Write an expression for $gf(x)$ in its simplest form.

Sol : $f(x) = 6x + 8$, $g(x) = \frac{x-2}{3}$

(i) $$g\left(\frac{1}{2}\right) = \frac{\frac{1}{2}-2}{3} = \frac{1-4}{2\times 3} = -\frac{3}{6} = -\frac{1}{2}$$

$$\therefore\ gg\left(\frac{1}{2}\right) = g\left[g\left(\frac{1}{2}\right)\right]$$

$$= g\left(-\frac{1}{2}\right) = \frac{-\frac{1}{2}-2}{3} = -\frac{1-4}{2\times 3} = -\frac{5}{6}$$

(ii) $gf(x) = g[f(x)] = g[6x + 8]$

$$= \frac{6x+8-2}{3} = \frac{6x+6}{3}$$

$$= \frac{6(x+1)}{3} = 2(x+1)$$

10. Write the domain of the following real functions

(i) $f(x) = \frac{2x+1}{x-9}$ **(ii) $p(x) = \frac{-5}{4x^2+1}$**

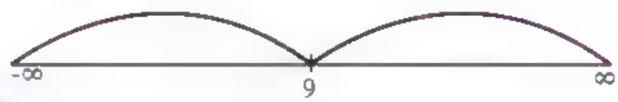
(iii) $g(x) = \sqrt{x-2}$ **(iv) $h(x) = x + 6$**

Sol :

(i) $f(x) = \frac{2x+1}{x-9}$

If $x - 9 = 0$, then

$x = 9$



$\therefore$ The domain is all values of x that make the expression defined.

i.e., $(-\infty, 9) \cup (9, \infty)$

i.e., $(x / x \neq 9) \Rightarrow R - \{9\}$

(ii) $p(x) = \frac{-5}{4x^2+1}$

Here, the expression is defined for all real values of 'x'.

i. e., $x \in R$

(iii) $g(x) = \sqrt{x-2}$

g (x) is defined real only when $x \geq 2$

(iv) $h(x) = x + 6$

h (x) is defined for all real values of 'x'.

i.e., $x \in R$.

CREATIVE QUESTIONS

I. Multiple Choice Questions

Cartesian Product

1. If A = {1, 2}, B = {0, 1}, then A × B is

(1) {(1, 0), (1, 1), (2, 0), (2, 1)}
(2) {(1, 0), (2, 1)}
(3) {(1, 1), (1, 2), (0, 1), (0, 2)}
(4) None of these [Ans: (1)]

Sol :

$$A = \{1, 2\},\quad B = \{0, 1\}$$
$$A \times B = \{(1, 0), (1, 1), (2, 0), (2, 1)\}$$

2. If the set A has 'p' elements, B has 'q' elements, then the number of elements in A × B is

(1) p + q (2) p + q + 1
(3) pq (4) p^2 [Ans: (3)]

Sol :

Number of elements in A = p
Number of elements in B = q
Number of elements in A × B = pq

3. If A, B, C are any three sets, then $A \times (B \cup C)$ is equal to

(1) $(A\times B)\cup(A\times C)$ (2) $(A\cup B)\cup(A\cup C)$
(3) Both (a) and (b) (4) None of these [Ans: (1)]

4. Let A = {a, b, c, d}, B = {b, c, d, e}, then $n\{(A\times B)\cap(B\times A)\}$ =

(1) 3
(2) 6
(3) 9
(4) None of these [Ans: (3)]

Sol :

A = {a, b, c, d}, B = {b, c, d, e}

$A\times B = \{(a,b),(a,c),(a,d),(a,e),(b,b),(b,c),(b,d),(b,e),(c,b),(c,c),(c,d),(c,e),(d,b),(d,c),(d,d),(d,e)\}$

$B\times A = \{(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d),(e,a),(e,b),(e,c),(e,d)\}$

$(A \times B) \cap (B \times A) = \{(b, b), (b, c), (b, d), (c, b), (c, c), (c, d), (d, b), (d, c), (d, d)\}$

$n\{(A \times B) \cap (B \times A)\} = 9$

Relations

6. If A is the set of even numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is

(1) 2^9 (2) 9^2
(3) 3^2 (4) 2^{9-1} [Ans: (1)]

Sol :

$$A = \{2, 4, 6\} \Rightarrow n(A) = 3$$
$$B = \{2, 3, 5\} \Rightarrow n(B) = 3$$
$$\therefore\ n(A \times B) = 3 \times 3 = 9$$

Number of relations from A to B = 2^9

7. Let N be the set of all natural numbers and let R be a relation on N defined as $R = \{(x, y) / x\in N, y\in N$ and x + 3y = 15}. Then R as set of ordered pairs is

(1) {(3, 4), (5, 3), (9, 2), (13, 2)}
(2) {(3, 5), (2, 7), (9, 2), (12, 1)}
(3) {(3, 4), (6, 3), (9, 2), (12, 1)}
(4) {(4, 5), (7, 3), (4, 5), (4, 2)} [Ans: (3)]

Sol :

$R = \{(x, y), x\in N, y\in N$ and $x + 3y = 15\}$
$x + 3y = 15$
When x = 3, y = 4, then 3 + 3(4) = 15
When x = 6, y = 3, then 6 + 3(3) = 15
When x = 9, y = 2, then 9 + 3(2) = 15
When x = 12, y = 1, then 12 + 3(1) = 15
$\therefore$ R = {(3, 4), (6, 3), (9, 2), (12, 1)}

8. A relation R is defined from {2, 3, 4, 5} to {3, 6, 7, 10} by : x R y ⇔ x is relatively prime to y. Then, domain of R is

(1) {2, 3, 5} (2) {3, 5}
(3) {2, 3, 4} (4) {2, 3, 4, 5} [Ans: (4)]

Sol :

Since x is relatively prime to y,
Domain = {2, 3, 4, 5}

9. Let R be a relation from set A to a set B, then

(1) R = $A\cup B$ (2) $A\cap B$
(3) $R\subseteq A\times B$ (4) $R\subseteq B\times A$ [Ans: (3)]

Sol :

Relation is a subset of Cartesian Product.

Functions

10. If f (x) = $2x^2$ + bx + c and f (0) = 3 and f (2) = 1, then f (1) is equal to

(1) – 2 (2) 0
(3) 1 (4) 2 [Ans: (2)]

Sol :

$$f(x) = 2x^2 + bx + c$$
$$f(0) = 3$$
$$\Rightarrow 2(0) + b(0) + c = 3 \Rightarrow c = 3$$
$$f(2) = 1$$
$$\Rightarrow 2(2)^2 + b(2) + c = 1 \Rightarrow 2b + c = -7$$
$$2b + 3 = -7$$
$$b = \frac{-10}{2} = -5$$
$$\therefore f(x) = 2x^2 - 5x + 3$$
$$f(1) = 2(1)^2 - 5(1) + 3$$
$$= 2 - 5 + 3 = 0$$

11. Let A = {x, y, z} and B = {a, b, c, d}. Which one of the following is not a function and is not a relation from A to B?

(1) {(x, a), (x, c)} (2) {(y, c), (y, d)}
(3) {(z, a), (z, d)} (4) {(z, b), (y, b), (a, d)}
[Ans: (4)]

Sol :

A = {x, y, z}, B = {a, b, c, d}
{(z, b), (y, b), (a, d)} is not a function from A to B as $a \notin A$.

12. The domain of the function 'f' given by
$f(x) = \frac{x^2+2x+1}{x^2-x-6}$

(1) R – {3, – 2} (2) R – {– 3, 2}
(3) R – {3, 2} (4) R – {– 3, – 2} [Ans: (1)]

Sol :

$$f(x) = \frac{x^2+2x+1}{x^2-x-6}$$
$$= \frac{x^2+2x+1}{(x-3)(x+2)}$$

f(x) becomes undefined when x = 3 (or) x = – 2.
Domain = R – {3, – 2}

13. Which of the following are functions?

(1) $\{(x, y) : y^2 = x, x, y \in R\}$
(2) $\{(x, y) : y = |x|, x, y \in R\}$
(3) $\{(x, y) : x^2 + y^2 = 1, x, y \in R\}$
(4) $\{(x, y) : x^2 - y^2 = 1, x, y \in R\}$ [Ans: (2)]

14. If $x \neq 1$ and $f(x) = \frac{x+1}{x-1}$ is a real function, then f (f(f(2))) is

(1) 1 (2) 2
(3) 3 (4) 4 [Ans: (3)]

Sol :

$$f(x) = \frac{x+1}{x-1}$$
$$f(2) = \frac{2+1}{2-1} = \frac{3}{1} = 3$$
$$f[f(2)] = f(3) = \frac{3+1}{3-1} = \frac{4}{2} = 2$$
$$f[f(f(2))] = f(2) = \frac{2+1}{2-1} = 3$$

15. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2, (x \neq 0)$ then f(2) = ?

(1) $\frac{-7}{4}$ (2) $\frac{5}{2}$
(3) – 1 (4) None of these [Ans: (1)]

Sol :

Given $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$

Let x = 2

$$2f(2) - 3f\left(\frac{1}{2}\right) = 4 \quad \text{... (1)}$$

Let $x = \frac{1}{2}$

$$2f\left(\frac{1}{2}\right) - 3f(2) = \frac{1}{4} \quad \text{... (2)}$$
$$\Rightarrow f\left(\frac{1}{2}\right) = \frac{\frac{1}{4}+3f(2)}{2}$$
$$= \frac{1+12f(2)}{8}$$

Substituting in (1), we get

$$f(2) = \frac{-7}{4}$$

Composition of Functions

16. Let $f\left(x+\frac{1}{x}\right) = x^2+\frac{1}{x^2}, x \neq 0,$ then f (x) is equal to

(1) x^2-2 (2) x^2-1

(3) $f\left(-\frac{a}{a+1}\right)$ (4) f (a) **[Ans: (1)]**

Sol :

Given $f\left(x+\frac{1}{x}\right) = x^2+\frac{1}{x^2}$

Let us find $\left(x+\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}+2x.\frac{1}{x}$

$= x^2+\frac{1}{x^2}+2$

Subtracting 2 from this, we get $x^2+\frac{1}{x^2}$

$\therefore$ $f(x) = x^2-2$

17. If f(x) = x – 2, g(x) = $\sqrt{x^2+1}$, then $(g\circ f)$(x) = ?

(1) $\sqrt{x^2+1}-2$ (2) $\sqrt{x^2+4x+5}$

(3) x^2-1 (4) x^2-4x+5 **[Ans: (2)]**

Sol :

$$(g\circ f)(x) = g[f(x)] = g[x-2] = \sqrt{(x-2)^2+1} = \sqrt{x^2-4x+4+1} = \sqrt{x^2-4x+5}$$

18. Given f(2) = 3, g(3) = 2 and g(2) = 5, then $(f\circ g)$(3) =

(1) 2 (2) 3

(3) 4 (4) 5 **[Ans: (2)]**

Sol :

$$(f\circ g)(3) = f[g(3)] = f[2] = 3$$

19. Given f = {(– 2, 1), (0, 3), (4, 5)}, g = {(1, 1), (3, 3), (4, 5)} then, Domain and range of $g\circ f$.

(1) D = {3, 0}, R = {– 2, 1}

(2) D = {3, – 2}, R = {1, 5}

(3) D = {– 2, 0}, R = {1, 3}

(4) D = {– 2, 1}, R = {0, 3} **[Ans: (3)]**

Sol :

$(g\circ f)(-2) = g[f(-2)] = g(1) = 1$

$(g\circ f)(0) = g[f(0)] = g(3) = 3$

$(g\circ f)(4) = g[f(4)] = g(5) =$ undefined

$\therefore g\circ f = \{(-2, 1), (0, 3)\}$

Hence the Domain = {– 2, 0},

Range = {1, 3}

II. Very Short Answer Questions

1. If $\left(\frac{x}{3}+1, y-\frac{2}{3}\right) = \left(\frac{5}{3},\frac{1}{3}\right)$, find the values of 'x' and 'y'

Sol :

$\frac{x}{3}+1 = \frac{5}{3}$, $y-\frac{2}{3} = \frac{1}{3}$

$\frac{x}{3} = \frac{5}{3}-1$, $y = \frac{1}{3}+\frac{2}{3}$

$\frac{x}{3} = \frac{5-3}{3}$, $y = \frac{3}{3}$

$\frac{x}{3} = \frac{2}{3}$, $y = 1$

$x = 2$

2. If the Set 'A' has 3 elements and the Set B = {3, 4, 5}, find the number of elements in (A × B).

Sol :

'A' has 3 elements and 'B' has 3 elements, then 'A × B' will have 9 elements.

3. If G = {7, 8} and H = {5, 4, 2}, find G × H and H × G.

Sol :

G × H = {7, 8} × {5, 4, 2}

= {(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)}

H × G = {5, 4, 2} × {7, 8}

= {(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)}

4. If A = {– 1, 1}, find A × A × A.

Sol :

A × A = {– 1, 1} × {– 1, 1}

= {(– 1, – 1), (– 1, 1), (1, – 1), (1, 1)}

A × A × A = {(– 1, – 1), (– 1, 1), (1, – 1), (1, 1)} × {– 1, 1}

= {(– 1, – 1, – 1), (– 1, – 1, 1), (– 1, 1, – 1), (– 1, 1, 1), (1, – 1, – 1), (1, – 1, 1), (1, 1, – 1), (1, 1, 1)}

5. If A × B = {(a, x), (a, y), (b, x), (b, y)}, find 'A' and 'B'.

Sol :

A = Set of all first entries = {a, b}

B = Set of all second entries = {x, y}

6. Let A = {3, 5} and B = {7, 11}. Let R = {(a, b); a ∈ A, b ∈ B, a – b is odd}. Show that R is an empty relation from A into B.

Sol :

Since a ∈ A and b ∈ B,

$$a - b = (3-7), (3-11), (5-7), (5-11)$$
$$= -4, -8, -2, -6 \text{ None of them is an odd number.}$$

Therefore, R is an empty relation.

7. What is the number of relations on A if R is a relation on a finite set A having 'n' elements?

Sol :

Set A has 'n' elements.

∴ Number of elements in $A \times A = n \times n = n^2$

The number of possible subsets of $A \times A = 2^{n^2}$

Since, each subset of A × A is a relation on A, the total number of relations on A is 2^{n^2}.

8. Let set A = {January, February, April, June, September, October, November, December} and set B = {28, 29, 30, 31}. Let R be a relation from A to B defined by R = {(a, b) ∈ A × B : 'a' month has 'b' number of days}. Write a subset of relation R connected with 'Teachers Day'.

Sol :

R = {(a, b) ∈ A × B, 'a' month has 'b' number of days}

∴ R = {(January, 31), (February, 28), (February, 29), (April, 30), (June, 30), (September, 30), (October, 31), (November, 30), December, 31)}

∴ The subset of relation R connected with Teachers Day is {(September, 30}

9. A function 'f' is defined by f (x) = 2x – 5. Write down the values of f (0), f (7) and f (– 3).

Sol :

$$f(x) = 2x - 5$$
$$f(0) = 2(0) - 5 = -5$$
$$f(7) = 2(7) - 5 = 14 - 5 = 9$$
$$f(-3) = 2(-3) - 5 = -6 - 5 = -11$$

10. If A = {1, 2, 3} and B = {a, b, c} then do

i) {(1, a), (2, b), (2, c), (3, c)} and

ii) {(2, b), (3, b)} represent a function A → B?

Sol :

(i) The two ordered pairs (2, b) and (2, c) have the same first co-ordinate. Therefore, (1, a), (2, b), (2, c), (3, c) does not represent a function A → B.

(ii) Since one element 1 of A is not associated with some element of B. i.e., 1 is not the first co-ordinate of any ordered pair. So {(2, b), (3, b)} does not represent a function from A → B.

11. Let A = {1, 2, 3} and B = {4, 5}. Let f = {(1, 4), (1, 5), (2, 4), (3, 5)}. Is 'f' a function from A into B?

Sol :

The first elements of two ordered pairs (1, 4) and (1, 5) in 'f' are same.

Therefore f = {(1, 4), (1, 5), (2, 4), (3, 5)} is not a function from A to B.

12. Find the domain of $f(x) = \dfrac{x}{x^2 - 5x + 6}$.

Sol :

$$f(x) = \frac{x}{x^2 - 5x + 6} = \frac{x}{(x-3)(x-2)}$$

When x = 2 or 3, f(x) becomes undefined.

∴ Domain of f = R – {2, 3}

13. Let f : R → R defined by $f(x) = \begin{cases} 3x - 2, & x < 0 \\ 1, & x = 0 \\ 4x + 1, & x > 0 \end{cases}$. Find f(– 3), f(0) and f(5).

Sol :

$$f(-3) = 3(-3) - 2 = -11 \text{ as } -3 < 0$$
$$f(0) = 1 \text{ as } x = 0$$
$$f(5) = 4(5) + 1 = 20 + 1 = 21 \text{ as } x > 0$$

14. Let a function f : R → A be defined as

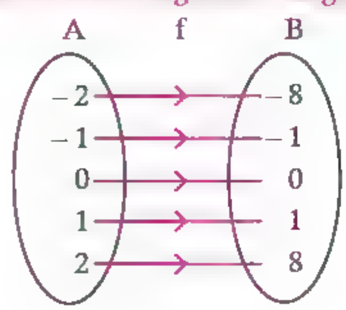
$$f(x) = \begin{cases} 1, & \textit{if } x \textit{ is rational} \\ -1, & \textit{if } x \textit{ is irrational, where } x \in R \end{cases}$$

Find $f\left(\frac{1}{3}\right)$ and $f(\sqrt{5})$.

Sol :

As $\frac{1}{3}$ is a rational number, $f\left(\frac{1}{3}\right) = 1$ and as $\sqrt{5}$ is an irrational number, $f(\sqrt{5}) = -1$.

15. Consider the following arrow diagram.



i) Is it a function? If so, write the equation form.

ii) List the elements of this function.

Sol :

Yes, it is a function and $f(x) = x^3$

Elements of $f = \{(-2,-8),(-1,-1),(0,0),(1,1),(2,8)\}$

16. If $f(x) = \frac{x-1}{x+1}$, then show that $f\left(\frac{1}{x}\right) = -f(x)$

Sol :

$$f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-1}{\frac{1}{x}+1} = \frac{\frac{1-x}{x}}{\frac{1+x}{x}} = \frac{1-x}{1+x}$$

$$= -\frac{(x-1)}{x+1} = -f(x)$$

Hence proved.

17. Given $f(x) = 3x + 2$, $g(x) = x + 5$ find $f[g(x)]$ and $g[f(x)]$.

Sol :

$$f[g(x)] = f[x+5] = 3(x+5)+2$$
$$= 3x+15+2 = 3x+17$$
$$g[f(x)] = g[3x+2] = 3x+2+5 = 3x+7$$

18. If $f(x) = 2x - 1$, then find $f \circ f$.

Sol :

$$f(x) = 2x-1$$
$$f \circ f = f[f(x)]$$
$$= f(2x-1) = 2(2x-1)-1$$
$$= 4x-2-1 = 4x-3$$

19. Given $f(x) = x^2 + 6$ and $g(x) = 2x + 1$, find $(g \circ f)(2)$.

Sol :

$$g \circ f = g[f(x)]$$
$$= g[x^2+6]$$
$$= 2(x^2+6)+1$$
$$= 2x^2+12+1 = 2x^2+13$$

Now, $(g \circ f)(2) = 2(2)^2+13$

$$= 2(4)+13 = 8+13 = 21$$

III. Short Answer Questions:

1. If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, then find $(A\times B)\cup(A\times C)$.

Sol :

$$A\times B = \{1,2,3\}\times\{3,4\}$$
$$= \{(1,3),(1,4),(2,3),(2,4),(3,3),(3,4)\}$$
$$A\times C = \{1,2,3\}\times\{4,5,6\}$$
$$= \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$$

$\therefore (A\times B)\cup(A\times C) = \{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,3),(3,4),(3,5),(3,6)\}$

2. If $A = \{1, 2, 4\}$, $B = \{2, 4, 5\}$, $C = \{2, 5\}$, find $(A - B) \times (B - C)$

Sol :

$$A - B = \{1,2,4\} - \{2,4,5\}$$
$$= \{1\}$$
$$B - C = \{2,4,5\} - \{2,5\}$$
$$= \{4\}$$
$$\therefore (A-B)\times(B-C) = \{1\}\times\{4\}$$
$$= \{(1,4)\}$$

3. If $R_1 = \{(x, y) / y = 2x + 7$. where $x \in R$ and $-5 \le y \le 5\}$ is a relation, then find the domain and range of R_1.

Sol :

$$R_1 = \{(x,y) / y = 2x+7, x\in R, -5\le y\le 5\}$$
$$x = \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\}$$
$$y = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

When $x = -6$, $y = -5$

$x = -5$, $y = -3$

$x = -4$, $y = -1$

$x = -3$, $y = 1$

$x = -2$, $y = 3$

$x = -1$, $y = 5$

$\therefore R = \{(-6,-5),(-5,-3),(-4,-1),(-3,1),(-2,3),(-1,5)\}$

Domain $= \{-6,-5,-4,-3,-2,-1\}$

Range $= \{-5,-3,-1,1,3,5\}$

4. Let R be a relation in N defined by $R = \{(1 + x, 1 + x^2) / x \le 4, x \in N\}$ Find

(i) R

(ii) domain of R

(iii) range of R

Sol :

$$R = \{(1+x, 1+x^2)/x\le 4, x\in N\}$$

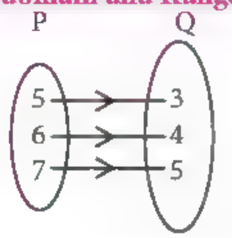
as $x \le 4$, $x \in N$, x takes the values 1, 2, 3, 4

$\therefore R = \{(2,2),(3,5),(4,10),(5,17)\}$

Domain of R $= \{2, 3, 4, 5\}$

Range of R $= \{2, 5, 10, 17\}$

5. The figure shows a relationship between the Sets P and Q. Write this relation in
(i) Set builder form
(ii) Roster form
(iii) What is its domain and Range?



Sol :

(i) Set builder form $R = \{(x, y) / y = x - 2$ for $x = 5, 6, 7\}$

(ii) Roster form $R = \{(5, 3), (6, 4), (7, 5)\}$

(iii) Domain of $R = \{5, 6, 7\}$

Range of $R = \{3, 4, 5\}$

6. Find the domain and range for the functions

(a) $f(x) = x^2 + 2$, (b) $f(t) = \frac{1}{t+2}$

Sol :

(a) $f(x) = x^2 + 2$ is defined for all real values of 'x'.
Hence the domain of f (x) is "all real values of x".
Since 'x^2' is always positive, $x^2 + 2$ is never less than 2.
$\therefore$ The range of f (x) is "all real numbers $f(x) \geq 2$ ".

(b) $f(t) = \frac{1}{t+2}$

Domain: The function is not defined for $t = -2$
Hence, the domain of f (t) is "all real numbers except – 2"
Range: "all real numbers except zero".

7. Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function from Z to Z defined by $f(x) = ax + b$, for some integers a, b. Determine a, b.

Sol :

$f(x) = ax + b$

for $x = 1, f(x) = 1$

$\therefore \quad a(1) + b = 1 \Rightarrow a + b = 1 \quad \dots (1)$

for $x = 2, f(x) = 3$

$\therefore \quad a(2) + b = 3 \Rightarrow 2a + b = 3 \quad \dots (2)$

Solving (1) and (2)

We get, $a = 2, b = -1$

$\therefore \quad f(x) = 2x - 1$

Also for $x = 0, f(0) = -1$ and
when $x = -1, f(-1) = -3$.

8. Find the domain and Range of the real function 'f' defined by $f(x) = \frac{4-x}{x-4}$.

Sol :

When $x = 4$, f(x) is not defined as $f(4) = \frac{0}{0}$. So, the domain is the set of all real numbers except $x = 4$ i.e., $R - \{4\}$.

and $f(x) = \frac{4-x}{x-4} = -\frac{(x-4)}{x-4} = -1, x \neq 4$

$\therefore$ Range of $f(x) = \frac{4-x}{x-4}$ is -1.

Domain $= R - \{4\}$, Range $= \{-1\}$

9. Express the following function as set of ordered pairs and determine the range: $f : X \to R$, $f(x) = x^3 + 1$, where $X = \{-1, 0, 3, 9, 7\}$.

Sol :

$f : X \to R, f(x) = x^3 + 1$ where $X = \{-1, 0, 3, 9, 7\}$

$f(-1) = (-1)^3 + 1 = -1 + 1 = 0$

$f(0) = (0)^3 + 1 = 0 + 1 = 1$

$f(3) = (3)^3 + 1 = 27 + 1 = 28$

$f(9) = (9)^3 + 1 = 729 + 1 = 730$

$f(7) = (7)^3 + 1 = 343 + 1 = 344$

$\therefore \quad f = \{(-1, 0), (0, 1), (3, 28), (9, 730), (7, 344)\}$ and

Range of $f = \{0, 1, 28, 730, 344\}$

10. If $A = \{-1, 0, 2, 5, 1\}$ and $f : A \to Z$ is defined by $f(x) = x^2 - x - 2$. Find the range of 'f'.

Sol :

Given $A = \{-1, 0, 2, 5, 1\}$

$f(x) = x^2 - x - 2$

$f(-1) = (-1)^2 - (-1) - 2 = 0$

$f(0) = (0)^2 - 0 - 2 = -2$

$f(2) = (2)^2 - 2 - 2 = 0$

$f(5) = (5)^2 - 5 - 2 = 18$

$f(1) = (1)^2 - 1 - 2 = -2$

$\therefore$ Range of $f = \{-2, 0, 18\}$

11. If $f(x) = x^2$, find $\frac{f(1.1) - f(1)}{(1.1-1)}$

Sol :

$f(x) = x^2$

$f(1.1) = (1.1)^2 = 1.21$

$f(1) = (1)^2 = 1$

$\therefore \quad \frac{f(1.1) - f(1)}{(1.1-1)} = \frac{1.21-1}{1.1-1}$

$= \frac{0.21}{0.1} \times \frac{100}{100} = \frac{21}{10} = 2.1$

12. If $f(x) = x^2 - 2$ and $g(x) = 2x - 3$, then Evaluate

(i) $f(-3)$ (ii) $g(5)$ (iii) $f(a+1)$

Sol :

(i) $f(-3)$

$$f(x) = x^2 - 2$$
$$f(-3) = (-3)^2 - 2 = 9 - 2 = 7$$

(ii) $g(5)$

$$g(x) = 2x - 3$$
$$g(5) = 2(5) - 3 = 10 - 3 = 7$$

(iii) $f(a+1)$

$$f(x) = x^2 - 2$$
$$f(a+1) = (a+1)^2 - 2$$
$$= a^2 + 2a + 1 - 2 = a^2 + 2a - 1$$

13. Given $f(x) = 4x + 2$, Evaluate $\frac{f(a+h) - f(a)}{h}$.

Sol :

$$f(x) = 4x + 2$$
$$f(a+h) = 4(a+h) + 2$$
$$= 4a + 4h + 2$$
$$f(a) = 4a + 2$$
$$\therefore \frac{f(a+h) - f(a)}{h} = \frac{4a + 4h + 2 - 4a - 2}{h}$$
$$= \frac{4h}{h} = 4$$

14. Given $f(x) = 3x + 7$, $g(x) = x + 1$, $h(x) = 2x - 3$, Find $f(2) + h(1)$ and $f(0) + g(0) - h(0)$.

Sol :

$$f(2) + h(1) = 3(2) + 7 + 2(1) - 3$$
$$= 6 + 7 + 2 - 3 = 12$$
$$f(0) + g(0) - h(0) = 3(0) + 7 + 0 + 1 - (2(0) - 3)$$
$$= 0 + 7 + 0 + 1 - 0 + 3 = 11$$

15. If $f(x) = 2x^4 + x^2 + 1$ and $g(x) = \sqrt{x}$. Find $f \circ g$.

Sol :

$$f \circ g = f[g(x)] = f[\sqrt{x}]$$
$$= 2(\sqrt{x})^4 + (\sqrt{x})^2 + 1 = 2x^2 + x + 1$$

16. If $f(x) = -2x + 9$ and $g(x) = -4x^2 + 5x - 3$, find $f \circ g$ and $g \circ f$.

Sol :

$$f \circ g = f[g(x)]$$
$$= f[-4x^2 + 5x - 3]$$
$$= -2(-4x^2 + 5x - 3] + 9$$
$$= 8x^2 - 10x + 15$$
$$g \circ f = g[f(x)]$$
$$= g[-2x + 9]$$
$$= -4(-2x + 9)^2 + 5(-2x + 9) - 3$$
$$= -4(4x^2 - 36x + 81) - 10x + 45 - 3$$
$$= -16x^2 + 144x - 324 - 10x + 42$$
$$= -16x^2 + 134x - 282$$

17. If $g(x) = \sqrt[3]{x-4}$ and $h(x) = x^3 + 4$, find $(h \circ g)(-15)$.

Sol :

$$h \circ g = h[g(x)]$$
$$= h[\sqrt[3]{x-4}]$$
$$= (\sqrt[3]{x-4})^3 + 4$$
$$= (x-4)^{3 \times \frac{1}{3}} + 4$$
$$= x - 4 + 4 = x$$
$$\therefore (h \circ g)(-15) = -15$$

IV. Long Answer Questions

1. If $A = \{x / x^2 - 4x + 3 = 0\}$, $B = \{x / x^2 - x - 6 = 0\}$, $C = \{x / x^3 - 4x = 0\}$, find (i) $A \times B$ (ii) $A \times C$ (iii) $(A - B) \times C$

Sol :

$$A = \{x / x^2 - 4x + 3 = 0\}$$
$$= \{x / (x-3)(x-1) = 0\}$$
$$= \{x / x = 1, 3\}$$
$$B = \{x / x^2 - x - 6 = 0\}$$
$$= \{x / (x+2)(x-3) = 0\}$$
$$= \{x / x = -2, 3\}$$
$$C = \{x / x^3 - 4x = 0\}$$
$$= \{x / x(x^2 - 4) = 0\}$$
$$= \{x / x = 0, -2, 2\}$$
$$A = \{1, 3\},\ B = \{-2, 3\},\ C = \{-2, 0, 2\}$$

(i) $A \times B = \{1, 3\} \times \{-2, 3\}$

$= \{(1, -2), (1, 3), (3, -2), (3, 3)\}$

(ii) $A \times C = \{1, 3\} \times \{-2, 0, 2\}$

$= \{(1, -2), (1, 0), (1, 2), (3, -2), (3, 0), (3, 2)\}$

(iii) $A - B = \{1, 3\} - \{-2, 3\} = \{1\}$

$(A - B) \times C = \{1\} \times \{-2, 0, 2\}$

$= \{(1, -2), (1, 0), (1, 2)\}$.

2. If A and B be two non-empty sets, then show that $A \times B = B \times A$ if $A = B$.

Sol :

Let $A \times B = B \times A$ and $x \in A$

$x \in A \Rightarrow (x, b) \in A \times B\ \forall\ b \in B$

$\Rightarrow (x, b) \in (B \times A)$

$\because$ A × B = B × A

$\Rightarrow \quad x \in B$

$\Rightarrow \quad A \subseteq B$

By the definition of Cartesian Product and let y be an arbitrary element of B

$y \in B \Rightarrow (a, y) \in A \times B \; \forall \; a \in A$

$\Rightarrow (a, y) \in (B \times A)$

$\because$ A × B = B × A

By the definition of Cartesian Product

$\Rightarrow \quad y \in A$

$B \subseteq A$

From these two results, A = B

Conversely, let A = B, then

A × B = A × A and B × A = A × A

$\therefore$ A × B = B × A

Hence A × B = B × A if A = B.

3. In the given ordered pairs (4, 6), (8, 4), (3, 3), (9, 11), (6, 3), (3, 0), (2, 3). Find the following relations. Also, find the domain and range.

(i) is two less than (ii) is less than

(iii) is greater than (iv) is equal to

Sol :

(i) R_1 is the set of all ordered pairs whose first component is two less than the second component.

$\therefore \quad R_1 = \{(4, 6), (9, 11)\}$

Domain of R_1 = Set of all first components = {4, 9}

Range of R_1 = Set of all second components = {6, 11}

(ii) R_2 is the set of all ordered pairs whose first component is less than the second component.

$\therefore \quad R_2 = \{(4, 6), (9, 11), (2, 3)\}$

Domain of R_2 = {4, 9, 2}

Range of R_2 = {6, 11, 3)

(iii) R_3 is the set of all ordered pairs whose first component is greater than the second component.

$\therefore \quad R_3 = \{(8, 4), (6, 3), (3, 0)\}$

Domain of R_3 = {8, 6, 3}

Range of R_3 = {4, 3, 0}

(iv) R_4 is the set of all ordered pairs whose first component is equal to the second component.

$\therefore \quad R_4 = \{(3, 3)\}$

Domain of R_4 = {3} and Range of R_4 = {3}.

4. Let A = {2, 3, 4, 5} and B = {8, 9, 10, 11}. Let R be the relation "is factor of" from A and B.

i) Write R in the roster form. Also find Domain and Range of R.

ii) Draw an arrow diagram to represent the relation.

Sol :

(i) Given, that the Relation R consists of elements (a, b) where 'a' is a factor of 'b'.

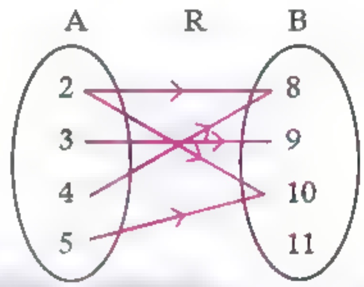
Therefore, Relation 'R' in the roster form is

R = {(2, 8), (2, 10), (3, 9), (4, 8), (5, 10)}

Domain of R = {2, 3, 4, 5}

Range of R = {8, 10, 9}

(ii) Arrow diagram representing 'R'



5. Given f (x) = 3x + 7, g (x) = – x + 8, h (x) = x^2 + 3x – 1. Prove that Composition of functions is associative.

Sol :

$$\begin{aligned}
f \circ g &= f[g(x)] \\
&= f[-x+8] \\
&= 3(-x+8)+7 \\
&= -3x+24+7 = -3x+31 \\
(f \circ g) \circ h &= (f \circ g)[h(x)] \\
&= (f \circ g)[x^2+3x-1] \\
&= -3(x^2+3x-1)+31 \\
&= -3x^2-9x+3+31 \\
&= -3x^2-9x+34 \qquad \text{... (1)} \\
g \circ h &= g[h(x)] \\
&= g[x^2+3x-1] \\
&= -(x^2+3x-1)+8 \\
&= -x^2-3x+1+8 = -x^2-3x+9 \\
f \circ (g \circ h) &= f[g(h(x))] = f[g(x^2+3x-1)] \\
&= f[-x^2-3x+9] \\
&= 3(-x^2-3x+9)+7 \\
&= -3x^2-9x+27+7
\end{aligned}$$

$= -3x^2 - 9x + 34$ ... (2)

From (1) and (2)

$(f \circ g) \circ h = f \circ (g \circ h)$

$\therefore$ Composition of functions is Associative.

6. If 'f' and 'g' are two real valued functions defined as $f(x) = 2x + 1$, $g(x) = x^2 + 1$, then find

(i) $f + g$ (ii) $f - g$ (iii) fg (iv) $\frac{f}{g}$

Sol :

(i) $$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= 2x + 1 + x^2 + 1\\ &= x^2 + 2x + 2\end{aligned}$$

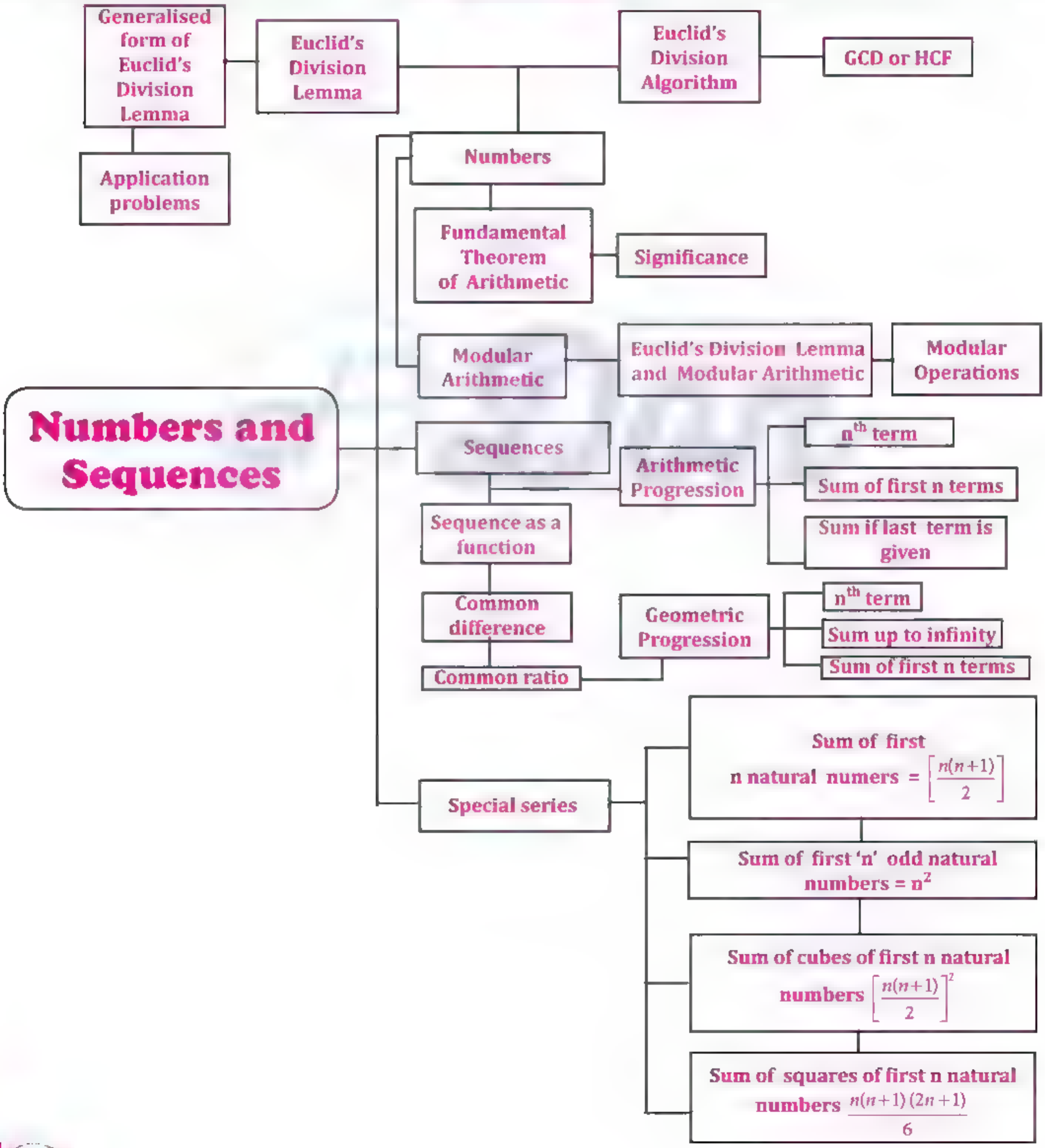
(ii) $$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= (2x+1) - (x^2+1)\\ &= 2x + 1 - x^2 - 1 = 2x - x^2\\ &= -x^2 + 2x\end{aligned}$$

(iii) $$\begin{aligned}(fg)(x) &= [f(x)]\,[g(x)]\\ &= (2x+1)(x^2+1)\\ &= 2x^3 + x^2 + 2x + 1\end{aligned}$$

(iv) $$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \text{provided } g(x) \neq 0\\ &= \frac{2x+1}{x^2+1}\end{aligned}$$

UNIT 2 NUMBERS AND SEQUENCES

MIND MAP



EUCLID'S DIVISION LEMMA

Key Points

Theorem 1: Euclid's Division Lemma

Let a and b ($a > b$) be any two positive integers. Then there exist unique integers q and r such that $a = bq + r$, $0 \leq r < b$.

Generalized form of Euclid's Division Lemma.

If a and b are any two integers then there exist unique integers q and r such that $a = bq + r$. where $0 \leq r < |b|$

Theorem 2: (Euclid's Division Algorithm)

If a and b are positive integers such that $a = bq + r$, then every common divisor of a and b is a common divisor of b and r and vice-versa.

Algorithm

An Algorithm means a series of mathematical step by step procedure of calculating successively on the results of earlier steps till the desired answer is obtained.

We use Euclid's Division Algorithm to find out H.C.F. of two positive integers easily.

Euclid's Division Algorithm

To find the Highest common factor of two positive integers a and b, where $a > b$

Step 1: Using Euclid's Division Lemma, $a = bq + r$; $0 \leq r < b$. where q $\rightarrow$ quotient, r $\rightarrow$ remainder. If $r = 0$, then H.C.F. $(a, b) = b$.

Step 2 : Otherwise applying Euclid's Division Lemma divide b by r to get $b = rq + r_1$, $0 \leq r_1 < r$.

Step 3 : If $r_1 = 0$ then r is the Highest common factor of a and b.

Step 4 : Otherwise using Euclid's Division Lemma repeat the process until we get $r = 0$. In that case, corresponding divisor is H.C.F. of (a, b).

Theorem 3:

If a, b are two positive integers with $a > b$ then G.C.D. of (a, b) = G.C.D. of $(a - b, b)$.

H.C.F. of three numbers

Let a, b, c be the given positive integers.

(i) First find H.C.F. (a, b) call it as d. i.e., $d = (a, b)$

(ii) Find H.C.F. of d and c.

This will be the H.C.F. of three numbers a, b, and c.

Worked Examples

2.1 **We have 34 cakes. Each box can hold 5 cakes only. How many boxes we need to pack and how many cakes are unpacked?**

Sol :

We see that 6 boxes are required to pack 30 cakes with 4 cakes left over. This distribution of cakes can be understood as follows.

34	=	5	×	6	+	4
Total number of cakes ↓ Dividend a	=	Number of cakes in each box ↓ Divisor b	×	Number of boxes ↓ Quotient q	+	Number of cakes left over ↓ Remainder r

2.2 Find the quotient and remainder when a is divided by b in the following cases

(i) $a = -12, b = 5$

(ii) $a = 17, b = -3$

(iii) $a = -19, b = -4$.

Sol :

(i) $a = -12, b = 5$

By Euclid's Division Lemma

$a = bq + r$, where $0 \le r < |b|$

$-12 = 5 \times (-3) + 3 \qquad 0 \le r < |5|$

Therefore, Quotient $q = -3$, Remainder $r = 3$

(ii) $a = 17, \quad b = -3$

By Euclid's Division Lemma

$a = bq + r$, where $0 \le r < |b|$

$17 = (-3) \times (-5) + 2, \qquad 0 \le r < |-3|$

Therefore Quotient $q = -5$

Remainder $r = 2$

(iii) $a = -19, b = -4$

By Euclid's Division Lemma

$a = bq + r$, where $0 \le r < |b|$

$-19 = (-4) \times (5) + 1 \qquad 0 \le r < |-4|$

Therefore Quotient $q = 5$, Remainder $r = 1$

2.3 Show that the square of an odd integer is of the form $4q + 1$, for some integer q.

Sol : Let x be any odd integer. Since any odd integer is one more than an even integer, we have $x = 2k + 1$, for some integer k.

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4k(k+1) + 1 \\ &= 4q + 1, \text{ where } q = k(k+1) \end{aligned}$$

is some integer.

2.4 If the Highest Common Factor of 210 and 55 is expressible in the form 55x - 325, find x.

Sol : Using Euclid's Division Algorithm, we have

$$\begin{aligned} 210 &= 55 \times 3 + 45 \\ 55 &= 45 \times 1 + 10 \\ 45 &= 10 \times 4 + 5 \\ 10 &= 5 \times 2 + 0 \end{aligned}$$

The remainder is zero.

So, the last divisor 5 is the Highest Common Factor (H.C.F.) of 210 and 55.

Since,

H.C.F. is expressible in the form $55x - 325 = 5$.

$\Rightarrow \quad 55x = 330$

Hence $x = 6$

2.5 Find the greatest number that will divide 445 and 572 leaving remainders 4 and 5 respectively.

Sol : Since the remainders are 4, 5 respectively the required number is the H.C.F. of the number $445 - 4 = 441$, $572 - 5 = 567$.

Hence, we will determine the H.C.F. of 441 and 567. Using Euclid's Division Algorithm, we have

$$\begin{aligned} 567 &= 441 \times 1 + 126 \\ 441 &= 126 \times 3 + 63 \\ 126 &= 63 \times 2 + 0 \end{aligned}$$

Therefore, H.C.F. of 441, 567 = 63 and so the required number is 63.

2.6 Find the H.C.F. of 396, 504, 636.

Sol : To find H.C.F. of three given numbers, first we have to find the H.C.F. of first two numbers.

To find H.C.F. of 396 and 504

Using Euclid's division algorithm,

we get $504 = 396 \times 1 + 108$

The remainder is $108 \neq 0$

Again applying Euclid's division algorithm

$396 = 108 \times 3 + 72$

The remainder is $72 \neq 0$

Again applying Euclid's division algorithm

$108 = 72 \times 1 + 36$

The remainder is $36 \neq 0$

Again applying Euclid's division algorithm

$72 = 36 \times 2 + 0$

Here the remainder is zero. Therefore, H.C.F. of 396, 504 = 36. To find the H.C.F. of 636 and 36.

Using Euclid's division algorithm,

we get $636 = 36 \times 17 + 24$

The remainder is $24 \neq 0$

Again Applying Euclid's algorithm

$36 = 24 \times 1 + 12$

The remainder is $12 \neq 0$

Again applying Euclid's division algorithm

$24 = 12 \times 2 + 0$

Here the remainder is zero. Therefore,

H.C.F. of 636, 36 = 12

Therefore, Highest Common Factor of 396, 504 and 636 is 12.

Progress Check

1. Find q and r for the following pairs of integers a and b satisfying a = bq + r

(i) a = 13, b = 3 (ii) a = 18, b = 4

(iii) a = 21, b = – 4 (iv) a = – 32, b = – 12

(v) a = – 31, b = 7

Ans :

(i) a = 13, b = 3

By Euclid's Division Lemma

$a = bq + r$, where $0 \le r < |b|$

$13 = 3(4) + 1,\ 0 \le r < |3|$

$\therefore$ Quotient q = 4

Remainder r = 1

(ii) a = 18, b = 4

By Euclid's Division Lemma

$a = bq + r$ where $0 \le r < |b|$

$18 = 4 \times 4 + 2,\ 0 \le r < |4|$

$\therefore$ Quotient q = 4

Remainder r = 2

(iii) a = 21, b = – 4

By Euclid's Division Lemma

$a = bq + r$ where $0 \le r < |b|$

$21 = (-4) \times (-5) + 1,\ 0 \le r < |-4|$

$\therefore$ Quotient q = – 5

Remainder r = 1

(iv) a = – 32, b = – 12

By Euclid's Division Lemma

$a = bq + r$, where $0 \le r < |b|$

$-32 = (-12) \times 3 + (4),\ 0 \le r < |-12|$

$\therefore$ Quotient q = 3

Remainder r = 4

(v) a = – 31, b = 7

By Euclid's Division Lemma

$a = bq + r$ where $0 \le r < |b|$

$-31 = 7 \times (-5) + 4,\ 0 \le r < |7|$

$\therefore$ Quotient q = – 5

Remainder r = 4

2. Euclid's division algorithm is a repeated application of division Lemma until we get______.

Ans : Zero remainder.

3. The H.C.F. of two equal positive integers k, [redacted] is ______

Ans : k

Thinking Corner

1. When a positive integer is divided by 3

(i) What are the possible remainders?

(ii) In which form it can be written?

Ans :

(i) Let the positive integer be a.

By Euclid's Division Lemma

$a = bq + r$, where $0 \le r < |b|$

So, $a = 3q + r,\ 0 \le r < |3|$

$a = 3q + r,\ r = 0, 1, 2$

Possible remainders are 0, 1 and 2.

(ii) The positive integer can be written as

$\therefore a = 3q;\ a = 3q + 1,\ a = 3q + 2$

Exercise 2.1

1. Find all positive integers when divided by 3 leaves remainder 2.

Sol : Let the required positive integer be a.

Given a is divided by 3 leaves remainder 2.

By Euclid's Division Lemma a and b are any positive integers, then there exist unique integer q and r such that

$a = bq + r$ where $0 \le r < |b|$

Here $a = 3q + r$ where $0 \le r < |3|$

$r = 0, 1, 2.$

Considering $r = 2$

$a = 3q + 2$, where $q \ge 0$

i.e., $a = 3q + 2,\ q = 0, 1, 2, 3, \ldots$

$a = 3(0) + 2,\ 3(1) + 2,\ 3(2) + 2,\ 3(3) + 2, \ldots$

$a = 2, 5, 8, 11, \ldots$

$\therefore$ The required numbers are 2, 5, 8, 11, ...

2. A man has 532 flower pots. He wants to arrange them in rows such that each row contains 21 flower pots. Find the number of completed rows and how many flower pots are left over.

Sol :

Given number of flower pots = 532

By Euclid's Division Lemma

$a = bq + r$ where $0 \le r < |b|$

$532 = 21 \times 25 + 7$

where $0 \le r < |21|$

$\therefore$ Number of completed rows = 25

Number of flower pots left over = 7

3. Prove that the product of two consecutive positive integers is divisible by 2.

Sol : Let n, n+1 be two consecutive positive integers.

Let $b = 2$.

By Euclid's Division Lemma

$n = bq + r$ where $0 \le r < |b|$

$n = 2q + r$ where $0 \le r < |2|$ i.e., $r = 0, 1$

$\therefore n = 2q$ or $n = 2q + 1$

Case 1:

If $n = 2q$ then $n + 1 = 2q + 1$

Their product $n(n+1) = 2q \times (2q + 1)$

$= (2q)(2q) + 2q$

$= 4q^2 + 2q$

$= 2q(2q + 1)$

$= 2m$ where $m = q(2q + 1)$ and m is positive.

$\therefore n(n + 1)$ is divisible by 2

Case 2:

If $n = 2q + 1$ then $n + 1 = (2q + 1) + 1$

$= 2q + 2$

$\therefore n(n+1) = (2q + 1)(2q + 2)$

$= 4q^2 + 2q + 4q + 2$

$= 4q^2 + 6q + 2$

$= 2(2q^2 + 3q + 1)$

$= 2p$, where $p = 2q^2 + 3q + 1$ and p is positive.

$\therefore n(n + 1)$ is divisible by 2.

Thus we conclude that,

Product of two consecutive positive integer is divisible by 2.

4. When the positive integer a, b and c are divided by 13, the respective remainders are 9, 7 and 10. Show that a + b + c is divisible by 13.

Sol : Given the positive integers a, b and c divided by 13 leaves remainders 9, 7, 10 respectively.

By Euclid's division Lemma we have

$a = 13q_1 + 9$

$b = 13q_2 + 7$

$c = 13q_3 + 10$

$a + b + c = 13q_1 + 9 + 13q_2 + 7 + 13q_3 + 10$

$= 13q_1 + 13q_2 + 13q_3 + 26$

$= 13q_1 + 13q_2 + 13q_3 + 13(2)$

$= 13[q_1 + q_2 + q_3 + 2] = 13m$

where $m = q_1 + q_2 + q_3 + 2$ and m is positive integer.

which is divisible by 13.

$\therefore a + b + c$ is divisible by 13.

5. Prove that square of any integer leaves the remainder either 0 or 1 when divided by 4.

Sol : All the integers 'a' must be either even or odd.

If it is even then $a = 2q$.

If it is odd then $a = 2q + 1$

Case 1:

If $a = 2q$

$a^2 = (2q)^2$

$a^2 = 4q^2$, remainder 0 when divided by 4.

Case 2:

If $a = 2q + 1$

$a^2 = (2q + 1)^2$

$= 4q^2 + 4q + 1$

$= 4q(q + 1) + 1$

$a^2 = 4m + 1$ where $m = q(q + 1)$ is an integer.

It is of the form $bq + 1$ where 1 is the remainder when divided by 4.

$\therefore$ The square of any integer leaves the remainder either 0 or 1 when divided by 4.

6. Use Euclid's Division Algorithm to find the Highest Common Factor (H.C.F) of

(i) 340 and 412 (ii) 867 and 255

(iii) 10224 and 9648 (iv) 84, 90 and 120

Sol :

(i) To find the H.C.F. of (340, 412),

$412 > 340$ = H.C.F. (412, 340)

Using Euclid's division algorithm we have

$412 = 340 \times 1 + 72$

The remainder $72 \ne 0$

Again applying Euclid's division algorithm

$340 = 72 \times 4 + 52$

The remainder $52 \ne 0$

Again applying Euclid's division algorithm

$72 = 52 \times 1 + 20$

The remainder $20 \ne 0$

Again applying Euclid's division algorithm

$52 = 20 \times 2 + 12$

The remainder $12 \ne 0$

$\therefore$ Again applying Euclid's division algorithm

$20 = 12 \times 1 + 8$

The remainder $8 \ne 0$

Again applying Euclid's Division Algorithm

$12 = 8(1) + 4$

The remainder $4 \ne 0$,

Again applying Euclid's Division Algorithm

$8 = 4(2) + 0$

Now the remainder $= 0$

$\therefore$ H.C.F. (340, 412) $= 4$

(ii) 867 and 255

Here $867 > 255$

Applying repeatedly Euclid's Division Algorithm until getting remainder zero, we have.

$$867 = 255(3) + 102$$
$$255 = 102(2) + 51$$
$$102 = 51(2) + 0$$

The remainder $= 0$

$\therefore$ H.C.F. (867, 255) $= 51$

(iii) 10224 and 9648

Here $10224 > 9648$

Applying repeatedly Euclid's Division Algorithm until getting remainder zero,

we have,
$$10224 = 9648(1) + 576$$
$$9648 = 576(16) + 432$$
$$576 = 432(1) + 144$$
$$432 = 144(3) + 0$$

The remainder $= 0$

$\therefore$ H.C.F. (10224, 9648) $= 144$

(iv) 84, 90 and 120

First we will find the H.C.F. of 84 and 90.

H.C.F. (90, 84)

Applying Euclid's Division Algorithm until we get remainder zero.

$$90 = 84(1) + 6$$
$$84 = 6(14) + 0$$

Remainder $= 0$

$\therefore$ H.C.F. (90, 84) $= 6$

Now finding H.C.F. (120, 6) we have

$$120 = 6(20) + 0$$

Remainder $= 0$

$\therefore$ H.C.F. is 6.

So H.C.F. of 84, 90 and 120 is 6.

7. Find the largest number which divides 1230 and 1926 leaving remainder 12 in each case.

Sol : Since the remainder is 12 in each case, the required number is H.C.F. of the numbers $1230 - 12 = 1218$ and $1926 - 12 = 1914$. Now to find H.C.F. of 1218 and 1914, we use Euclid's Division Algorithm, we have,

$$1914 = 1218 \times 1 + 696$$
$$1218 = 696 \times 1 + 522$$
$$696 = 522 \times 1 + 174$$
$$522 = 174 \times 3 + 0$$

$\therefore$ H.C.F. of 1218, 1914 $= 174$

$\therefore$ The required number is 174.

8. If d is the Highest Common Factor of 32 and 60, find x and y satisfying $d = 32x + 60y$.

Sol : Using Euclid's Division Algorithm, we have

$$60 = 32 \times 1 + 28 \quad \dots (1)$$
$$32 = 28 \times 1 + 4 \quad \dots (2)$$
$$28 = 4 \times 7 + 0$$

The remainder is zero.

$\therefore$ The last divisor 4 is the H.C.F. of 60 and 32.

Given $d = 32x + 60y$

From (2), we have $4 = 32 - 28 \times 1 \quad \dots (3)$

Also from (1) we have $28 = 60 - 32 \times 1 \quad \dots (4)$

Substituting (4) in (3), we have,

$$4 = 32 - (60 - 32 \times 1) \times 1$$
$$4 = 32 - 60 + 32 \times 1$$

$4 = 32(2) + 60(-1)$ which is of the form

$d = 32x + 60y$ where $x = 2$ and $y = -1$

$\therefore x = 2; y = -1$

9. A positive integer when divided by 88 gives the remainder 61. What will be the remainder when the same number is divided by 11?

Sol : Let the positive integer be 'n'

So $n = 88(p) + 61$, where p be an integer

$$n = 88(p) + (5 \times 11 + 6)$$
$$n = 8 \times 11 \times p + 5 \times 11 + 6$$
$$n = 11(8p + 5) + 6$$

Dividing both the sides by 11, we get

$$\frac{n}{11} = (8p+5) + \frac{6}{11}$$

$\therefore$ When the same number 'n' is divided by 11 the remainder will be 6.

10. Prove that two consecutive positive integers are always coprime.

Sol : Let the two consecutive positive integers be n and $n + 1$.

Here $n + 1 > n$

By Euclid's Division Algorithm, we have

$n + 1 = n \times q + 1$ for some integer q.

Remainder $= 1$

$\therefore$ Again applying Euclid' s Division Algorithm

$$n = 1 \times (n) + 0$$

Here the remainder $= 0$

$\therefore$ H.C.F. is the last divisor 1.

$\therefore$ H.C.F. $(n, n + 1) = 1$

i.e., H.C.F. of two consecutive positive integers $= 1$

$\therefore$ Two consecutive positive integers are always coprime.

FUNDAMENTAL THEOREM OF ARITHMETIC

Key Points

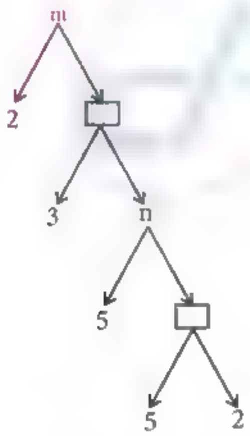
Theorem: (Fundamental Theorem of Arithmetic)

"Every natural number except 1 can be factorized as a product of primes and this factorization is unique except for the order in which the prime factors are written".

- For a composite number N, we decompose it uniquely in the form
  $N = p_1^{q_1} \times p_2^{q_2} \times p_3^{q_3} \times \ldots \times p_n^{q_n}$ where $p_1, p_2, p_3, \ldots, p_n$ are primes and $q_1, q_2, q_3, \ldots, q_n$ are natural numbers.
- Thus every composite number can be written uniquely as the product of power of primes.
- If a prime number p divides ab then either p divides a or p divides b. That is p divides atleast one of them.

Worked Examples

2.7 In the given factor tree, find the numbers m and n.



Sol :

Value of the first box from bottom

$= 5 \times 2 = 10$

Value of $n = 5 \times 10 = 50$

Value of the second box from bottom

$= 3 \times 50 = 150$

Value of $m = 2 \times 150 = 300$

Thus, the required numbers are

$m = 300, n = 50$

2.8 Can the number 6^n, n being a natural number end with the digit 5? Give reason for your answer.

Sol :

Since $6^n = (2 \times 3)^n = 2^n \times 3^n$

2 is a factor of 6^n

So, 6^n is always even.

But any number whose last digit is 5 is always odd.

Hence, 6^n cannot end with the digit 5.

2.9 Is $7 \times 5 \times 3 \times 2 + 3$ a composite number? Justify your answer.

Sol :

Yes, the given number is a composite number.

Because,

$7 \times 5 \times 3 \times 2 + 3 = 3 \times (7 \times 5 \times 2 + 1) = 3 \times 71$

Since the given number can be factorized in terms of two primes, it is a composite number.

2.10 'a' and 'b' are two positive integers such that $a^b \times b^a = 800$. Find 'a' and 'b'.

Sol :

The number 800 can be factorized as

$800 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^5 \times 5^2$

Hence, $a^b \times b^a = 2^5 \times 5^2$

The implies that $a = 2$ and $b = 5$ (or)

$a = 5$ and $b = 2$.

Progress Check

1. **Every natural number except _____ can be expressed as _____.**

   Ans : 1, product of primes.

2. **In how many ways a composite number can be written as product of power of primes?**

   Ans : In a unique way.

3. The number of divisors of any prime number is ______.

Ans : 2

4. Let m divides n. Then GCD and LCM of m, n are _____ and ______.

Ans : m and n

5. The HCF of numbers of the form 2^m and 3^n is _____.

Ans : 1

Thinking Corner

1. Is 1 a prime number?

Ans : No, 1 is not a prime number.

Because the definition of a prime number is "a positive integer that has exactly two positive divisors (1 and itself) is a prime number".

But 1 has only one positive divisor 1 itself.

∴ So it is not a prime.

2. Can you think of positive integers a, b such that $a^b = b^a$?

Ans : Take a = 2 and b = 4

We have $2^4 = 4^2$

Exercise 2.2

1. For what values of natural number n, 4^n can end with the digit 6?

Sol : for some natural number n,

$4^n = (2)^{2n}$

So 2 is a factor of 4^n

By fundamental theorem of arithmetic, we know the factorization of 4^n is unique

∴ Only factor of 4^n is 2, even number of times.

But 4^n always end with 4 or 6.

If n is odd then 4^n end with 4.

If n is even, then 4^n end with the digit 6.

2. If m, n are natural numbers, for what values of m, does $2^n \times 5^m$ ends in 5?

Sol : Consider $2^n \times 5^m$

Since the product has 2 as a factor

$2^n \times 5^m$ is even [∵ *n is natural*]

But if a number ends with the digit 5, then the number is an odd number

It is impossible.

For no value of m, $2^n \times 5^m$ ends in 5.

3. Find the HCF of 252525 and 363636.

Sol :

5	252525
5	50505
3	10101
7	3367
13	481
	37

3	363636
3	121212
2	40404
2	20202
3	10101
7	3367
13	481
	37

$252525 = 3^1 \times 5^2 \times 7^1 \times 13^1 \times 37^1$

$363636 = 2^2 \times 3^3 \times 7^1 \times 13^1 \times 37^1$

$\text{H.C.F.} = 3^1 \times 7^1 \times 13^1 \times 37^1$

$= 3 \times 3367$

$= 10101$

4. If $13824 = 2^a \times 3^b$ then find a and b.

Sol :

2	13824
2	6912
2	3456
2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	9
	3

The number 13824 can be factorized as

$2^a \times 3^b = 13824 = 2^9 \times 3^3$

∴ a = 9 and b = 3.

5. If $p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4} = 113400$ where p_1, p_2, p_3, p_4 are primes in ascending order and x_1, x_2, x_3, x_4 are integers, find the value of p_1, p_2, p_3, p_4 and x_1, x_2, x_3, x_4.

Sol :

Given $113400 = p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4}$

$$\begin{array}{r|l} 2 & 113400 \\ \hline 2 & 56700 \\ \hline 2 & 28350 \\ \hline 3 & 14175 \\ \hline 3 & 4725 \\ \hline 3 & 1575 \\ \hline 3 & 525 \\ \hline 5 & 175 \\ \hline 5 & 35 \\ \hline 7 & 7 \\ \hline & 1 \end{array}$$

The number 113400 can be factorized as

$113400 = 2^3 \times 3^4 \times 5^2 \times 7^1$ where

2, 3, 5, 7 are primes in ascending order.

$\therefore$ Comparing

$p_1^{x_1} \times p_2^{x_2} \times p_3^{x_3} \times p_4^{x_4} = 2^3 \times 3^4 \times 5^2 \times 7^1$

$p_1 = 2;\ p_2 = 3;\ p_3 = 5;\ p_4 = 7$

$x_1 = 3;\ x_2 = 4;\ x_3 = 2;\ x_4 = 1.$

6. Find the L.C.M. and H.C.F. of 408 and 170 by applying the fundamental theorem of Arithmetic.

Sol : By fundamental theorem, every composite number can be expressed as a product of primes.

$$\begin{array}{r|l} 2 & 170 \\ \hline 5 & 85 \\ \hline & 17 \end{array} \qquad \begin{array}{r|l} 2 & 408 \\ \hline 2 & 204 \\ \hline 2 & 102 \\ \hline 3 & 51 \\ \hline & 17 \end{array}$$

$\therefore$ Factorizing 408 and 170 we get

$408 = 2^3 \times 3^1 \times 17^1$

$170 = 2^1 \times 5^1 \times 17^1$

H.C.F. of 408 and 170 $= 2^1 \times 17^1 = 34$

Also we know that H.C.F. $\times$ L.C.M.

= product of two numbers

$34 \times$ L.C.M. $= 408 \times 170$

$$\text{L.C.M.} = \frac{408 \times 170}{34} = 2040$$

$\therefore$ H.C.F. (408, 170) = 34; L.C.M. (408, 170) = 2040

7. Find the greatest number consisting 6 digits which is exactly divisible by 24, 15, 36?

Sol :

$$\begin{array}{r|l} 3 & 24,\ 15,\ 36 \\ \hline 2 & 8,\ 5,\ 12 \\ \hline 2 & 4,\ 5,\ 6 \\ \hline & 2,\ 5,\ 3 \end{array}$$

L.C.M. $= 3 \times 2 \times 2 \times 2 \times 5 \times 3 = 360$

Greatest number of 6 digit is 999999

L.C.M. of 24, 15 and 36 = 360

$$\begin{array}{r|l} & 2777 \\ \hline 360 & 999999 \\ & 720 \\ \hline & 2799 \\ & 2520 \\ \hline & 2799 \\ & 2520 \\ \hline & 2799 \\ & 2520 \\ \hline & 279 \end{array}$$

On dividing 999999 by 360 remainder obtained is 279.

$\therefore$ Greatest number of 6 digit, divisible by 24, 15 and 36 = 999999 – 279 = 999720

Hence the required number is = 999720

8. What is the smallest number that when divided by 35, 56 and 91 leaves remainder 7 in each case?

Sol :

$$\begin{array}{r|l} 7 & 35,\ 56,\ 91 \\ \hline & 5,\ 8,\ 13 \end{array}$$

The required number = L.C.M. of (35, 56, 91) + 7

$\therefore$ L.C.M. of 35, 56, 91 $= 7 \times 5 \times 8 \times 13$

= 3640

The required number = 3640 + 7

= 3647

9. Find the least number that is divisible by the first ten natural numbers.

Sol :

The required number is the L.C.M. of first ten natural numbers

i.e., L.C.M. of

(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)

$$\begin{array}{r|l} 5 & 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\ \hline 2 & 1, 2, 3, 4, 1, 6, 7, 8, 9, 2 \\ \hline 3 & 1, 1, 3, 2, 1, 3, 7, 4, 9, 1 \\ \hline 2 & 1, 1, 1, 2, 1, 1, 7, 4, 3, 1 \\ \hline & 1, 1, 1, 1, 1, 1, 7, 2, 3, 1 \end{array}$$

L.C.M. is $5 \times 2 \times 3 \times 2 \times 7 \times 2 \times 3 = 2520$

$\therefore$ The least number divisible by first ten natural numbers is 2520.

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MODULAR ARITHMETIC

Key Points

Modular Arithmetic

1. Modular arithmetic is a system of arithmetic for integers where numbers wrap around a certain value.
2. Modular arithmetic process cyclically.
3. The German mathematician Carl Friedrich Gauss developed modular arithmetic. He is hailed as Prince of mathematicians.

Congruence modulo

1. If b – a = kn for some integer k. Then it can be written as $a \equiv b \pmod{n}$. Here n is called modulus.
2. If $a \equiv b \pmod{n}$ then a – b is divisible by n. Example: $61 \equiv 5 \pmod{7}$. Here 61 – 5 = 56 is divisible by 7.
3. When a positive integer is divided by n, then the possible remainders are 0, 1, 2......., n – 1.
4. The equation n = mq + r through Euclid's Division lemma can also be written as $n \equiv r \pmod{m}$
5. Two integers a and b are congruent modulo m, written as $a \equiv b \pmod{m}$ if they leave the same remainder when divided by m.
6. While solving congruent equations we get infinitely many solutions.

Theorem:

1. a, b, c and d are integers and m is a positive integer such that if $a \equiv b \pmod{m}$ and then $c \equiv d \pmod{m}$ then

   (i) $(a + c) = (b + d) \pmod{m}$

   (ii) $(a - c) = (b - d) \pmod{m}$

   (iii) $(a \times c) = (b \times d) \pmod{m}$

Theorem:

If $a \equiv b \pmod{m}$ then

(i) $ac \equiv bc \pmod{m}$

(ii) $a \pm c \equiv b \pm c \pmod{m}$ for some integer c.

Worked Examples

2.11 Find the remainders when 70004 and 778 is divided by 7.

Sol :

Since 70000 is divisible by 7.

$$70000 \equiv 0 \pmod{7}$$
$$70000 + 4 \equiv 0 + 4 \pmod{7}$$
$$70004 \equiv 4 \pmod{7}$$

Therefore, the remainder when 70004 is divided by 7 is 4.

Since 777 is divisible by 7

$$777 \equiv 0 \pmod{7}$$
$$777 + 1 \equiv 0 + 1 \pmod{7}$$
$$778 \equiv 1 \pmod{7}$$

Therefore, the remainder when 778 is divided by 7 is 1

2.12 Determine the value of d such that $15 \equiv 3 \pmod{d}$.

Sol :

$15 \equiv 3 \pmod{d}$ means $15 - 3 = kd$, for some integer k.

$12 = kd$

$\Rightarrow$ d divides 12.

The divisors of 12 are 1, 2, 3, 4, 6, 12. But d

should be larger than 3 and so the possible values for d are 4, 6, 12.

2.13 **Find the least positive value of x such that**

(i) $67+x\equiv 1 \pmod 4$ **(ii)** $98 \equiv (x+4) \pmod 5$

Sol :

(i) $67 + x \equiv 1 \pmod 4$

$\Rightarrow 67 + x - 1 = 4n$, for some integer n

$66 + x = 4n$

$\Rightarrow 66 + x$ is a multiple of 4.

Therefore, the least positive value of x must be 2, since 68 is the nearest multiple of 4 more than 66.

(ii) $98 \equiv (x + 4) \pmod 5$

$\Rightarrow 98 - (x + 4) = 5n$, for some integer n.

$94 - x = 5n$

$\Rightarrow 94 - x$ is a multiple of 5

Therefore, the least positive value of x must be 4 Since $94 - 4 = 90$ is the nearest multiple of 5 less than 94

2.14 **Solve $8x \equiv 1 \pmod{11}$**

Sol :

$8x\equiv 1 \pmod{11}$ can be written as $8x - 1 = 11k$, for some integer k.

$$x = \frac{11k+1}{8}$$

When we put $k = 5, 13, 21, 29,....$ then $k + 1$ is divisible by 8.

$$x = \frac{11\times 5+1}{8} = 7$$

$$x = \frac{11\times 13+1}{8} = 18$$

Therefore, the solutions are 7, 18, 29, 40, ...

2.15 **Compute x, such that $10^4 \equiv x \pmod{19}$**

Sol : $10^2 = 100 \equiv 5 \pmod{19}$

$10^4 = (10^2)^2 \equiv 5^2 \pmod{19}$

$10^4 \equiv 25 \pmod{19}$

$10^4 \equiv 6 \pmod{19}$

(since $25 \equiv 6 \pmod{19}$)

Therefore, $x = 6$.

2.16 **Find the number of integer solutions of $3x\equiv 1 \pmod{15}$**

Sol : $3x \equiv 1 \pmod{15}$ can be written as

$3x - 1 = 15k$ for some integer k

$3x = 15k + 1$

$$x = \frac{15k+1}{3}$$

$$x = 5k+\frac{1}{3}$$

Since 5k is an integer, $5k+\frac{1}{3}$ cannot be an integer.

So there is no integer solution.

2.17 **A man starts his journey from Chennai to Delhi by train. He starts at 22.30 hours on Wednesday. If it takes 32 hours of travelling time and assuming that the train is not late, when will he reach Delhi?**

Sol :

Starting time 22.30, Travelling time 32 hours.

Here we use modulo 24.

The reaching time is

$22.30 + 32 \pmod{24} \equiv 54.30 \pmod{24}$

$\equiv 6.30 \pmod{24}$

Since $32 = (1 \times 24)$(Thursday) + 8(Friday)

Thus, he will reach Delhi at 6.30 hours on Friday.

2.18 **Kala and Vani are friends. Kala says "Today is my birthday" and she asked Vani "When will you celebrate your birthday". Vani replied "Today is Monday and I celebrated my brithday 75 days ago". Find the day when Vani celebrated her birthday?**

Sol :

Let us associate the numbers 0, 1, 2, 3, 4, 5, 6 to represent the weekdays from Sunday to Saturday respectively.

Vani says today is Monday. So the number for Monday is 1. Since Vani's birthday was 75 days ago, we have to subtract 75 from 1 and take the modulo 7, since a week contain 7 days.

$-74 \pmod 7 \equiv -4 \pmod 7 \equiv 7-4 \pmod 7 \equiv 3 \pmod 7$

(Since, $-74 - 3 = -77$ is divisible by 7)

Thus, $1 - 75 \equiv 3 \pmod 7$

The day for the number 3 is Wednesday.

Therefore, Vani's birthday must be on Wednesday.

Progress Check

1. Two integers a and b are congruent modulo n if

Ans : They differ by an integer multiple of n or if $b - a = kn$ for some integer k.

2. The set of all positive integers which leave remainder 5 when divided by 7 are _____

Ans : $n \equiv 5 \pmod 7$

3. The positive values of k such that $(k-3) \equiv 5 \pmod{11}$ are _____

Ans : 19, 30, 41, 52, ...

4. If $59 \equiv 3 \pmod 7$, $46 \equiv 4 \pmod 7$ then $105 \equiv$ _____ (mod 7), $13 \equiv$ _____ (mod 7), $413 \equiv$ _____ (mod 7), $368 \equiv$ _____ (mod 7)

Ans : 7, –1, 0, 4

5. The remainder when $7 \times 13 \times 19 \times 23 \times 29 \times 31$ is divided by 6 is _____

Ans : 1

Thinking Corner

1. How many integers exist which leave a remainder of 2 when divided by 3?

Ans : Infinitely many numbers exist.

Exercise 2.3

1. Find the least positive value of x such that

(i) $71 \equiv x \pmod 8$ (ii) $78 + x \equiv 3 \pmod 5$

(iii) $89 \equiv (x+3) \pmod 4$

(iv) $96 \equiv \frac{x}{7} \pmod 5$ (v) $5x \equiv 4 \pmod 6$

Sol :

(i) $71 \equiv x \pmod 8$

$71 - x = 8n$ for some integer n

$71 - x$ is a multiple of 8

$\therefore$ The least positive value of x must be 7.

(ii) $78 + x = 3 \pmod 5$

$78 + x - 3 = 5n$ for some integer n.

$75 + x = 5n$ for some integer n.

$75 + x$ is a multiple of 5

$\therefore$ least positive value of $x = 5$

(iii) $89 \equiv (x+3) \pmod 4$

$89 - (x+3) = 4n$ for some integer n

$89 - x - 3 = 4n$ for some integer n.

$86 - x - 4n$, for some integer n.

$86 - x$ is a multiple of 4.

$\therefore$ x= 2 is the least positive value.

(iv) $96 \equiv \frac{x}{7} \pmod 5$

$\left(96 - \frac{x}{7}\right) = 5n$ for some integer n.

$\frac{672 - x}{7} = 5n$ for some integer n.

$\therefore$ Least positive value of x = 7

(v) $5x \equiv 4 \pmod 6$

$5x - 4 = 6n$ for some integer n

$5x - 4$ is a multiple of 6.

$x = 2$ is the least positive x.

2. If x is congruent to 13 modulo 17 then $7x - 3$ is congruent to which number modulo 17?

Sol :

Given $x \equiv 13 \pmod{17}$

$7x \equiv 13 \times 7 \pmod{17}$

$7x - 3 \equiv (91 - 3) \pmod{17}$

$7x - 3 \equiv 88 \pmod{17}$

$7x - 3 \equiv 3 \pmod{17}$

$7x - 3$ is congruent to 3 modulo 17.

3. Solve $5x \equiv 4 \pmod 6$

Sol :

$5x \equiv 4 \pmod 6$

$5x - 4 = 6k$ for some integer k.

$$x = \frac{6k+4}{5}$$

When we put k = 1, 6, 11, 16...

then $6k + 4$ is divisible by 5

$$\therefore \quad x = \frac{6(1)+4}{5} = \frac{10}{5} = 2$$

$$x = \frac{6(6)+4}{5} = \frac{40}{5} = 8$$

$$x = \frac{6(11)+4}{5} = \frac{70}{5} = 14$$

$$x = \frac{6(16)+4}{5} = \frac{100}{5} = 20$$

Therefore, the solutions are 2, 8, 14, 20,...

4. Solve $3x - 2 \equiv 0 \pmod{11}$

Sol :

$3x - 2 \equiv 0 \pmod{11}$

$3x - 2 = 11k$ for some k

$3x = 11k + 2$

$$x = \frac{11k+2}{3}$$

When we put k = 2, 5, 8, 11,..., $11k + 2$ is divisible by 3.

$$\therefore x = \frac{11(2)+2}{3} = \frac{24}{3} = 8$$

$$x = \frac{11(5)+2}{3} = \frac{57}{3} = 19$$

$$x = \frac{11(8)+2}{3} = \frac{90}{3} = 30$$

$$x = \frac{11(11)+2}{3} = \frac{123}{3} = 41$$

$\therefore$ The solutions are 8, 19, 30, 41,.......

5. What is the time 100 hours after 7 a.m.?

Sol : Starting from 7 o' clock 100 hours.

we use modulo 24.

7 o' clock a.m. + 100 (modulo 24) = 7 o' clock + 4 hrs

= 11 o' clock a.m.

$\therefore$ 100 hrs after 7 o' clock a.m. is 11 o' clock a.m.

6. What is the time 15 hours before 11 p.m?

Sol : We have to calculate the time from 11 o' clock p.m.

It is 15 hours before 11 o' clock pm.

Subtracting 15 from 11.

$$\begin{aligned} 11-15 &\equiv -4 \bmod 12 \\ &\equiv 12-4 \pmod{12} \\ &\equiv 8 \pmod{12} \\ 11-15 &\equiv 8 \pmod{12} \end{aligned}$$

$\therefore$ It is 8 'o' clock a.m.

7. Today is Tuesday. My uncle will come after 45 days. In which day my uncle will be coming?

Sol :

Starting from Tuesday we have to calculate the day after 45 days.

The number for Tuesday is 2.

$$\begin{aligned} 2+45 \pmod 7 &\equiv 47 \pmod 7 \\ &\equiv 5 \pmod 7 \end{aligned}$$

Number 5 stands for Friday.

$\therefore$ Uncle will be coming on Friday.

8. Prove that $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n.

Sol :

7 is divisible by 7

$$\begin{aligned} \therefore 7 &\equiv 0 \pmod 7 \\ 7+2 &\equiv 0+2 \pmod 7 \end{aligned}$$

[$\because$ Adding the constant 2]

$$\begin{aligned} 9 &\equiv 2 \pmod 7 \\ 9^n &\equiv 2^n \pmod 7 \\ 6\times 9^n &\equiv 6\times 2^n \pmod 7 \end{aligned}$$

[$\because$ multiplying by the constant 6]

$$2^n + 6\times 9^n \equiv 2^n + 6\times 2^n \pmod 7$$

[$\because$ Adding 2^n]

$$\begin{aligned} &\equiv 2^n(1+6) \pmod 7 \\ &\equiv 2^n \times 7 \pmod 7 \\ &\equiv 2^n \times 0 \pmod 7 \\ &\equiv 0 \pmod 7 \end{aligned}$$

$\therefore$ When $2^n + 6 \times 9^n$ is divided by 7 we get remainder 0.

Hence $2^n + 6 \times 9^n$ is always divisible by 7 for any positive integer n.

9. Find the remainder when 2^{81} is divided by 17.

Sol :

$$\begin{aligned} \text{First take } 2^5 &\equiv 15 \pmod{17} \\ (2^5)^2 &\equiv 15^2 \pmod{17} \\ &\equiv 4 \pmod{17} \\ 2^{10} &\equiv 4 \pmod{17} \\ (2^{10})^4 &\equiv 4^4 \pmod{17} \\ 2^{40} &\equiv 1 \bmod 17 \\ (2^{40})^2 &\equiv 1^2 \pmod{17} \\ 2.2^{80} &\equiv 2\times 1 \pmod{17} \\ 2^{81} &\equiv 2 \bmod 17 \end{aligned}$$

Remainder when 2^{81} is divided by 17 is 2

10. The duration of flight travel from Chennai to London through British Airlines is approximately 11 hours. The airplane begins its journey on Sunday at 23:30 hours. If the time to Chennai is four and half hours ahead to that of London's time, then find the time at London, when the flight lands at London Airport?

Sol :

Starting time from Chennai = 23.30 hrs

Travelling time = 11 hrs

Here we use modulo 24.

$$\begin{aligned} \therefore \text{ Reaching time } &= 23.30 + 11 \pmod{24} \\ &= 34.30 \pmod{24} \\ &= 10.30 \pmod{24} \end{aligned}$$

$$\text{Since } 11 = \underbrace{0 \times 24}_{\text{Sunday}} + \underbrace{11}_{\text{Monday}}$$

It reaches London on Monday at 10.30 a.m

Chennai time = 4.30 hrs + London time

London time = Chennai time – 4.30 a.m

= 10.30 – 4.30 = 6 a.m

$\therefore$ The flight will land at London Airport on Monday at 6 a.m

SEQUENCES

Key Points

Definition

1. A real valued sequence is a function defined on the set of natural numbers and taking real values.
2. Each element in the sequence is called 'a term' of the sequence.
3. General form of a sequence is $a_1, a_2, a_3 \ldots\ldots a_n, \ldots\ldots$
4. If the number of elements in the sequence is finite then it is called a finite sequence.
5. If the number of elements in a sequence is infinite then it is called an infinite sequence.
6. A sequence can be considered as a function defined on the set of natural numbers N.
7. A sequence is a function $f : N \to R$, where R is the set of all real numbers.

Worked Examples

2.19 Find the next three terms of the sequences

(i) $\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \ldots$ (ii) 5, 2, –1, –4,.... (iii) 1, 0.1, 0.01,....

Sol :

(i) $\frac{1}{2} \xrightarrow{+4} \frac{1}{6} \xrightarrow{+4} \frac{1}{10} \xrightarrow{+4} \frac{1}{14}, \ldots$

In the above sequence the numerators are same and the denominator is increased by 4. So the next three terms are

$$a_5 = \frac{1}{14+4} = \frac{1}{18}$$

$$a_6 = \frac{1}{18+4} = \frac{1}{22}$$

$$a_7 = \frac{1}{22+4} = \frac{1}{26}$$

(ii) $5 \xrightarrow{-3} 2 \xrightarrow{-3} -1 \xrightarrow{-3} -4, \ldots$

Here each term is decreased by 3. So the next three terms are – 7, – 10, – 13.

(iii) $1 \xrightarrow{\div 10} 0.1 \xrightarrow{\div 10} 0.01, \ldots$

Here each term is divided by 10. Hence, the next three terms are

$$a_4 = \frac{0.01}{10} = 0.001$$

$$a_5 = \frac{0.001}{10} = 0.0001$$

$$a_6 = \frac{0.0001}{10} = 0.00001$$

2.20 Find the general term for the following sequences.

(i) 3, 6, 9, ... (ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ (iii) $5, -25, 125, \ldots$

Sol :

(i) 3, 6, 9,...

Here the terms are multiples of 3. So the general term is

$a_n = 3n,\ n \in N$

(ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$

$a_1 = \frac{1}{2};\ a_2 = \frac{2}{3};\ a_3 = \frac{3}{4}$

We see that the numerator of n^{th} term is n, and the denominator is one more than the numerator. Hence, $a_n = \frac{n}{n+1}, n \in N$

(iii) 5, – 25, 125,...

The terms of the sequence have + and – sign alternatively and also they are in powers of 5.

So the general term $a_n = (-1)^{n+1}5^n, n \in N$

2.21 The general term of a sequence is defined as

$$a_n = \begin{cases} n(n+3);\ n \in N \text{ is odd} \\ n^2+1;\ n \in N \text{ is even} \end{cases}$$

Find the eleventh and eighteenth terms.

Sol :

To find a_{11}, since 11 is odd, we put n = 11 in

$a_n = n(n+3) \Rightarrow a_{11} = 11 \times 14$

Thus, the eleventh term $a_{11} = 154$

To find a_{18}, since 18 is even,

we put n = 18 in $a_n = n^2 + 1$

Thus, the eighteenth term $a_{18} = 18^2 + 1 = 325$

2.22 Find the first five terms of the following sequence.

$$a_1 = 1, a_2 = 1, a_n = \frac{a_{n-1}}{a_{n-2}+3}; n \geq 3, n \in N$$

Sol : The first two terms of this sequence are given by $a_1 = 1, a_2 = 1$. The third term a_3 depends on the first and second terms.

$$a_3 = \frac{a_{3-1}}{a_{3-2}+3} = \frac{a_2}{a_1+3} = \frac{1}{1+3} = \frac{1}{4}$$

Similarly the fourth term a_4 depends upon a_2 and a_3.

$$a_4 = \frac{a_{4-1}}{a_{4-2}+3} = \frac{a_3}{a_2+3} = \frac{\frac{1}{4}}{1+3} = \frac{\frac{1}{4}}{4} = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

In the same way, the fifth term a_5 can be calculated as

$$a_5 = \frac{a_{5-1}}{a_{5-2}+3} = \frac{a_4}{a_3+3} = \frac{\frac{1}{16}}{\frac{1}{4}+3} = \frac{1}{16} \times \frac{4}{13} = \frac{1}{52}$$

Therefore, the first five terms of the sequence are

$1, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{25}$

Progress Check

1. Fill in the blanks for the following sequences.

(i) 7, 13, 19, _____,...

(ii) 2, _____, 10, 17, 26, ...

(iii) 1000, 100, 10, 1, _____,...

Ans : (i) 25, (ii) 5, (iii) 1/10

2. A sequence is a function defined on the set of _____

Ans : Natural Numbers.

3. The n^{th} term of a sequence 0, 2, 6, 12, 20, ... can be expressed as _____

Ans : $a_n = n(n-1)$

4. Say True or False

(i) All sequences are functions

Ans : True.

(ii) All functions are sequences

Ans : False.

Exercise 2.4

1. Find the next three terms of the following sequence.

(i) 8, 24, 72, ... **(ii) 5, 1, – 3**

(iii) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, ...$

Sol :

(i) 8, 24, 72, ...

$$8 \xrightarrow{\times 3} 24 \xrightarrow{\times 3} 72 \xrightarrow{\times 3} 216 \xrightarrow{\times 3} 648 \xrightarrow{\times 3} 1944$$

Each term is obtained by multiplying the previous term by 3.

∴ Next three terms are 216, 648, 1944.

(ii) 5, 1, –3

$$5 \xrightarrow{-4} 1, \xrightarrow{-4} -3, ...$$

Here each term is obtained by subtracting 4 from the previous term.

∴ Next three terms are $a_4 = -3 - 4 = -7$

$a_5 = -7 - 4 = -11$

$a_6 = -11 - 4 = -15$

(iii) $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, ...$

$$\frac{1}{4} \xrightarrow{+1} \frac{2}{9} \xrightarrow{+1} \frac{3}{16}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ 2^2 & 3^2 & 4^2 \end{array}$$

This sequence is generated by $a_n = \frac{n}{(n+1)^2}$

The numerator is increased by 1 in successive terms.

The denominators are $2^2, 3^2, 4^2$,...

$$\therefore a_4 = \frac{4}{5^2} = \frac{4}{25}$$

$$a_5 = \frac{5}{6^2} = \frac{5}{36}$$

$$a_6 = \frac{6}{7^2} = \frac{6}{49}.$$

∴ The next three terms are $\frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

2. Find the first four terms of the sequences whose n^{th} terms are given by

(i) $a_n = n^3 - 2$

(ii) $a_n = (-1)^{n+1} n(n+1)$ (iii) $a_n = 2n^2 - 6$

Sol :

(i) $a_n = n^3 - 2$

$\therefore$ The first term $a_1 = 1^3 - 2 = 1 - 2 = -1$

Second term $a_2 = 2^3 - 2 = 8 - 2 = 6$

Third term $a_3 = 3^3 - 2 = 27 - 2 = 25$

Fourth term $a_4 = 4^3 - 2 = 64 - 2 = 62$

$\therefore$ First four terms are $-1, 6, 25, 62$

(ii) $a_n = (-1)^{n+1} n(n+1)$

n^{th} term is given by $a_n = (-1)^{n+1} n(n+1)$

The first term $a_1 = (-1)^{1+1} 1(1+1) = 2$

Second term $a_2 = (-1)^{2+1} 2(2+1) = -6$

Third term $a_3 = (-1)^{3+1} 3(3+1) = 12$

Fourth term $a_4 = (-1)^{4+1} 4(4+1) = -20$

First four terms are $2, -6, 12, -20$.

(iii) $a_n = 2n^2 - 6$

n^{th} term is given by $a_n = 2n^2 - 6$

$\therefore$ First term $a_1 = 2(1)^2 - 6 = -4$

Second term $a_2 = 2(2)^2 - 6 = 2$

Third term $a_3 = 2(3)^2 - 6 = 12$

Fourth term $a_4 = 2(4)^2 - 6 = 26$

$\therefore$ First four terms are $-4, 2, 12, 26$.

3. Find the n^{th} term of the following sequences.

(i) 2, 5, 10, 17,... (ii) $0, \frac{1}{2}, \frac{2}{3}, \ldots$

(iii) 3, 8, 13, 18,...

Sol :

(i) 2, 5, 10, 17, ...

$\Rightarrow 1^2 + 1, 2^2 + 1, 3^2 + 1, 4^2 + 1, \ldots..$

Here the every term is obtained by adding 1 to its square.

$\therefore$ The general term $a_n = n^2 + 1$

(ii) $0, \frac{1}{2}, \frac{2}{3}, \ldots$

We see that the numerators of n^{th} term is $n - 1$ and the denominators of n^{th} term is n.

$\therefore \; a_n = \frac{n-1}{n}$

(iii) 3, 8, 13, 18, ...

$a_1 = 3$; $a_2 = 8$; $a_3 = 13$; $a_4 = 18, \ldots$

$a_1 = 3$; $a_2 = 3 + 5$; $a_3 = 8 + 5$; $a_4 = 13 + 5, \ldots$

We see that n^{th} term is the sum of previous term and 5

$\therefore a_n = a_{n-1} + 5$

Another Method

$a_1 = 3$; $a_2 = 8$; $a_3 = 13$; $a_4 = 18, \ldots$

$a_1 = 5 - 2$; $a_2 = 2(5) - 2$; $a_3 = 3(5) - 2$; $a_4 = 4(5) - 2, \ldots$

$a_n = 5n - 2$

4. Find the indicated terms of the sequences whose n^{th} terms are given by

(i) $a_n = \frac{5n}{n+2}$; a_6 and a_{13}

(ii) $a_n = -(n^2 - 4)$; a_4 and a_{11}

Sol :

(i) $a_n = \frac{5n}{n+2}$; a_6 and a_{13}

Given $a_n = \frac{5n}{n+2}$

To find a_6, put $n = 6$

$\therefore a_6 = \frac{5 \times 6}{6+2} = \frac{30}{8} = \frac{15}{4}$

To find a_{13}, put $n = 13$

$a_{13} = \frac{5 \times 13}{13+2} = \frac{65}{15} = \frac{13}{3}$

$\therefore \; a_6 = \frac{15}{4}$ and $a_{13} = \frac{13}{3}$

(ii) $a_n = -(n^2 - 4)$; a_4 and a_{11}

Given $a_n = -(n^2 - 4)$

To find a_4 put $n = 4$

$a_4 = -(4^2 - 4) = -(16 - 4) = -12$

To find a_{11} put $n = 11$

$a_{11} = -(11^2 - 4) = -(121 - 4) = -117$

$\therefore a_4 = -12$; $a_{11} = -117$

5. Find a_8 and a_{15} whose n^{th} term is

$$a_n = \begin{cases} \dfrac{n^2-1}{n+3}; & n \text{ is even, } n \in N \\ \dfrac{n^2}{2n+1}; & n \text{ is odd, } n \in N \end{cases}$$

Sol : To find a_8, since $n = 8$ is even, put $n = 8$ in

$$a_n = \frac{n^2-1}{n+3}$$

$$a_8 = \frac{8^2-1}{8+3} = \frac{64-1}{11} = \frac{63}{11}$$

To find a_{15}, since n = 15 is odd

put n = 15 in $a_n = \frac{n^2}{2n+1}$

$$a_{15} = \frac{15^2}{2(15)+1} = \frac{225}{30+1} = \frac{225}{31}$$

$\therefore a_8 = \frac{63}{11}$ and $a_{15} = \frac{225}{31}$

6. If $a_1 = 1$, $a_2 = 1$ and $a_n = 2a_{n-1} + a_{n-2}$, $n \geq 3$, $n \in N$. Then find the first six terms of the sequence.

Sol : Given first two terms of the sequence.

$a_1 = 1$ and $a_2 = 1$

For $n \geq 3$ $a_n = 2a_{n-1} + a_{n-2}$

When n = 3 $a_3 = 2a_{3-1} + a_{3-2} = 2a_2 + a_1$

$a_3 = 2 + 1 = 3$

When n = 4, $a_4 = 2a_{4-1} + a_{4-2} = 2a_3 + a_2$

$a_4 = 2 \times 3 + 1 = 6 + 1 = 7$

When n = 5 $a_5 = 2a_{5-1} + a_{5-2} = 2a_4 + a_3$

$a_5 = 2 \times 7 + 3 = 14 + 3 = 17$

When n = 6 $a_6 = 2a_{6-1} + a_{6-2} = 2a_5 + a_4$

$a_6 = 2(17) + 7 = 34 + 7 = 41$

$\therefore$ First 6 terms of the sequence are 1, 1, 3, 7, 17, 41

ARITHMETIC PROGRESSION

Key Points

Definition:

Let a and d be real numbers. Then the numbers of the form a, a + d, a + 2d, a + 3d, a + 4d, ... is said to form Arithmetic progression denoted by A. P. The number 'a' is called the first term and d is called the common difference.

1. Arithmetic progression is a sequence whose successive terms differ by a constant number. Example: 2, 4, 6, 8 ...
2. The difference between any two consecutive terms of an A.P. is always constant. The constant value is called the **common difference.**
3. If there are finite number of terms in an A.P. then it is called Finite Arithmetic Progression.
4. If there are infinitely many terms in an A.P. then it is called Infinite Arithmetic Progression.

Terms and common difference of an A.P.

1. n^{th} term of an A.P. is denoted by t_n and $t_n = a + (n-1)d$.
2. To find the common difference we subtract first term from the second, second term from the third and so on.

   $\therefore$ Common difference $= t_2 - t_1 = t_3 - t_2 = ...$
3. The common difference 'd' of an A.P. can be positive, negative or zero.
4. An Arithmetic progression having a common difference of zero is called a **Constant Arithmetic Progression.** Example: –1, –1, –1,.....
5. If finite A.P. whose first term is a and last term l, then the number of terms in the A.P. is given by

$$l = a + (n-1)d$$

$$\Rightarrow n = \frac{l-a}{d} + 1$$

6. All the sequences are functions but all the functions are not sequences.
7. If every term is added or subtracted by a constant then the resulting sequence is also an A.P.
8. If every term is multiplied or divided by a non zero number then the resulting sequence is also an A.P

9. If the sum of three consecutive terms of an A.P. is given then they can be taken as a – d, a, a + d. Here common difference = d.
10. If the sum of four consecutive terms of an A.P. is given they can be taken as a – 3d, a – d, a + d and a + 3d. Here common difference = 2d
11. Three non zero numbers a, b, c are in A.P. if and only if 2b = a + c.

Formulae:

1. Sum up to n terms $S_n = \frac{n}{2}[2a + (n-1)d]$
2. If last term is given then $S_n = \frac{n}{2}(a + l)$
3. Sum of first n odd natural numbers = n^2
4. Sum of first n even natural numbers = n (n + 1)

Worked Examples

2.23 Check whether the following sequences are in A.P. or not?

(i) x + 2, 2x + 3, 3x + 4,...

(ii) 2, 4, 8, 16,...

(iii) $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}$,...

Sol :

To check that the given sequence is in A.P., it is enough to check if the differences between the consecutive terms are equal or not.

(i) $t_2 - t_1 = (2x + 3) - (x + 2) = x + 1$

$t_3 - t_2 = (3x + 4) - (2x + 3) = x + 1$

$t_2 - t_1 = t_3 - t_2$

Thus, the differences between consecutive terms are equal. Hence the sequence x + 2, 2x + 3, 3x + 4,... is in A.P.

(ii) $t_2 - t_1 = 4 - 2 = 2$

$t_3 - t_2 = 8 - 4 = 4$

$t_2 - t_1 \neq t_3 - t_2$

Thus, the differences between consecutive terms are not equal. Hence 2, 4, 8, 16,... are not in A.P.

(iii) $t_2 - t_1 = 5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$

$t_3 - t_2 = 7\sqrt{2} - 5\sqrt{2} = 2\sqrt{2}$

$t_4 - t_3 = 9\sqrt{2} - 7\sqrt{2} = 2\sqrt{2}$

Thus the difference between consecutive terms are equal. Hence the terms of the sequence $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2}$,... are in A.P.

2.24 Write an A.P. whose first term is 20 and common difference is 8.

Sol :

First term = a = 20; Common difference = d = 8

Arithmetic progression a, a + d, a + 2d, a + 3d,...

In this case, we get 20, 20 + 8, 20 + 2(8), 20 + 3 (8),...

So, the required A.P. is 20, 28, 36, 44,...

2.25 Find the 15^{th}, 24^{th} and n^{th} term (general term) of an A.P. given by 3, 15, 27, 39,...

Sol :

We have, first term = a = 3 and

common difference = d = 15 – 3 = 12.

We know that n^{th} term (general term) of an A.P. with first term a and common difference d is given by $t_n = a + (n - 1)d$

$t_{15} = a + (15 - 1)d = a + 14d = 3 + 14(12) = 171$

(Here a = 3 and d = 12)

$t_{24} = a + (24 - 1)d = a + 23d = 3 + 23(12) = 279$

The n^{th} (general term) term is given by

$t_n = a + (n - 1)d$

Thus, $t_n = 3 + (n - 1)12$

$t_n = 12n - 9$

2.26 Find the number of terms in the A.P. 3, 6, 9, 12,..., 111.

Sol : First term a = 3; Common difference d = 6 – 3 = 3; Last term l = 111.

We know that, $n = \left(\frac{l-a}{d}\right) + 1$

$n = \left(\frac{111-3}{3}\right) + 1 = 37$

Thus the A.P. contains 37 terms.

2.27 Determine the general term of an A.P. whose 7^{th} term is – 1 and 16^{th} term is 17.

Sol :

Let the A.P. be $t_1, t_2, t_3, t_4, \ldots t_n, \ldots$

It is given that $t_7 = -1$ and $t_{16} = 17$

$$a + (7-1)d = -1 \text{ and } a + (16-1)d = 17$$

$$a + 6d = -1 \quad \ldots(1)$$

$$a + 15d = 17 \quad \ldots(2)$$

Subtracting equation (1) from equation (2),

we get $9d = 18 \Rightarrow d = 2$

putting $d = 2$ in equation (1), we get $a + 12 = -1$

so, $a = -13$

Hence, General term $t_n = a + (n-1)d$

$= -13 + (n-1) \times 2 = 2n - 15$

2.28 If l^{th}, m^{th} and n^{th} terms of an A.P. are x, y, z respectively, then show that

(i) $x(m-n) + y(n-l) + z(l-m) = 0$

(ii) $(x-y)n + (y-z)l + (z-x)m = 0$

Sol :

(i) Let a be the first term and d be the common difference.

It is given that $t_l = x$, $t_m = y$, $t_n = z$

Using the general term formula

$$a + (l-1)d = x \quad \ldots(1)$$

$$a + (m-1)d = y \quad \ldots(2)$$

$$a + (n-1)d = z \quad \ldots(3)$$

We have , $x(m-n) + y(n-l) + z(l-m)$

$= a[(m-n) + (n-l) + (l-m)] + d[(m-n)(l-1) + (n-l)(m-1) + (l-m)(n-1)]$

$= a[0] + d[lm - ln - m + n + mn - lm - n + l + ln - mn - l + m]$

$= a(0) + d(0) = 0$

(ii) On subtracting equation (2) from equation (1), equation (3) from equation (2), and equation (1) from equation (3), we get

$$x - y = (l-m)d$$

$$y - z = (m-n)d$$

$$z - x = (n-l)d$$

$(x-y)n + (y-z)l + (z-x)m = [(l-m)n + (m-n)l + (n-l)m]d$

$= [ln - mn + lm - nl + nm - lm]d = 0$

2.29 In an A.P., sum of four consecutive terms is 28 and their sum of their squares is 276. Find the four numbers.

Sol : Let us take the four terms in the form of $(a-3d), (a-d), (a+d)$ and $(a+3d)$.

Since sum of the four terms is 28,

$$a - 3d + a - d + a + d + a + 3d = 28$$

$$4a = 28 \Rightarrow a = 7$$

Similarly, since sum of their squares is 276,

$$(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 276$$

$$a^2 - 6ad + 9d^2 + a^2 - 2ad + d^2 + a^2 + 2ad + d^2 + a^2 + 6ad + 9d^2 = 276$$

$$4a^2 + 20d^2 = 276 \Rightarrow 4(7)^2 + 20d^2 = 276.$$

$$d^2 = 4 \Rightarrow d = \pm 2$$

If $d = 2$ then the four numbers are

$7 - 3(2), 7 - 2, 7 + 2, 7 + 3(2)$

That is the four numbers are 1, 5, 9 and 13.

If $a = 7, d = -2$ then the four numbers are

13, 9, 5 and 1.

Therefore, the four consecutive terms of the A.P. are 1, 5, 9 and 13.

2.30 A mother divides ₹ 207 into three parts such that the amounts are in A.P. and gives it to her three children. The product of the two least amounts that the children had ₹ 4623. Find the amount received by each child.

Sol :

Let the amount received by the children be in the form of A.P. given by $a - d, a, a + d$.

Since, sum of the amount is ₹ 207, we have

$$(a-d) + a + (a+d) = 207$$

$$3a = 207 \Rightarrow a = 69$$

It is given that product of the two least amounts is 4623

$$(a-d)a = 4623$$

$$(69-d)69 = 4623$$

$$d = 2$$

Therefore, Amount given by the mother to her three children are ₹ (69 – 2), ₹ 69, ₹ (69 + 2).

That is, ₹ 67, ₹ 69 and ₹ 71.

Progress Check

1. The difference between any two consecutive terms of an A.P. is _____

Ans : always constant.

2. If a and d are the first term and common difference of an A.P. then the 8^{th} term is _____

Ans : $t_8 = a + 7d$

3. If t_n is the n^{th} term of an A.P. then $t_{2n} - t_n$ is ___

Ans : nd

4. The common difference of a constant A.P. is ___

Ans : Zero.

5. If a and l are first and last terms of an A.P. then the number of terms is _______.

Ans : $n = \frac{l-a}{d} + 1$

6. If every term of an A.P. is multiplied by 3, then the common difference of the new A.P. is _____

Ans : 3 × old common difference.

7. Three numbers a, b and c will be in A.P. if and only if _____

Ans : 2b = a + c

Thinking Corner

1. If t_n is the n^{th} term of an A.P. then the value of $t_{n+1} - t_{n-1}$ is _____

Ans : $t_{n+1} - t_{n-1} = a + ((n+1) - 1)d - [a + ((n-1) - 1)d]$

$= [a + d(n+1) - d] - [a + d(n-1) - d]$

$= [a + nd + d - d] - [a + nd - d - d]$

$= a + nd - a - nd + 2d$

$= 2d.$

Exercise 2.5

1. Check whether the following sequences are in A.P.

(i) $a-3, a-5, a-7, \ldots$ **(ii)** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

(iii) 9, 13, 17, 21, 25,...

(iv) $\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots$

(v) 1, –1, 1, –1, 1, –1,....

Sol :

(i) $a-3, a-5, a-7, \ldots$

$$\begin{aligned} t_2 - t_1 &= a - 5 - [a-3] \\ &= a - 5 - a + 3 \\ &= -2 \\ t_3 - t_2 &= a - 7 - (a-5) \\ &= a - 7 - a + 5 \\ &= -2 \\ t_2 - t_1 &= t_3 - t_2 \end{aligned}$$

∴ The difference between consecutive terms are equal.

∴ $a-3, a-5, a-7, \ldots$ are in A.P.

(ii) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

$$t_2 - t_1 = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = \frac{-1}{6}$$

$$t_3 - t_2 = \frac{1}{4} - \frac{1}{3} = \frac{3-4}{12} = \frac{-1}{12}$$

$$t_4 - t_3 = \frac{1}{5} - \frac{1}{4} = \frac{4-5}{20} = \frac{-1}{20}$$

$\therefore t_2 - t_1 \neq t_3 - t_2 \neq t_4 - t_3$

∴ The difference between consecutive terms are not equal.

∴ $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ are not in A.P.

(iii) 9, 13, 17, 21, 25,...

$$\begin{aligned} t_2 - t_1 &= 13 - 9 = 4 \\ t_3 - t_2 &= 17 - 13 = 4 \\ t_4 - t_3 &= 21 - 17 = 4 \\ t_5 - t_4 &= 25 - 21 = 4 \end{aligned}$$

$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = t_5 - t_4 = 4.$

The difference between consecutive terms are equal.

∴ 9, 13, 17, 21, 25, ... are in A.P.

(iv) $\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots$

$$t_2 - t_1 = 0 - \left(\frac{-1}{3}\right) = \frac{1}{3}$$

$$t_3 - t_2 = \frac{1}{3} - 0 = \frac{1}{3}$$

$$t_4 - t_3 = \frac{2}{3} - \frac{1}{3} = \frac{2-1}{3} = \frac{1}{3}$$

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \frac{1}{3}$$

∴ The differences between consecutive terms are equal.

∴ $\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots$ are in A.P.

(v) 1, –1, 1, –1, 1, –1,...

$$\begin{aligned} t_2 - t_1 &= (-1) - 1 = -2 \\ t_3 - t_2 &= 1 - (-1) = 1 + 1 = 2 \\ t_4 - t_3 &= -1 - 1 = -2 \\ t_5 - t_4 &= 1 - (-1) = 1 + 1 = 2 \end{aligned}$$

Here $t_2 - t_1 \neq t_3 - t_2$

∴ The difference between consecutive terms are not equal.

∴ 1, –1, 1, –1, 1, –1,... are not in A.P.

2. First term a and common difference d are given below. Find the corresponding A.P.

(i) a = 5, d = 6, **(ii) a = 7, d = – 5**

(iii) $a = \frac{3}{4}, d = \frac{1}{2}$

Sol :

(i) a = 5, d = 6

First term a = 5; common difference d = 6.

A.P.is given by a, a + d, a + 2d, a+ 3d,...

In this case 5, 5 + 6, 5 + 2(6), 5 + 3 (6),...

$\Rightarrow$ 5, 11, 17, 23,...

$\therefore$ The required A.P. is 5, 11, 17, 23,....

(ii) a = 7; d = – 5

A.P is given by a, a + d, a + 2d, a + 3d,...

In this case 7, 7 + (– 5), 7 + 2(– 5), 7 + 3 (– 5),...

$\Rightarrow$ 7, 2, – 3, – 8,...

$\therefore$ The required A.P. is 7, 2, –3, – 8,...

(iii) $a = \frac{3}{4}, d = \frac{1}{2}$

A.P. is given by a, a + d, a + 2d, a + 3d,...

In this case $\frac{3}{4}, \frac{3}{4}+\frac{1}{2}, \frac{3}{4}+2\left(\frac{1}{2}\right), \frac{3}{4}+3\left(\frac{1}{2}\right), \ldots$

$\Rightarrow \frac{3}{4}, \frac{3+2}{4}, \frac{3+4}{4}, \frac{3+6}{4}, \ldots$

$\Rightarrow \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \ldots.$

$\therefore$ The required A.P. is $\frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \ldots$

3. Find the first term and common difference of the Arithmetic progressions whose n^{th} terms are given below.

(i) $t_n = -3 + 2n$ (ii) $t_n = 4 - 7n$

Sol :

(i) $t_n = -3 + 2n$

Given the n^{th} term of the A.P. is $t_n = -3 + 2n$

Put n = 1 $\Rightarrow$ $t_1 = -3 + 2(1) = -3 + 2$

$a = t_1 = -1$

Put n = 2 $\Rightarrow$ $t_2 = -3 + 2(2) = -3 + 4$

$t_2 = 1$

Common difference $d = t_2 - t_1 = 1 - (-1)$

$= 1 + 1 = 2$

$\therefore$ First term a = – 1; Common difference d = 2

(ii) $t_n = 4 - 7n$

Given the n^{th} term of the A.P. is $t_n = 4 - 7n$

Put n = 1 $\Rightarrow$ $t_1 = 4 - 7(1) = 4 - 7 = -3$

$a = t_1 = -3$

$t_2 = 4 - 7(2) = 4 - 14 = -10$

Common difference $d = t_2 - t_1 = -10 - (-3)$

$= -10 + 3 = -7$

First term a = – 3; Common difference d = – 7

4. Find the 19^{th} term of an A.P. – 11, –15, – 19, ...

Sol : Given the A.P. – 11, – 15, – 19,....

Here First term a = – 11

Common difference $d = t_2 - t_1 = -15 - (-11)$

$= -15 + 11$

$d = -4$

$\therefore$ n^{th} term of an A.P. is $t_n = a + (n-1)d$

19^{th} term $(t_{19}) = -11 + (19-1)(-4)$

$= -11 + 18(-4)$

$= -11 + (-72) = -83$

$\therefore$ 19^{th} term of –11, – 15, – 19,...-is – 83.

5. Which term of an A.P. 16, 11, 6, 1,... is – 54?

Sol : Given the A.P. 16, 11, 6, 1, ...

Here a = 16, d = 11 – 16 = – 5

We have the n^{th} term of an A.P. $t_n = a + (n-1)d$

Put $t_n = -54$

$-54 = 16 + (n-1)(-5)$

$-54 - 16 = (n-1)(-5)$

$\frac{-70}{-5} = n - 1$

$14 = n - 1$

$14 + 1 = n$

$n = 15$

$\therefore$ – 54 is the 15^{th} term of 16, 11, 6, 1,...

6. Find the middle term (s) of an A.P. 9, 15, 21, 27,..., 183.

Sol : Given A.P. is 9, 15, 21, 27,... 183

a = 9, d = 15 – 9 = 6

Let last term $l = 183$

We have $l = a + (n-1)d$

$183 = 9 + (n-1)(6)$

$183 - 9 = (n-1)6$

$174 = (n-1)6$

$\frac{174}{6} = n - 1$

$29 = n - 1$

$n = 29 + 1 = 30$

$\therefore$ Number of terms in the A.P. are 30.

30 is an even number.

$\therefore \frac{30^{th}}{2}, \left(\frac{30}{2}+1\right)^{th}$ terms are middle term.

15^{th}, 16^{th} terms are middle terms

$t_{15} = 9 + (15-1)(6)$

$= 9 + 14(6) = 9 + 84$

$t_{15} = 93$

$t_{16} = 9 + (16-1)(6)$

$= 9 + 15(6) = 9 + 90$

$t_{16} = 99$

Middle terms are $t_{15} = 93$ and $t_{16} = 99$

7. If nine times ninth term is equal to the fifteen times fifteenth term, show that six times twenty fourth term is zero.

Sol : We know that n^{th} term of an A.P. is

$t_n = a + (n - 1)d$

Given 9 times 9^{th} term = 15 times 15^{th} term

$$9 \times t_9 = 15 \times t_{15}$$
$$9[a + (9 - 1)d] = 15[a + (15 - 1)d]$$
$$9(a + 8d) = 15(a + 14d)$$
$$9a + 72d = 15a + 210d$$
$$15a + 210d - 9a - 72d = 0$$
$$6a + 138d = 0$$
$$6(a + 23d) = 0$$
$$6[a + (24 - 1)d] = 0$$
$$6 \times t_{24} = 0$$

$\therefore$ 6 times 24^{th} term = 0

8. If 3 + k, 18 – k, 5k + 1 are in A.P. then find k

Sol :

Given 3 + k, 18 – k, 5k + 1 are in A.P.

$t_1 = 3 + k, t_2 = 18 - k, t_3 = 5k + 1$

$\therefore$ Difference between the consecutive terms must be equal.

i.e., $t_2 - t_1 = t_3 - t_2$

$$(18 - k) - (3 + k) = (5k + 1) - (18 - k)$$
$$18 - k - 3 - k = 5k + 1 - 18 + k$$
$$15 - 2k = 6k - 17$$
$$15 + 17 = 6k + 2k$$
$$32 = 8k$$
$$k = \frac{32}{8} = 4$$

9. Find x, y and z given that the numbers x, 10, y, 24, z are in A.P.

Sol :

Let $t_1 = x,\ t_2 = 10,\ t_3 = y, t_4 = 24, t_5 = z$.

Given that x, 10, y, 24, z are in A.P.

Then difference between consecutive terms are equal.

i.e., $t_2 - t_1 = t_3 - t_2 = t_4 - t_3$

$10 - x = 24 - y$ = common difference 'd'

Also $t_4 - t_2 = 2d$

$$24 - 10 = 2d$$
$$14 = 2d$$
$$d = \frac{14}{2} = 7$$

$$10 - x = 24 - y = 7$$

If $10 - x = 7;\quad x = 10 - 7 = 3$

If $24 - y = 7;\quad y = 24 - 7 = 17$

Also $z - 24 = 7;\quad z = 24 + 7 = 31$

$\therefore x = 3; y = 17; z = 31.$

10. In a theatre, there are 20 seats in the front row and 30 rows were allotted. Each successive row contains two additional seats than its front row. How many seats are there in the last row?

Sol :

Let the number of seats in the front row

$t_1 = 20$

Number of seats in the second row

$t_2 = 20 + 2 = 22$

$\therefore$ Let the numbers of seats in 30^{th} row be t_{30}.

Since the additional seats in each row is 2.

$t_1, t_2, ...t_{30}$ form an A.P.

i.e., 20, 22, 24,... upto 30 terms

Here $a = 20; d = 2$

$t_n = a + (n - 1)d$ form an A.P.

$$t_{30} = 20 + (30 - 1)(2) = 20 + 29 \times 2$$
$$= 20 + 58 = 78$$

$\therefore$ In the last row there are 78 seats.

11. Sum of three consecutive terms in an A.P. is 27 and their product is 288. Find the three terms.

Sol :

Let the three consecutive terms be a – d, a, a +d.

Given their sum is 27

$$(a - d) + a + (a + d) = 27$$
$$a - d + a + a + d = 27$$
$$3a = 27$$
$$a = \frac{27}{3} = 9$$

$$\text{Product} = 288$$
$$(a - d)\,a\,(a + d) = 288$$
$$a(a^2 - d^2) = 288$$
$$9(9^2 - d^2) = 288$$
$$81 - d^2 = \frac{288}{9}$$
$$81 - d^2 = 32$$
$$d^2 = 81 - 32 = 49$$
$$d \times d = 7 \times 7$$
$$d = 7$$

a = 9 and d = 7

$\therefore$ The terms are $a - d, a, a + d$

$\Rightarrow 9 - 7, 9, 9 + 7 \Rightarrow 2, 9, 16$

$\therefore$ The required terms are 2, 9, 16

12. The ratio of 6th and 8th term of an A.P. is 7 : 9. Find the ratio of 9th to 13th term.

Sol :

Given in an A.P. 6th term : 8th term = 7 : 9

$a + (6 - 1)d : a + (8 - 1)d = 7 : 9$

$a + 5d : a + 7d = 7 : 9$

Product of the extremes = product of the means.

$$9(a + 5d) = 7(a + 7d)$$
$$9a + 45d = 7a + 49d$$
$$9a - 7a = 49d - 45d$$
$$2a = 4d$$
$$a = 2d$$

To find the ratio of 9th term : 13th term

$$a + (9 - 1)d : a + (13 - 1)d = a + 8d : a + 12d$$
$$= 2d + 8d : 2d + 12d$$
$$= 10d : 14d$$
$$= 5 : 7.$$

The ratio of 9th term to 13th term is 5 : 7.

13. In a winter season let us take the temperature of Ooty from Monday to Friday to be in A.P. The sum of temperatures from Monday to Wednesday is 0°C and the sum of the temperatures from Wednesday to Friday is 18°C. Find the temperature of each of the five days.

Sol :

Given the temperatures are in A.P.

Let the temperatures of Monday, Tuesday, Wednesday, Thursday and Friday be t_1, t_2, t_3, t_4 and t_5 respectively.

Given $t_1 + t_2 + t_3 = 0\,°C$

$a + a + d + a + 2d = 0\,°C$

$3a + 3d = 0$ ...(1)

And $t_3 + t_4 + t_5 = 18\,°C$

$a + 2d + a + 3d + a + 4d = 18\,°C$

$3a + 9d = 18\,°C$ ...(2)

Subtracting (1) from (2)

$$6d = 18\,°C$$
$$d = \frac{18}{6}$$
$$d = 3\,°C$$

Put $d = 3\,°C$ in (1)

$$3a + 3 \times 3 = 0$$
$$3a = -9$$
$$a = \frac{-9}{3} = -3\,°C$$

$\therefore$ $t_1 = -3\,°C$, $t_2 = (-3) + 3 = 0\,°C$,

$t_3 = -3 + 2(3) = 3\,°C$,

$t_4 = -3 + 3(3) = 6\,°C$, $t_5 = -3 + 4(3) = 9\,°C$.

$\therefore$ Five days temperatures are $-3\,°C$, $0\,°C$, $3\,°C$, $6\,°C$, $9\,°C$

14. Priya earned ₹ 15,000 in the first month. Thereafter her salary increases by ₹ 1500 per year. Her expenses are ₹ 13,000 during the first year and the expenses increases by ₹ 900 per year. How long will it take her to save ₹ 20,000 per month?

Sol :

Let the first month earning be

t_1 = ₹ 15,000

Increase per year d = ₹ 1,500

Expenses for first year = ₹ 13,000

Expenses increase per year = ₹ 900

$\therefore$ Saving for every year will be

(15000 – 13000), (16500 – 13900), (18000 – 14800),...

ie. 2000, 2600, 3200, ...

Since $t_2 - t_1 = t_3 - t_2$, this sequence form an A.P.

$a = 2000$, $d = 2600 - 2000 = 600$

Let the nth years saving be ₹ 20,000 then

$$a + (n - 1)d = t_n$$
$$2000 + (n - 1)(600) = 20{,}000$$
$$(n - 1)\,600 = 20{,}000 - 2000$$
$$(n - 1)\,600 = 18000$$
$$n - 1 = \frac{18000}{600} = 30$$
$$n = 30 + 1 = 31$$

$\therefore$ To save ₹ 20000, it takes 31 years.

SERIES

Key Points

1. The sum of the terms of a sequence is called series.
2. If $a_1, a_2, a_3, \ldots, a_n$ be the sequence then the series of real numbers is defined as $a_1 + a_2 + a_3 + \ldots$
3. If a series have finite number of terms then it is called a finite series.
4. If a series have infinite number of terms then it is called an infinite series.

Sum to 'n' terms of an A.P.

1. Sum upto n terms of the series a + (a + d) + (a + 2d) + ... is $S_n = \frac{n}{2}[2a + (n-1)d]$
2. If l is the last term then $S_n = \frac{n}{2}(a + l)$

A series whose terms are in Arithmetic progression is called Arithmetic series.

Let a, a + d, a + 2d, a + 3d,.... be the Arithmetic Progression.

The sum of first n terms of a Arithmetic Progression denoted by S_n is given by

$$S_n = a + (a + d) + (a + 2d) + \ldots + [a + (n-1)d] \quad \ldots(1)$$

Rewriting the above in reverse order.

$$S_n = [a + (n-1)d] + [a + (n-2)d] + \ldots + (a + d) + a \quad \ldots(2)$$

Adding (1) and (2), We get

$$2S_n = [a + a + (n-1)d] + [a + d + a + (n-2)d] + \ldots + [a + (n-2)d + (a + d)] + [a + (n-1)d + a]$$

$$= [2a + (n-1)d] + [2a + (n-1)d + \ldots + [2a + (n-1)d] \text{ (n terms)}$$

$$2S_n = n \times [2a + (n-1)d] \Rightarrow S_n = \frac{n}{2}[2a + (n-1)d]$$

Note

If the first term a, and the last term l (n^{th} term) are given then

$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + a + (n-1)d]$ Since, $l = a + (n-1)d$ We have

$$S_n = \frac{n}{2}[a + l]$$

Worked Examples

2.31 **Find the sum of first 15 terms of the A.P. $8, 7\frac{1}{4}, 6\frac{1}{2}, 5\frac{3}{4}, \ldots$**

Sol : Here the first term a = 8,

common difference $d = 7\frac{1}{4} - 8 = -\frac{3}{4}$,

Sum of first n terms of an A.P.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_{15} = \frac{15}{2}\left[2 \times 8 + (15-1)\left(-\frac{3}{4}\right)\right]$$

$$S_{15} = \frac{15}{2}\left[16 - \frac{21}{2}\right] = \frac{165}{4}$$

2.32 Find the sum of 0.40 + 0.43 + 0.46 + .. + 1.

Sol : Here the value of n is not given. But the last term is given. From this, we can find the value of n.

Given $a = 0.40,\ l = 1$.

We find $d = 0.43 - 0.40 = 0.03$

Therefore, $n = \left(\frac{l-a}{d}\right)+1$

$$= \left(\frac{1-0.40}{0.03}\right)+1 = 21$$

Sum of first n terms of an A.P.

$$S_n = \frac{n}{2}[a+l]$$

Here, n = 21, Therefore,

$$S_{21} = \frac{21}{2}[0.40+1] = 14.7$$

So, the sum of 21 terms of the given series is 14.7.

2.33 How many terms of the series 1 + 5 + 9 + ... must be taken so that their sum is 190?

Sol : Here we have to find the value of n, such that $S_n = 190$.

First term a = 1, common difference d = 5 – 1 = 4.

Sum of first n terms of an A.P

$$S_n = \frac{n}{2}[2a+(n-1)d] = 190$$

$$\frac{n}{2}[2\times1+(n-1)\times 4] = 190$$

$$n[4n-2] = 380$$

$$2n^2 - n - 190 = 0$$

$$(n-10)(2n+19) = 0$$

But n = 10 as $n = -\frac{19}{2}$ is impossible.

Therefore, n = 10.

2.34 The 13th term of an A.P. is 3 and the sum of first 13 terms is 234. Find the common difference and the sum of first 21 terms.

Sol :

Given the 13th term = 3, so $t_{13} = a + 12d = 3$ ...(1)

Sum of first 13 terms = 234

$$\Rightarrow S_{13} = \frac{13}{2}[2a+12d] = 234$$

$$2a + 12d = 36 \quad ...(2)$$

Solving (1) and (2) we get,

$$a = 33,\ d = \frac{-5}{2}$$

Therefore, common difference is $\frac{-5}{2}$

Sum of first 21 terms

$$S_{21} = \frac{21}{2}\left[2\times33+(21-1)\times\left(-\frac{5}{2}\right)\right]$$

$$= \frac{21}{2}[66-50) = 168.$$

2.35 In an A.P. the sum of first n terms is $\frac{5n^2}{2}+\frac{3n}{2}$. Find the 17th term.

Sol :

The 17th term can be obtained by subtracting the sum of first 16 terms from the sum of first 17 terms.

$$S_{17} = \frac{5\times(17)^2}{2}+\frac{3\times17}{2} = \frac{1445}{2}+\frac{51}{2} = 748$$

$$S_{16} = \frac{5\times(16)^2}{2}+\frac{3\times16}{2} = \frac{1280}{2}+\frac{48}{2} = 664$$

Now, $t_{17} = S_{17} - S_{16} = 748 - 664 = 84$

2.36 Find the sum of all natural numbers between 300 and 600 which are divisible by 7.

Sol : The natural numbers between 300 and 600 which are divisible by 7 are 301, 308, 315, ... 595.

The sum of all natural numbers between 300 and 600 is 301 + 308 + 315 + ...+ 595.

The terms of the above series are in A.P.

First term a = 301; common difference d = 7;

Last term $l = 595$.

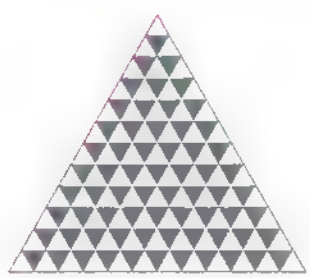
$$n = \left(\frac{l-a}{d}\right)+1 = \left(\frac{595-301}{7}\right)+1 = 43$$

Since, $S_n = \frac{n}{2}[a+l]$, we have

$$S_{43} = \frac{43}{2}[301+595] = 19264.$$

2.37 A mosaic is designed in the shape of an equilateral triangle, 12 ft on each side. Each tile in the mosaic is in the shape of an equilateral triangle of 12 inch side. The tiles are alternate in color as shown in the figure. Find the number of tiles of each colour and total number of tiles in the mosaic.

Sol :



Since the mosaic is in the shape of an equilateral triangle of 12 ft and the tile is in the shape of an equilateral triangle of 12 inch (1 ft), there will be 12 rows in the mosaic.

From the figure, it is clear that number of white tile in each row are 1, 2, 3, 4,...,12 which clearly forms an Arithmetic Progression.

Similarly the number of blue tiles in each row are 0, 1, 2, 3, ..., 11 which is also an Arithmetic Progression.

Number of white tiles $= 1+2+3+\ldots+12$

$$= \frac{12}{2}[1+12] = 78$$

Number of blue tiles $= 0+1+2+3+\ldots+11$

$$= \frac{12}{2}[0+11] = 66$$

The total number of tiles in the mosaic

$$= 78+66 = 144$$

2.38 **The houses of a street are numbered from 1 to 49. Senthil's house is numbered such that the sum of numbers of the houses prior to Senthil's house is equal to the sum of numbers of the houses following Senthil's house. Find Senthil's house number.**

Sol :

Let Senthil's house number be x.

It is given that $1+2+3+\ldots+(x-1)$

$$= (x+1)+(x+2)+\ldots+49$$

$$1+2+3\ldots+(x-1) = [1+2+3+\ldots+49] - [1+2+3+\ldots+x]$$

$$\frac{x-1}{2}[1+(x-1)] = \frac{49}{2}[1+49] - \frac{x}{2}[1+x]$$

$$\frac{x(x-1)}{2} = \frac{49\times 50}{2} - \frac{x(x+1)}{2}$$

$$x^2 - x = 2450 - x^2 - x \Rightarrow 2x^2 = 2450$$

$$x^2 = 1225 \Rightarrow x = 35$$

Therefore Senthil's house number is 35.

2.39 **The sum of first n, 2n and 3n terms of an A.P. are S_1, S_2, S_3 respectively. Prove that $S_3 = 3(S_2 - S_1)$.**

Sol :

If S_1, S_2, S_3 are sum of first n, 2n, 3n terms of an A.P. respectively then

$$S_1 = \frac{n}{2}[2a+(n-1)d],$$

$$S_2 = \frac{2n}{2}[2a+(2n-1)d], \quad S_3 = \frac{3n}{2}[2a+(3n-1)d]$$

Consider,

$$S_2 - S_1 = \frac{2n}{2}[2a+(2n-1)d] - \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}\big[[4a+2(2n-1)d] - [2a+(n-1)d]\big]$$

$$S_2 - S_1 = \frac{n}{2}\times[2a+(3n-1)d]$$

$$3(S_2 - S_1) = \frac{3n}{2}[2a+(3n-1)d]$$

$$3(S_2 - S_1) = S_3$$

Progress Check

1. **The sum of terms of a sequence is called _____.**

   Ans : series

2. **If a series have finite number of terms then it is called _____.**

   Ans : finite series.

3. **A series whose terms are in _____ is called Arithmetic series.**

   Ans : Arithmetic progression.

4. **If the first and last terms of an A.P. are given then the formula to find the sum is**

   Ans : $S_n = \frac{n}{2}(a+l)$

5. **State True or false.**

   (i) The n^{th} term of any A.P. is of the form pn + q where p and q are some constants.

   Ans : True

   (ii) The sum upto n^{th} term of any A.P. is of the form $pn^2 + qn + r$ where p, q, r are some constants.

   Ans : False

Thinking Corner

1. **The value of n must be positive. Why?**

   Ans : n denotes the number of terms of a series.

   ∴ n must be positive. If it is negative means less.

2. **What is the sum of first *n* odd natural numbers?**

   Ans : First n odd natural numbers are

   1, 3, 5, 7,... upto n terms.

   a = 1, d = 3 − 1 = 2;

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Now $1+3+5+7+... = \frac{n}{2}[2(1)+(n-1)(2)]$

$$= \frac{n}{2}[2+2n-2] = \frac{n}{2}[2n] = n^2$$

$\therefore$ Sum of first n odd natural numbers $= n^2$

3. What is the sum of first n even natural numbers?

Ans : Sum of n numbers $S_n = \frac{n}{2}[2a+(n-1)d]$

Let the first n even natural numbers be 2, 4, 6, 8, ... upto n terms.

$a = 2;\ d = 4-2 = 2$

$$S_n = \frac{n}{2}[2(2)+(n-1)(2)]$$

$$= \frac{n}{2}[4+2n-2] = \frac{n}{2}[2n+2]$$

$$= \frac{n}{2}\times 2(n+1) = n(n+1)$$

$\therefore$ Sum of first n even natural numbers $= n(n+1)$

Exercise 2.6

1. Find the sum of the following

(i) 3, 7, 11,... upto 40 terms

(ii) 102, 97, 92, ... upto 27 terms.

(iii) 6 + 13 + 20 + ...+ 97

Sol :

(i) 3, 7, 11, ... upto 40 terms.

We have sum of n terms of an A.P.

$$S_n = \frac{n}{2}\times[2a+(n-1)d]$$

Here $d = 7-3 = 11-7 = 4$

$\therefore t_2 - t_1 = t_3 - t_2$ and it forms an A.P.

where $a = 3$, $d = 4$.

$$\therefore S_{40} = \frac{40}{2}[2(3)+(40-1)(4)]$$

$$= 20[6+(39\times 4)] = 20[6+156]$$

$$= 20[162] = 3240$$

Sum upto 40 terms $= 3240$

(ii) 102, 97, 92, ...upto 27 terms

Here $t_2 - t_1 = 97-102 = -5$

$t_3 - t_2 = 92-97 = -5$

$t_2 - t_1 = t_3 - t_2$

$\therefore$ It is an A.P. with $a = 102$, $d = -5$

Sum upto n terms $S_n = \frac{n}{2}[2a+(n-1)d]$

$$S_{27} = \frac{27}{2}[2(102)+(27-1)(-5)]$$

$$= \frac{27}{2}[204+26(-5)]$$

$$= \frac{27}{2}[204-130]$$

$$= \frac{27}{2}\times 74$$

$$= 27\times 37$$

Sum upto 27 terms is $= 999$

(iii) $6+13+20+...+97$

Here $t_2 - t_1 = t_3 - t_2$

$\Rightarrow$ $13-6 = 20-13 = 7$

$\therefore$ It is an Arithmetic series

$\therefore$ Sum $S_n = \frac{n}{2}(a+l)$

$a = 6;\ l = 97$

$$a+(n-1)d = 97$$
$$6+(n-1)(7) = 97$$
$$(n-1)7 = 97-6$$
$$(n-1)7 = 91$$
$$n-1 = \frac{91}{7} = 13$$
$$n = 13+1 = 14$$

Now $S_n = \frac{14}{2}(6+97) = 7\times 103$

$6+13+20+...+97 = 721$

2. How many consecutive odd integers beginning with 5 will sum to 480?

Sol :

Let the consecutive odd integers beginning with 5 be, 5, 7, 9, 11,....

Here $a = 5$, $d = 2$

We know that sum of n consecutive integers

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$480 = \frac{n}{2}[2(5)+(n-1)(2)]$$

$$480 = \frac{2n}{2}[5+n-1]$$

$$480 = n(n+4)$$

$$n^2+4n-480 = 0$$

$$(n-20)(n+24) = 0$$

$$n = 20 \text{ (or) } -24$$

Number of terms cannot be negative.

$\therefore \quad n = 20$

20 consecutive odd integers beginning with 5 will sum to 480.

3. Find the sum of first 28 terms of an A.P. whose n^{th} term is $4n - 3$.

Sol : Given $n = 28$

$t_n = 4n - 3$

$t_1 = 4(1) - 3 = 4 - 3 = 1 \quad = a$

$t_2 = 4(2) - 3 = 8 - 3 = 5$

$d = t_2 - t_1 = 5 - 1 = 4$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$= \frac{28}{2}[2(1) + (28-1)(4)]$$

$$= 14[2 + 27(4)] = 14[2 + 108]$$

$$= 14 \times 110 = 1540$$

Sum of first 28 terms of the given A.P. = 1540

4. The sum of first n terms of a certain series is given as $2n^2 - 3n$. Show that the series is an A.P.

Sol :

Given sum of first n terms $S_n = 2n^2 - 3n$

$\therefore$ sum of first term $S_1 = 2(1)^2 - 3(1) = 2 - 3$

$= -1 = t_1$

Sum of first two terms $S_2 = 2(2)^2 - 3(2)$

$= 8 - 6 = 2$

$\therefore t_2 = S_2 - S_1$

$= 2 - (-1) = 2 + 1$

$t_2 = 3$

Sum of first three terms $S_3 = 2(3)^2 - 3(3)$

$= 18 - 9 = 9$

$\therefore t_3 = S_3 - S_2 = 9 - 2$

$t_3 = 7$

Here $t_2 - t_1 = 3 - (-1) = 3 + 1 = 4$

$t_3 - t_2 = 7 - 3 = 4$

$\therefore t_2 - t_1 = t_3 - t_2$

$\therefore$ The series is in A.P.

5. The 104^{th} term and 4^{th} term of an A.P. are 125 and 0. Find the sum of first 35 terms.

Given 104^{th} terms = 125

$t_{104} = 125$

$a + (104 - 1)d = 125$

$a + 103d = 125 \quad \ldots(1)$

4^{th} term is 0

i,e., $a + (4 - 1)d = 0$

$a + 3d = 0 \quad \ldots(2)$

$(1) - (2) \Rightarrow 100d = 125$

$$d = \frac{125}{100} = \frac{5}{4}$$

Put $d = \frac{5}{4}$ in (2)

$a + 3d = 0$

$$a + 3\left(\frac{5}{4}\right) = 0$$

$$a + \frac{15}{4} = 0$$

$$a = \frac{-15}{4}$$

$\therefore$ Sum of first n terms

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Sum of first 35 terms

$$S_{35} = \frac{35}{2}\left[2\left(\frac{-15}{4}\right) + (35-1)\left(\frac{5}{4}\right)\right]$$

$$= \frac{35}{2}\left[\frac{-15}{2} + 34\left(\frac{5}{4}\right)\right]$$

$$= \frac{35}{2}\left[\frac{-15}{2} + 17\frac{(5)}{2}\right]$$

$$= \frac{35}{2}\left[\frac{-15}{2} + \frac{85}{2}\right]$$

$$= \frac{35}{2}\left[\frac{70}{2}\right] = \frac{2450}{4} = \frac{1225}{2}$$

Sum of first 35 terms of the given series

$= 612.5$

6. Find the sum of all odd positive integers less than 450.

Sol :

Series of odd positive integers less then 450 is

$1 + 3 + 5 + 7 + \ldots\ldots + 449$

Here $a = 1$; $d = 2$; $l = 449$.

$$\text{Number of terms } n = \left(\frac{l-a}{d}\right) + 1$$

$$n = \frac{449-1}{2} + 1 = \frac{448}{2} + 1 = 224 + 1 = 225$$

Sum of first n odd integers = n^2

$\therefore$ Sum of first 225 odd integers = $225^2 = 225 \times 225$

$1 + 3 + 5 + \ldots.. + 449 = 50625$

7. Find the sum of all natural numbers between 602 and 902 which are not divisible by 4.

Sol :

$$\left\{\begin{array}{l}\textit{Sum of natural numbers between}\\ \textit{602 and 902 not divisible by 4}\end{array}\right\} =$$

$$\left\{\begin{array}{l}\textit{Sum of all natural}\\ \textit{numbers between}\\ \textit{602 and 902}\end{array}\right\} - \left\{\begin{array}{l}\textit{Sum of all natural}\\ \textit{numbers between}\\ \textit{602 and 902 divisible}\\ \textit{by 4}\end{array}\right\}$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$\therefore$ Sum of Natural numbers upto n $= \dfrac{n(n+1)}{2}$

$603 + 604 + ... + 901 = (1 + 2 + ... + 901) - (1 + 2 + 3 + ... + 602)$

$$= \frac{901 \times (901+1)}{2} - \frac{602 \times (602+1)}{2}$$

$$= \frac{901 \times 902}{2} - \frac{602 \times 603}{2}$$

$= 4,06,351 - 1,81,503$

$= 2,24,848$

Again numbers divisible by 4 between 602 and 902 are $604 + 608 + ... + 900$

$$n = \left(\frac{l-a}{d}\right) + 1 = \left(\frac{900 - 604}{4}\right) + 1$$

$$= \left(\frac{296}{4}\right) + 1 = 74 + 1 = 75$$

$$S_n = \frac{75}{2}[2(604) + (75-1)(4)]$$

$$= \frac{75}{2} \times 2[604 + (74 \times 2)]$$

$= 75 \times 752 = 56400$

Required sum $= 224848 - 56400$

$= 1,68,448$

8. Raghu wish to buy a Laptop. He can buy it by paying ₹ 40,000 cash or by giving it in 10 installments as ₹ 4800 in the first month, ₹ 4750 in the second month, ₹ 4700 in the third month and so on. If he pays the money in this fashion, Find

(i) Toal amount paid in 10 installments

(ii) How much extra amount that he has to pay than the cost?

Sol :

(i) If paid in cash, cost of laptop = ₹ 40000

The amount he pays in installments are

₹ 4800 + ₹ 4750 + ₹ 4700 + ...10 months.

$\therefore$ This form an Arithmetic series with

a = ₹ 4800

$d = 4750 - 4800 = -50$

$n = 10$

$$S_n = \frac{10}{2}[2(4800) + (10-1)(-50)]$$

$$= \frac{10 \times 2}{2}[4800 + 9 \times (-25)]$$

$= 10\,[4800 - 225]$

$= 10 \times 4575$

= ₹ 45750

$\therefore$ Total amount paid in 10 installments

= ₹ 45750

(ii) Extra amount he pays in installments

= ₹ 45750 − ₹ 40000

= 5750

He pays ₹ 5750 extra by installments.

9. A man repays a loan of ₹ 65000 by paying ₹ 400 in the first month and then increasing the payment by ₹ 300 every month. How long will it take him to clear the loan?

Sol :

Total amount to repay = ₹ 65000

He pays ₹ 400 in the first installment and increasing the payment by ₹ 300 every month.

This form an A.P.

i.e., 400, 700, 1000, ...

$a = 400, d = 300$

$\therefore$ Sum upto n terms

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$65000 = \frac{n}{2}[2(400) + (n-1)300]$$

$$65000 = \frac{n}{2} \times 2\,[400 + (n-1)150]$$

$= n\,[400 + 150n - 150]$

$= n\,[150n + 250]$

$65000 = 150n^2 + 250n$

Divided by 50

$1300 = 3n^2 + 5n$

$$3n^2 + 5n - 1300 = 0$$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{5^2 - 4(3)(-1300)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{25 + 15600}}{6} = \frac{-5 \pm \sqrt{15625}}{6}$$

$$= \frac{-5 \pm 125}{6}$$

$$= \frac{-5 - 125}{6}, \frac{-5 + 125}{6}$$

$$n = \frac{-130}{6}, \frac{120}{6}$$

$$n = \frac{-130}{6}, 20$$

n cannot be negative

$\therefore n = 20$.

He will clear the loan by 20 months.

10. A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two bricks less than the previous step.

(i) How many bricks are required for the top most step?

(ii) How many bricks are required to build the staircase?

Sol : Total number of steps n = 30.

Bottom step requires a = 100 bricks.

Each successive steps requires 2 less than the previous step.

$\therefore$ Number of bricks in each step form an A.P.

100, 98, 96, ...upto 30 terms.

a = 100, d = – 2, n = 30

(i) Number of bricks required for the topmost step is l

$$l = a + (n - 1)\,d$$
$$= 100 + (30 - 1)(-2)$$
$$= 100 + 29(-2)$$
$$= 100 - 58 = 42$$

$\therefore$ Topmost step requires 42 bricks.

(ii) Number of bricks required to build the staircase, $S_n = \frac{n}{2}(l + a) = \frac{30}{2}(100 + 42)$

$$= 15 \times 142 = 2130$$

Total number of bricks required = 2130

11. If $S_1, S_2, S_3, \ldots S_m$ are the sums of n terms of 'm' A.P.'s whose first terms are 1, 2, 3,... m and whose common differences are 1, 3, 5,... (2m – 1) respectively, then show that $(S_1 + S_2 + S_3 + \ldots + S_m) = \frac{1}{2} mn(mn + 1)$.

Sol :

S_1 is the sum of n terms of an A.P. with a = 1 and d = 1.

S_2 is the sum of n terms of an A.P. with a= 2 and d = 3.

S_m is the sum of n terms of an A.P. with a = m and d = 2m – 1.

$$\therefore S_n = \frac{n}{2}[2a + (n-1)\,d]$$

$$S_1 = \frac{n}{2}[2(1) + (n-1)(1)] = \frac{n}{2}[2 + n - 1]$$

$$S_1 = \frac{n}{2}[n + 1]$$

$$S_2 = \frac{n}{2}[2(2) + (n-1)(3)] = \frac{n}{2}[4 + 3n - 3]$$

$$S_2 = \frac{n}{2}[3n + 1]$$

Similarly, $S_m = \frac{n}{2}[2(m) + (n-1)(2m-1)]$

$$= \frac{n}{2}[2m + 2mn - n - 2m + 1]$$

$$S_m = \frac{n}{2}[2mn - n + 1]$$

Now we find $S_1 + S_2 + \ldots\ldots + S_m$ where S_1 is the first term $a = \frac{n}{2}(n+1)$ and

S_m = last term $l = \frac{n}{2}[2mn - n + 1]$

Here number of terms = m.

$$\therefore S_n = \frac{n}{2}(l + a)$$

$$S_1 + S_2 + S_3 + \ldots\ldots S_m = \frac{m}{2}\left[\frac{n}{2}[2mn - n + 1] + \frac{n}{2}(n+1)\right]$$

$$= \frac{m}{2} \times \frac{n}{2}[2mn - n + 1 + n + 1]$$

$$= \frac{1}{4} \times mn(2mn + 2) = \frac{1}{4} mn \times 2(mn + 1)$$

$$S_1 + S_2 + S_3 + \ldots\ldots S_m = \frac{1}{2} mn(mn + 1)$$

12. Find the sum $\left[\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\frac{5a-3b}{a+b}+...\text{to 12 terms}\right]$

Sol :

Here $t_2 - t_1 = \frac{3a-2b}{a+b} - \frac{a-b}{a+b}$

$= \frac{3a-2b-a+b}{a+b} = \frac{2a-b}{a+b}$

$t_3 - t_2 = \frac{5a-3b}{a+b} - \frac{3a-2b}{a+b}$

$= \frac{5a-3b-3a+2b}{a+b} = \frac{2a-b}{a+b}$

$\therefore \; t_2 - t_1 = t_3 - t_2$

$\therefore$ The sequence form an A.P. with $d = \frac{2a-b}{a+b}$

$\therefore$ Sum of $S_n = \frac{n}{2}[2a+(n-1)d]$

$$S_{12} = \frac{12}{2}\left[2\left(\frac{a-b}{a+b}\right)+(12-1)\left(\frac{2a-b}{a+b}\right)\right]$$

$$= 6\left[2\left(\frac{a-b}{a+b}\right)+11\left(\frac{2a-b}{a+b}\right)\right]$$

$$= 6\times\left[\left(\frac{2a-2b}{a+b}\right)+\frac{22a-11b}{a+b}\right]$$

$$= 6\left[\frac{2a-2b+22a-11b}{a+b}\right]$$

$$= 6\left[\frac{[24a-13b]}{a+b}\right]$$

Sum $S_{12} = \frac{6}{a+b}\left[24a-13b\right]$

GEOMETRIC SEQUENCE OR GEOMETRIC PROGRESSION

Key Points

Definition:

1. A Geometric progression is a sequence in which each term is obtained by multiplying a fixed non-zero number to the proceeding term except the first term.
2. The fixed number is called common ratio. It is denoted by r.
3. The numbers a, ar, ar^2, ..., ar^{n-1} ... is called a geometric progression,where 'a' is the first term $r \neq 0$ is the common ratio.
4. The general term or n^{th} term of a G.P. is $t_n = ar^{n-1}$.
5. The ratio between any two consecutive terms of a G.P. is always constant and that constant is the common ratio. i.e., $r = \frac{t_2}{t_1}$
6. When the product of three consecutive terms of a G.P. are given, we take them as $\frac{a}{r}, a, ar$.
7. When the product of four consecutive terms are given for a G.P, we take them as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
8. When a non-zero constant is multiplied or divided in each term of a G.P, the resulting sequence is also a G.P.
9. Three non-zero number a, b, c are in G.P. if and only if $b^2 = ac$.

Formulae:

1. Sum up to n terms of a GP, $S_n = \frac{a(r^n-1)}{r-1}, r \neq 1$. S_n = na, if r = 1

2. The sum of infinite G.P. $a + ar + ar^2 + \ldots = \frac{a}{1-r}, -1 < r < 1$.

3. $\sum_{K=1}^{n} K = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

4. $\sum_{K=1}^{n} (2K-1) = 1 + 2 + 3 + \ldots + (2n-1) = n^2$

5. $\sum_{K=1}^{n} K^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

6. $\sum_{K=1}^{n} K^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[n\frac{(n+1)}{2}\right]^2$

Worked Examples

2.40 Which of the following sequences form a Geometric Progression?

(i) 7, 14, 21, 28,...

(ii) $\frac{1}{2}, 1, 2, 4, \ldots$, (iii) 5, 25, 50, 75,...

Sol : To check if a given sequence form a G.P. we have to see if the ratio between successive terms are equal.

(i) 7, 14, 21, 28,...

$\frac{t_2}{t_1} = \frac{14}{7} = 2; \quad \frac{t_3}{t_2} = \frac{21}{14} = \frac{3}{2};$

$\frac{t_4}{t_3} = \frac{28}{21} = \frac{4}{3}$

Since the ratios between successive terms are not equal, the sequence 7, 14, 21. 28,... is not a Geometric Progression.

(ii) $\frac{1}{2}, 1, 2, 4, \ldots$

$\frac{t_2}{t_1} = \frac{1}{\frac{1}{2}} = 2; \quad \frac{t_3}{t_2} = \frac{2}{1} = 2; \quad \frac{t_4}{t_3} = \frac{4}{2} = 2$

Here the ratios between successive terms are equal. Therefore the sequence $\frac{1}{2}, 1, 2, 4, \ldots$ is a Geometric Progression with common ratio r = 2.

(iii) 5, 25, 50, 75,...

$\frac{t_2}{t_1} = \frac{25}{5} = 5; \quad \frac{t_3}{t_2} = \frac{50}{25} = 2; \quad \frac{t_4}{t_3} = \frac{75}{50} = \frac{3}{2}$

Since the ratios between successive terms are not equal, the sequence 5, 25, 50, 75,... is not a Geometric Progression.

2.41 Find the geometric progression whose first term and common ratios are given by (i) a = – 7, r = 6 (ii) a = 256, r = 0.5

Sol :

(i) The general form of Geometric progression is a, ar, ar^2,...

$a = -7, ar = -7 \times 6 = -42, ar^2 = -7 \times 6^2 = -252$

Therefore the required Geometric Progression is – 7, – 42, – 252,...

(ii) The general form of Geometric progression is a, ar, ar^2,...

$a = 256, ar = 256 \times 0.5 = 128, ar^2 = 256 \times (0.5)^2 = 64$

Therefore the required Geometric progression is 256, 128, 64,...

2.42 Find the 8th term of the G.P. 9, 3, 1, ...

Sol :

To find the 8th term we have to use the nth term formula $t_n = ar^{n-1}$

First term a = 9, Common ratio $r = \frac{t_2}{t_1} = \frac{3}{9} = \frac{1}{3}$

$t_8 = 9 \times \left(\frac{1}{3}\right)^{8-1} = 9 \times \left(\frac{1}{3}\right)^7 = \frac{1}{243}$

Therefore the 8th term of the G.P. is $\frac{1}{243}$.

2.43 In a Geometric progression, the 4th term is $\frac{8}{9}$ and the 7th term is $\frac{64}{243}$. Find the Geometric Progression.

Sol :

4th term, $t_4 = \frac{8}{9} \Rightarrow ar^3 = \frac{8}{9}$ ...(1)

7th term $t_7 = \frac{64}{243} \Rightarrow ar^6 = \frac{64}{243}$ ...(2)

Dividing (2) by (1) we get, $\frac{ar^6}{ar^3} = \frac{\frac{64}{243}}{\frac{8}{9}}$

$$\Rightarrow r^3 = \frac{8}{27} \Rightarrow r = \frac{2}{3}$$

Substituting the value of r in (1) we get

$$a \times \left[\frac{2}{3}\right]^3 = \frac{8}{9} \Rightarrow a = 3$$

Therefore the Geometric Progression is

$a, ar, ar^2, \ldots$ That is $3, 2, \frac{4}{3}, \ldots$

2.44 **The product of three consecutive terms of a Geometric Progression is 343 and their sum is $\frac{91}{3}$. Find the three terms.**

Sol :

Since the product of 3 consecutive terms is given.

We can take them as $\frac{a}{r}, a, ar$.

Product of the terms = 343

$$\frac{a}{r} \times a \times ar = 343$$

$$a^3 = 7^3 \Rightarrow a = 7$$

Sum of the terms = $\frac{91}{3}$

Hence $a\left(\frac{1}{r} + 1 + r\right) = \frac{91}{3} \Rightarrow 7\left(\frac{1+r+r^2}{r}\right) = \frac{91}{3}$

$$3 + 3r + 3r^2 = 13r \Rightarrow 3r^2 - 10r + 3 = 0$$

$$(3r - 1)(r - 3) = 0 \Rightarrow r = 3 \text{ or } r = \frac{1}{3}$$

If $a = 7, r = 3$ then the three terms are $\frac{7}{3}, 7, 21$.

If $a = 7, r = \frac{1}{3}$ then the three terms are $21, 7, \frac{7}{3}$.

2.45 **The present value of a machine is ₹ 40,000 and its value depreciates each year by 10% . Find the estimated value of the machine in 6th year.**

The value of the machine at present is ₹ 40,000.
Since it is depreciated at the rate of 10% after one year the value of the machine is 90% of the initial value.

That is value of the machine at the end of the first year is $40{,}000 \times \frac{90}{100}$

After two years, the value of the machine is 90% of the value in the first year.

Value of the machine at the end of the 2nd year is $40{,}000 \times \left(\frac{90}{100}\right)^2$

Continuing this way, the value of the machine depreciates as

$$40000,\ 40000 \times \frac{90}{100},\ 40000 \times \left(\frac{90}{100}\right)^2 \ldots$$

This sequence is in the form of G.P. with first term 40,000 and common ratio $\frac{90}{100}$. For finding the value of the machine at the end of 5th year (i.e., in 6th year)

We need to find the sixth term of this G.P.

Thus, $n = 6, a = 40{,}000, r = \frac{90}{100}$.

Using $t_n = ar^{n-1}$, we have $t_6 = 40{,}000 \times \left(\frac{90}{100}\right)^{6-1}$

$$= 40000 \times \left(\frac{90}{100}\right)^5$$

$$t_6 = 40000 \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10} = 23619.6$$

Therefore the value of the machine in 6th year = ₹ 23619.60

Progress Check

1. A G.P. is obtained by multiplying _____ to the preceding term.

Ans : a constant or a fixed non-zero number.

2. The ratio between any two consecutive terms of the G.P. is ____ and it is called _____.

Ans : constant, common ratio

3. Fill in the blanks if the following are in G.P.

(i) $\frac{1}{8}, \frac{3}{4}, \frac{9}{2}, \_\_$ **(ii)** $7, \frac{7}{2}, \_\_$

(iii) $\_\_, 2\sqrt{2}, 4, \ldots$

Ans : (i) 27, (ii) $\frac{7}{4}$, (iii) 2

4. If first term = a, common ratio = r, t_9 = _____, t_{27} = _____

Ans : ar^8, ar^{26}

5. In a G.P. if $t_1 = \frac{1}{5}$ and $t_2 = \frac{1}{25}$ then the common ratio is ____

Ans : $\frac{1}{5}$

6. Three non-zero numbers a, b, c are in G.P. if and only if ____

Ans : $b^2 = ac$

Thinking Corner

1. Is the sequence $2, 2^2, 2^{2^2}, 2^{2^{2^2}}, ...$ is a G.P.?

Ans : $\frac{t_2}{t_1} = \frac{2^2}{2} = \frac{2 \times 2}{2} = 2$

$$\frac{t_3}{t_2} = \frac{2^{2^2}}{2^2} = \frac{2^2 \times 2^2}{2^2} = 2^2 = 4$$

$$\frac{t_4}{t_3} = \frac{2^{2^{2^2}}}{2^{2^2}} = \frac{2^{2^2} \times 2^{2^2}}{2^{2^2}} = 2^{2^2} = 16$$

$$\frac{t_2}{t_1} \neq \frac{t_3}{t_2} \neq \frac{t_4}{t_3}$$

$\therefore$ The sequence is not in G.P.

2. Split 64 into three parts such that the numbers are in G.P.

Ans :

Let the three numbers be ar^{n-1}, a, ar

Let a = 4 and r = 4

The three numbers are 1, 4, 16.

3. If a, b, c,... are in G.P. then 2a, 2b, 2c, ... are in ____

Ans : G.P.

4. If 3, x, 6.75 are in G.P. then x is ____

Ans : 4.5

Exercise 2.7

1. Which of the following sequences are in G.P.?

(i) 3, 9, 27, 81,...
(ii) 4, 44, 444, 4444,
(iii) 0.5, 0.05, 0.005,... **(iv)** $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, ...$,
(v) 1, –5, 25, –125,... **(vi)** 120, 60, 30, 18...
(vii) 16, 4, 1 $\frac{1}{4}$,...

Sol :

(i) 3, 9, 27, 81,...

Here $\frac{t_2}{t_1} = \frac{9}{3} = 3$

$$\frac{t_3}{t_2} = \frac{27}{9} = 3$$

$$\frac{t_4}{t_3} = \frac{81}{27} = 3$$

The ratios between successive terms are equal.

$\therefore$ The sequence 3, 9, 27, 81,... are in G.P.

(ii) 4, 44, 444, 4444

$$\frac{t_2}{t_1} = \frac{44}{4} = 11$$

$$\frac{t_3}{t_2} = \frac{444}{44} = \frac{111}{11}$$

$$\frac{t_4}{t_3} = \frac{4444}{444} = \frac{1111}{111}$$

The ratios between the successive terms are not equal. Therefore the sequence 4, 44, 444,... are not in G.P.

(iii) 0.5, 0.05, 0.005,...

$$\frac{t_2}{t_1} = \frac{0.05}{0.5} = \frac{0.5}{5} = \frac{5}{50} = \frac{1}{10}$$

$$\frac{t_3}{t_2} = \frac{0.005}{0.05} = \frac{5}{50} = \frac{1}{10}$$

The ratios between the successive terms are equal. $\therefore$ 0.5, 0.05, 0.005,... are in G.P.

(iv) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, ...$

$$\frac{t_2}{t_1} = \frac{1/6}{1/3} = \frac{1}{6} \times \frac{3}{1} = \frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{1/12}{1/6} = \frac{1}{12} \times \frac{6}{1} = \frac{1}{2}$$

The ratios between successive terms are equal

$\therefore$ $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, ...$ are in G.P.

(v) 1, – 5, 25, – 125,...

$$\frac{t_2}{t_1} = \frac{-5}{1} = -5$$

$$\frac{t_3}{t_2} = \frac{25}{-5} = -5$$

$$\frac{t_4}{t_3} = \frac{-125}{25} = -5$$

The ratios between successive terms are equal. Therefore, 1, – 5, 25, – 125 ...... are in G.P.

(vi) 120, 60, 30, 18, ...

$$\frac{t_2}{t_1} = \frac{60}{120} = \frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{30}{60} = \frac{1}{2}$$

$$\frac{t_4}{t_3} = \frac{18}{30} = \frac{9}{15} = \frac{3}{5}$$

The ratios between successive terms are not equal. Therefore, 120, 60, 30, 18,... are not a G.P.

(vii) $16, 4, 1, \frac{1}{4}, ...$

$$\frac{t_2}{t_1} = \frac{4}{16} = \frac{1}{4}$$

$$\frac{t_3}{t_2} = \frac{1}{4}$$

$$\frac{t_4}{t_3} = \frac{1/4}{1} = \frac{1}{4}$$

The ratios between successive terms are equal.

$\therefore$ $16, 4, 1, \frac{1}{4}$,.... in G.P.

2. Write the first three terms of the G.P. whose first term and the common ratio are given below.

(i) a = 6, r = 3 **(ii)** $a = \sqrt{2}, r = \sqrt{2}$

(iii) $a = 1000, r = \frac{2}{5}$

Sol :

(i) a = 6, r = 3

The three terms of G.P. are a, ar, ar^2

$\Rightarrow$ $6, 6 \times 3, 6(3)^2$

$\Rightarrow$ 6, 18, 54

$\therefore$ First three terms are 6, 18, 54.

(ii) $a = \sqrt{2}, r = \sqrt{2}$

Let the three terms of G.P. are a, ar, ar^2

$\sqrt{2}, \sqrt{2}\sqrt{2}, \sqrt{2}(\sqrt{2})^2$

$\sqrt{2}, 2, 2\sqrt{2}$

$\therefore$ First three terms of the G.P. $\sqrt{2}, 2, 2\sqrt{2}$

(iii) $a = 1000 \quad r = \frac{2}{5}$

Let the three terms of G.P. are a, ar, ar^2

$$1000, 1000\left(\frac{2}{5}\right), 1000\left(\frac{2}{5}\right)^2 \Rightarrow 1000, 400, 160$$

$\therefore$ First three terms of the G.P. are 1000, 400, 160.

3. In a G.P. 729, 243, 81,... find t_7.

Sol : n^{th} term of G.P. = ar^{n-1}

Here a = 729

$$r = \frac{t_2}{t_1} = \frac{243}{729}$$

$$r = \frac{1}{3}$$

$$t_7 = ar^{7-1} = ar^6 = 729\left(\frac{1}{3}\right)^6$$

$$= 729 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$$

$$t_7 = 1$$

4. Find x so that x + 6, x + 12 and x + 15 are consecutive terms of a Geometric progression.

Sol :

If the given numbers are consecutive terms of a G.P. then

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$

i.e., $$\frac{x+12}{x+6} = \frac{x+15}{x+12}$$

$$(x + 12)^2 = (x + 6)(x + 15)$$

$$x^2 + 24x + 144 = x^2 + 6x + 15x + 90$$

$$x^2 + 24x + 144 - x^2 - 6x - 15x - 90 = 0$$

$$24x - 21x + 144 - 90 = 0$$

$$3x + 54 = 0$$

$$3x = -54$$

$$x = \frac{-54}{3} = -18$$

5. Find the number of terms in the following G.P.

(i) 4, 8, 16, ..., 8192

(ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ..., \frac{1}{2187}$

Sol :

(i) 4, 8, 16, ...8192

$$a = 4, r = \frac{8}{4} = 2$$

n^{th} term of a G.P. $t_n = ar^{n-1}$

$$8192 = 4r^{n-1}$$

$$r^{n-1} = \frac{8192}{4}$$
$$r^{n-1} = 2048 = 2^{11}$$
$$n - 1 = 11 \quad [\because r = 2]$$
$$n = 11 + 1 = 12$$

2	2048
2	1024
2	512
2	256
2	128
2	64
2	32
2	16
2	8
2	4
	2

$\therefore$ Number of terms in the given G.P. is 12.

(ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots \frac{1}{2187}$

Here $a = \frac{1}{3}, r = \frac{t_2}{t_1} = \frac{1/9}{1/3} = \frac{1}{9} \times \frac{3}{1}$

$$r = \frac{1}{3}$$

n^{th} term of the G.P. $t_n = ar^{n-1}$

$$\frac{1}{2187} = \frac{1}{3} r^{n-1}$$
$$\frac{3}{2187} = r^{n-1}$$
$$\frac{1}{729} = r^{n-1}$$
$$\left(\frac{1}{3}\right)^6 = r^{n-1}$$

$n - 1 = 6 \quad \left[\because r = \frac{1}{3}\right]$

$n = 6 + 1 = 7$

$\therefore$ The number of terms in this G.P. is 7.

6 In a G.P. the 9^{th} term is 32805 and 6^{th} term is 1215. Find the 12^{th} term.

Sol :

9^{th} term $= ar^{9-1} = 32805 \Rightarrow ar^8 = 32805$ ...(1)

6^{th} term $= ar^{6-1} = 1215 \Rightarrow ar^5 = 1215$ ...(2)

$$\frac{(1)}{(2)} \Rightarrow \frac{ar^8}{ar^5} = \frac{32805}{1215}$$
$$r^3 = 27$$
$$r^3 = 3^3$$
$$r = 3$$

Put $r = 3$ in (2)

$$ar^5 = 1215$$
$$a(3^5) = 1215$$
$$a = \frac{1215}{3^5}$$
$$a = 5$$

$\therefore$ 12^{th} term $t_{12} = ar^{12-1}$

$= 5(3)^{11} = 5 \times 1,53,147$

12th term of the G.P. = 7,65,735

7 Find the 10^{th} term of a G.P. whose 8^{th} term is 768 and the common ratio is 2.

Sol :

n^{th} term of a G.P. $t_n = ar^{n-1}$

Given 8^{th} term $t_8 = 768$ and $r = 2$

$ar^{8-1} = 768$

$a(2)^7 = 768$

$$a = \frac{768}{2^7} = \frac{2^7 \times 2^1 \times 3}{2^7} = 6$$

2	768
2	384
2	192
2	96
2	48
2	24
2	12
2	6
	3

10^{th} term $t_{10} = ar^{10-1} = ar^9$

$= 6 \times 2^9 = 6 \times 512 = 3072$

10^{th} term of the G.P. is 3072.

8 If a, b, c are in A.P. then show that $3^a, 3^b, 3^c$ are in G.P.

Sol : Given a, b, c are in A.P.

$$\therefore t_2 - t_1 = t_3 - t_2$$
$$b - a = c - b$$
$$b + b = c + a$$
$$2b = a + c$$

If we multiply both the sides by same number value will not change.

$$\therefore 3^{2b} = 3^{a+c}$$
$$3^{b+b} = 3^{a+c}$$
$$3^b . 3^b = 3^a . 3^c$$
$$\frac{3^b}{3^a} = \frac{3^c}{3^b}$$

Thus $3^a, 3^b, 3^c$ are in G.P.

9. In a G.P. the product of three consecutive terms is 27 and the sum of product of terms taken two at a time is $\frac{57}{2}$. Find the three terms.

Sol :

Let the three terms be ar^{-1}, a, ar

$$\text{Given product} = 27$$
$$ar^{-1} \times a \times ar = 27$$
$$a^3 = 27$$
$$a^3 = 3^3$$
$$a = 3$$

Sum of the product taken two at a time $= \frac{57}{2}$

$$(ar^{-1} \times a) + (a \times ar) + (ar^{-1} \times ar) = \frac{57}{2}$$

$$a^2r^{-1} + a^2r + a^2 = \frac{57}{2}$$

$$a^2\left(\frac{1}{r} + r + 1\right) = \frac{57}{2}$$

$$3^2\left(\frac{1 + r^2 + r}{r}\right) = \frac{57}{2}$$

$$\frac{1 + r^2 + r}{r} = \frac{57}{2 \times 3 \times 3}$$

$$\frac{1 + r^2 + r}{r} = \frac{19}{6}$$

$$6 + 6r^2 + 6r = 19r$$
$$6r^2 + 6r - 19r + 6 = 0$$
$$6r^2 - 13r + 6 = 0$$
$$6r^2 - 9r - 4r + 6 = 0$$
$$3r(2r - 3) - 2(2r - 3) = 0$$
$$(2r - 3)(3r - 2) = 0$$
$$2r - 3 = 0 \text{ (or) } 3r - 2 = 0$$
$$2r = 3 \text{ (or) } 3r = 2$$
$$r = \frac{3}{2} \text{ (or) } = \frac{2}{3}$$

If $r = \frac{3}{2}$, the terms are $\frac{3}{3/2}, 3, 3\left(\frac{3}{2}\right) = 2, 3, \frac{9}{2}$

If $r = \frac{2}{3}$ the terms are $\frac{3}{2/3}, 3, 3\left(\frac{2}{3}\right) = \frac{9}{2}, 3, 2$

$\therefore$ The three terms of G.P. are $\frac{9}{2}, 3, 2$

10. A man joined a company as Assistant Manager. The company gave him a starting salary of ₹ 60,000 and agreed to increase his salary 5% annually. What will be his salary after 5 years?

Sol :

Initial salary for the first year = ₹ 60000

At the end of first year salary = $60000 \times \left(\frac{105}{100}\right)$

At the end of second year salary

$$= (60000)\left(\frac{105}{100}\right)^2$$

Continuing this way the salary increase will be like $60000, 60000\left(\frac{105}{100}\right), 60000\left(\frac{105}{100}\right)^2 \ldots$

This form a G.P. with $a = 60000$ and $r = \frac{105}{100}$

Salary after 5 years (6^{th} year) $= ar^{n-1} = ar^{6-1} = ar^5$

$$= 60000\left(\frac{105}{100}\right)^5$$

$$= 60000 \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100} \times \frac{105}{100}$$

$\therefore$ After 5 years his salary = ₹ 76577

11. Sivamani is attending an interview for a job and the company gave two offers to him.
Offer A : ₹ 20,000 to start with followed by a guaranteed annual increase of 6% for the first 5 years.
Offer B : ₹ 22,000 to start with followed by a guaranteed annual increase of 3% for the first 5 years. What is his salary in the 4^{th} year with respect to the offers A and B?

Sol : Offer A:

Starting Salary = ₹ 20,000

Increase in salary per year = 6% = $\frac{6}{100}$

$\therefore$ The salary after first year = $20,000 \times \frac{106}{100}$

$\therefore$ Salary after second year = $20000 \times \left(\frac{106}{100}\right)^2$

Continuing like this the salary will be in G.P.

with $a =$ ₹ 20000, $r = \frac{106}{100}$

$\therefore$ In the fourth year salary $= t_n = ar^{n-1}$

$$t_4 = ar^{4-1} = ar^3$$

$$= 20000 \times \left(\frac{106}{100}\right)^3 = 23820$$

$\therefore$ Salary in the fourth year = ₹ 23, 820 by offer A.

Offer B:

Starting salary = ₹ 22000

Annual Increment = 3% = $\frac{3}{100}$

$\therefore$ Salary at the end of first year = $22000 \times \frac{103}{100}$

Salary at second year = $22000\left(\frac{103}{100}\right)^2$

Continuing like this we get a G.P. with

$a =$ ₹ 22000, $r = \frac{103}{100}$

Salary at fourth year $= ar^{n-1}$

$$= 22000\left(\frac{103}{100}\right)^{4-1} = 22000\left(\frac{103}{100}\right)^{3}$$

Salary at 4th year = ₹ 24040 by offer B.

12. If a, b, c are three consecutive terms of an A.P. and x, y, z are three consecutive terms of a G.P. then prove that $x^{b-c} \times y^{c-a} \times z^{a-b} = 1$.

Sol : Given a, b, c are in A.P.

$$b - a = c - b$$
$$b + b = c + a$$
$$2b = c + a \quad ...(1)$$

Given x, y, z are three consecutive terms of a G.P.

$$\therefore x = a$$
$$y = ar$$
$$z = ar^2$$

Now $x^{b-c} \times y^{c-a} \times y^{a-b} = a^{b-c} \times (ar)^{c-a} \times (ar^2)^{a-b}$

$$= a^{b-c} \times a^{c-a} \times r^{c-a} \times a^{a-b}\, r^{2(a-b)}$$
$$= a^{b-c+c-a+a-b}\, r^{c-a+2a-2b}$$
$$= a^0\, r^{(a+c)-2b}$$
$$= 1.r^{2b-2b} \text{ [from (1) } a + c = 2b]$$
$$= 1.\, r^0$$
$$= 1 \times 1 = 1$$
$$= \text{RHS}$$

$\therefore x^{b-c} \times y^{c-a} \times z^{a-b} = 1$

Hence proved.

SUM TO n TERMS OF A G.P.

Key Points

1. A series whose terms are in Geometric progression is called Geometric series.
2. Sum upto n terms of a G.P. $S_n = \dfrac{a(r^n - 1)}{r-1}, r \neq 1$. $S_n = na$ if $r = 1$.
3. The sum of infinite G.P. is given by $a + ar + ar^2 + ar^3 + ... = \dfrac{a}{1-r}, -1 < r < 1$.

Worked Examples

2.46 Find the sum of 8 terms of the G.P. 1, –3, 9, – 27...

Sol : Here the first term a = 1,

common ratio $r = \dfrac{-3}{1} = -3 < 1$, Here n = 8.

Sum to n terms of a G.P. is

$$S_n = \frac{a(r^n - 1)}{r-1} \text{ if } r \neq 1$$

Hence. $S_8 = \dfrac{1\left((-3)^8 - 1\right)}{(-3)-1} = \dfrac{6561-1}{-4} = -1640$

2.47 Find the first term of a G.P. in which $S_6 = 4095$ and r = 4?

Sol :

Common ratio = 4 >1, Sum of first 6 terms

$$S_6 = 4095$$

Hence, $S_6 = \dfrac{a(r^n - 1)}{r-1} = 4095$

Since, r = 4, $\dfrac{a(4^6-1)}{4-1} = 4095$

$$\Rightarrow \frac{a(4096-1)}{3} = 4095 \Rightarrow a = 3$$

2.48 How many terms of the series 1 + 4 + 16 + ... make the sum 1365?

Sol : Let n be the number of terms to be added to get the sum 1365.

$$a = 1, r = \frac{4}{1} = 4 > 1$$

$$S_n = 1365 \Rightarrow \frac{a(r^n - 1)}{r-1} = 1365$$

$$\frac{1(4^n - 1)}{4-1} = 1365 \Rightarrow (4^n - 1) = 4095$$

$$4^n = 4096 \Rightarrow 4^n = 4^6$$
$$n = 6$$

2.49 Find the sum $3 + 1 + \frac{1}{3} + ... \infty$

Sol :

Here a = 3, $r = \dfrac{t_2}{t_1} = \dfrac{1}{3}$

Sum of infinite terms $= \dfrac{a}{1-r} = \dfrac{3}{1-\frac{1}{3}} = \dfrac{9}{2}$

2.50 **Find the rational form of the number 0.6666...**

Sol :

We can express the number 0.6666... as follows

0.6666...= 0.6 + 0.06 + 0.006 + 0.0006 + ...

We now see that numbers 0.6, 0.06, 0.006... form an G.P. whose first term a = 0.6

and common ratio $r = \frac{0.06}{0.6} = 0.1$.

Also $-1 < r = 0.1 < 1$.

Using the infinite G.P. formula,

we have $0.6666... = 0.6 + 0.06 + 0.006 + 0.0006 + ...$

$$= \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{2}{3}$$

Thus the rational number equivalent of

0.6666 ... is $\frac{2}{3}$

2.51 **Find the sum to n terms of the series $5 + 55 + 555 + ...$**

Sol : The series is neither Arithmetic nor Geometric series. So it can be split into two series and then find the sum.

5 + 55 + 555 + ... + n terms = 5 [1 + 11 + 111 + ...+ n terms]

$$= \frac{5}{9}\left[9 + 99 + 999 + ... + n \text{ terms}\right]$$

$$= \frac{5}{9}\left[(10-1) + (100-1) + (1000-1) + ... + n \text{ terms}\right]$$

$$= \frac{5}{9}\left[(10 + 100 + 1000 + ... + n \text{ terms}) - n\right]$$

$$= \frac{5}{9}\left[\frac{10(10^n-1)}{(10-1)} - n\right] = \frac{50(10^n-1)}{81} - \frac{5n}{9}$$

2.52 **Find the least positive integer n such that $1 + 6 + 6^2 + ... + 6^n > 5000$.**

Sol :

We want to find the least number of terms for which the sum must exceed 5000.

That is to find the least value of n such that $S_n > 5000$

We have $S_n = \frac{a(r^n-1)}{r-1} = \frac{1(6^n-1)}{6-1} = \frac{6^n-1}{5}$

$$S_n > 5000 \Rightarrow \frac{6^n-1}{5} > 5000$$

$$6^n - 1 > 25000 \Rightarrow 6^n > 25001$$

Since, $6^5 = 7776$ and $6^6 = 46656$

The least positive value of n is 6 such that $1 + 6 + 6^2 + ... + 6^n > 5000$.

2.53 **A person saved money every year half as much as he could in the previous year. If he had totally saved ₹ 7875 in 6 years then how much did he save in the first year?**

Sol :

Total amount saved in 6 years is $S_6 = 7875$

Since he saved half as much money as every year he saved in the previous year.

We have $r = \frac{1}{2} < 1$

Since, $\frac{a(1-r^n)}{1-r} = \frac{a\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}} = 7875.$

$$\frac{a\left(1-\frac{1}{64}\right)}{\frac{1}{2}} = 7875 \Rightarrow a \times \frac{63}{32} = 7875$$

$$a = \frac{7875 \times 32}{63} \Rightarrow a = 4000$$

The amount saved in the first year is ₹ 4000.

Progress Check

1. **A series whose terms are in Geometric progression is called _____**

   Ans : Geometric series.

2. **When r = 1, the formula for finding sum to n terms of a G.P. is ______**

   Ans : $S_n = na$

3. **When $r \neq 1$, the formula for finding sum to n terms of a G.P. is _____**

   Ans : $S_n = \frac{a(r^n-1)}{r-1}$

4. **Sum to infinite number of terms of a G.P. is ______**

   Ans : $S_\infty = \frac{a}{1-r}, -1 < r < 1$

5. **For what values of r, does the formula for infinite G.P. is valid?**

   Ans : For $-1 < r < 1$, the formula is valid.

6. Is the series 3 + 33 + 333 + ... a Geometric series?

Ans : In the series 3 + 33 + 333 + ...

$$\frac{t_2}{t_1} = \frac{33}{3} = 11$$

$$\frac{t_3}{t_2} = \frac{333}{33} = \frac{111}{11}$$

$$\frac{t_2}{t_1} \neq \frac{t_3}{t_2}$$

∴ It is not a Geometric series.

7. The value of r, such that $1 + r + r^2 + r^3 \ldots = \frac{3}{4}$ is

Ans : We have for an infinite series

$$S_\infty = \frac{a}{1-r} \quad -1 < r < 1$$

Here a = 1

$$\frac{3}{4} = \frac{1}{1-r}$$

$$3(1-r) = 4$$

$$3 - 3r = 4$$

$$-3r = 4 - 3$$

$$-3r = 1$$

$$r = \frac{-1}{3}$$

∴ The value of $r = \frac{-1}{3}, -1 < \frac{-1}{3} < 1$.

Exercise 2. 8

1. Find the sum of first n terms of the G.P

(i) $5, -3, \frac{9}{5}, -\frac{27}{25}, \ldots$ (ii) 256, 64, 16,...

Sol :

(i) $5, -3, \frac{9}{5}, -\frac{27}{25}, \ldots$

It is a geometric progression with

$a = 5, r = \frac{-3}{5} \neq 1$

Sum upto n terms $S_n = \frac{a(r^n - 1)}{r-1}$

$$= \frac{5\left[\left(\frac{-3}{5}\right)^n - 1\right]}{\frac{-3}{5} - 1}$$

$$= \frac{5\left[\left(\frac{-3}{5}\right)^n - 1\right]}{\frac{-3-5}{5}}$$

$$= \frac{5\left[\left(\frac{-3}{5}\right)^n - 1\right]}{\frac{-8}{5}}$$

$$= \frac{-25}{8}\left[\left(\frac{-3}{5}\right)^n - 1\right]$$

$$= \frac{25}{8}\left[1 - \left(\frac{-3}{5}\right)^n\right]$$

(ii) 256, 64, 16,....

Here $a = 256, r = \frac{64}{256} = \frac{1}{4} \neq 1$

Sum upto n terms

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{256\left[\left(\frac{1}{4}\right)^n - 1\right]}{\frac{1}{4} - 1} = \frac{256\left[\left(\frac{1}{4}\right)^n - 1\right]}{-\frac{3}{4}}$$

$$= \frac{-1024}{3}\left[\left(\frac{1}{4}\right)^n - 1\right] = \frac{1024}{3}\left[1 - \left(\frac{1}{4}\right)^n\right]$$

2. Find the sum of first six terms of the G.P. 5, 15, 45, ...

Sol :

We have to find 5 + 15 + 45 + ...upto 6 terms.

Sum upto n terms of a G.P. $S_n = \frac{a(r^n - 1)}{r-1}$

Here $a = 5, r = \frac{15}{5} = 3$

$$\therefore \text{Sum} = \frac{5(3^6 - 1)}{3-1} = \frac{5(729-1)}{2}$$

$$= \frac{5 \times 728}{2} = 5 \times 364$$

5 + 15 + 45 + ...upto 6 terms = 1820

3. Find the first term of the G.P. whose common ratio 5 and whose sum to first 6 terms is 46872.

Sol :

Given r = 5 and $S_6 = 46872$

Sum upto n terms of a G.P., $S_n = \frac{a(r^n - 1)}{r - 1}$

$$46872 = \frac{a(5^6 - 1)}{5 - 1}$$

$$46872 = a\frac{(15625 - 1)}{4} = a \times \frac{15624}{4}$$

$$46872 = a \times 3906$$

$$\frac{46872}{3906} = a$$

$$a = 12$$

∴ First term of the G.P., a = 12

4. Find the sum to infinity of (i) 9 + 3 + 1 + ... (ii) $21 + 14 + \frac{28}{3} + ...$

Sol :

(i) 9 + 3 + 1 + ...

Here $\frac{3}{9} = \frac{1}{3}$

∴ It is a geometric series with

$a = 9$ and $r = \frac{1}{3}, -1 < r < 1$

$$\therefore \text{Sum} = \frac{a}{1-r} = \frac{9}{1-\frac{1}{3}}$$

$$= \frac{9}{2/3} = 9 \times \frac{3}{2}$$

$$\therefore 9 + 3 + 1 + ... = \frac{27}{2} = 13.5$$

(ii) $21 + 14 + \frac{28}{3} + ...$

Here $\frac{14}{21} = \frac{2}{3} = \frac{t_2}{t_1}$

$$\frac{t_3}{t_2} = \frac{28/3}{14} = \frac{28}{3} \times \frac{1}{14} = \frac{2}{3}$$

$$\frac{t_2}{t_1} = \frac{t_3}{t_2}$$

It forms a geometric series with

$a = 21, \quad r = \frac{2}{3}, -1 < r < 1.$

$$\text{Sum} = \frac{a}{1-r} = \frac{21}{1-\frac{2}{3}}$$

$$= \frac{21}{1/3} = 21 \times \frac{3}{1}$$

$$21 + 14 + \frac{28}{3} + ... = 63.$$

5. If the first term of an infinite G.P. is 8 and its sum to infinity is $\frac{32}{3}$ then find the common ratio.

Sol : First term a = 8

$$S_\infty = \frac{32}{3}$$

Sum upto infinity of a G.P.,

$$S_\infty = \frac{a}{1-r}$$

$$\frac{32}{3} = \frac{8}{1-r}$$

$$32(1 - r) = 24$$

$$1 - r = \frac{24}{32}$$

$$1 - r = \frac{3}{4}$$

$$1 - \frac{3}{4} = r$$

$$\frac{1}{4} = r$$

Common ratio $r = \frac{1}{4}$

6. Find the sum to n terms of the series

(i) 0.4 + 0.44 + 0.444 + ... to n terms

(ii) 3 + 33 + 333 + ... to n terms

Sol :

(i) 0.4 + 0.44 + 0.444 + ... to n terms

Let S_n = 0.4 + 0.44 + 0.444 + ... to n terms.

= 4 [0.1 + 0.11 + 0.111 + ... to n terms]

(multiply and divide by 9)

$$= \frac{4}{9}[0.9 + 0.99 + 0.999 + ... \text{ to n terms}]$$

$$= \frac{4}{9}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + ...\text{to } n \text{ terms}\right]$$

$$= \frac{4}{9}\left[\frac{10-1}{10} + \frac{100-1}{100} + \frac{1000-1}{100} + ...n \text{ terms}\right]$$

$$=\frac{4}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\ldots n \text{ terms}\right]$$

$$=\frac{4}{9}\left[n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots \text{up to } n \text{ terms}\right)\right]$$

$$=\frac{4}{9}\left[n-\left[\frac{\frac{1}{10}\left(1-\left(\frac{1}{10}\right)^n\right)}{1-\frac{1}{10}}\right]\right]$$

$$=\frac{4}{9}\left[n-\frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{\frac{9}{10}}\right]$$

$$=\frac{4}{9}\left[n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]=\left[\frac{4n}{9}-\frac{4}{81}\left(1-\frac{1}{10^n}\right)\right]$$

$$0.4+0.44+0.444\ldots=\left[\frac{4n}{9}-\frac{4}{81}\left(1-\frac{1}{10^n}\right)\right]$$

$$=\frac{4}{9}n-\frac{4\left(1-\left(\frac{1}{10}\right)^n\right)}{81}$$

(ii) $3 + 33 + 333 + \ldots$ to n terms.

Let $S_n = 3 + 33 + 333 + \ldots$upto n terms.

$= 3\,[1 + 11 + 111 +$ upto n terms]

$=\frac{3}{9}\,[9 + 99 + 999 + \ldots$ upto n terms]

(multiply and divide by 9)

$=\frac{1}{3}\,[(10-1)+(100-1)+(1000-1)+\ldots$ n terms]

$=\frac{1}{3}\,[(10-1)+(10^2-1)+(10^3-1)+\ldots$ upto n terms]

$=\frac{1}{3}\,\{[10+10^2+10^3+\ldots$upto n terms$]-n\}$

$10 + 10^2 + \ldots$ is a G.P. with $a = 10$, $r = 10$.

$$\therefore\ S_n=\frac{a(r^n-1)}{r-1}$$

$$S_n=\frac{1}{3}\left\{\left[\frac{10(10^n-1)}{10-1}\right]-n\right\}$$

$$=\frac{1}{3}\left[\frac{10(10^n-1)}{9}-n\right]$$

$$=\frac{10}{27}(10^n-1)-\frac{n}{3}$$

$$3+33+333+\ldots \text{ to n terms}=\frac{10}{27}(10^n-1)-\frac{n}{3}$$

7. Find the sum of the Geometric series $3 + 6 + 12 + \ldots + 1536$.

Sol : $a = 3$, $r = \frac{6}{3} = 2$

n^{th} term of the G.P., $t_n = ar^{n-1}$

$$1536 = 3(2)^{n-1}$$
$$\frac{1536}{3} = 2^{n-1}$$
$$512 = 2^{n-1}$$
$$2^9 = 2^{n-1}$$
$$n-1 = 9$$
$$n = 9+1 = 10$$

$\therefore$ Number of terms in the series $n = 10$.

$$\therefore \text{Sum upto n terms } S_n=\frac{a(r^n-1)}{r-1}$$

$$S_{10}=\frac{3(2^{10}-1)}{2-1}$$

$$=\frac{3\times(1024-1)}{1}=3\times 1023$$

$3+6+12+\ldots+1536 = 3069$.

8. Kumar writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with the instruction that they continue the process similarly. Assuming that the process is unaltered and it costs ₹ 2 to mail one letter, find the amount spent on postage when 8^{th} set of letters is mailed.

Sol : Total letters in the first set = 4

Total letters in the second set = $4^2 = 16$

Total letters in the third set = $4^3 = 64$

So the sequence of letters is 4, 16, 64,....

Here $a = 4$, $r = \frac{16}{4} = 4$ and $n = 8$

$$S_n=\frac{a(r^n-1)}{r-1}=\frac{4(4^8-1)}{4-1}$$

$$=\frac{4}{3}(65536-1)=\frac{4}{3}\times 65535$$

$$= 87380$$

Since amount of postage per letter is ₹ 2

Total amount spend on postage = 87380 × 2

= ₹ 174760.

9. Find the rational form of the number $0.\overline{123}$.

Sol : We have $0.\overline{123} = 0.123123123....$

$= 0.123 + 0.000123 + 0.000000123 + ...$

$\frac{t_2}{t_1} = \frac{0.000123}{0.123} = \frac{0.123}{123} = 0.001$

$\therefore$ It is a G.P. with $a = 0.123$ and $r = 0.001$

Sum of infinity $= \frac{a}{1-r}$

$= \frac{0.123}{1-0.001} = \frac{0.123}{1-\frac{1}{1000}}$

$= \frac{0.123}{\frac{999}{1000}} = \frac{123}{999}$

$\therefore 0.\overline{123} = \frac{41}{333}$

10. If $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...n$ terms then prove that

$$(x-y)S_n = \left[\frac{x^2(x^n-1)}{x-1} - \frac{y^2(y^n-1)}{y-1}\right]$$

Sol : Given $S_n = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ... n$ terms)

$= \frac{x-y}{x-y}[(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ...n \text{ terms})]$

[multiply and divide by $(x - y)$]

$= \frac{1}{x-y}\left[(x-y)(x+y)+(x-y)(x^2+xy+y^2)+(x-y)(x^3+x^2y+xy^2+y^3+...n)\right]$

$= \frac{1}{x-y}[(x^2 - y^2) + (x^3 - y^3) + (x^4 - y^4) + ...n \text{ terms}]$

$= \frac{1}{x-y}[x^2 + x^3 + x^4 + ...n \text{ terms}] - [y^2 + y^3 + y^4 + ... n \text{ terms}]$

$$S_n = \frac{1}{x-y}\left[\frac{x^2(x^n-1)}{x-1} - \frac{y^2(y^n-1)}{y-1}\right]$$

$$\left[\because S_n = \frac{a(r^n-1)}{r-1}\right]$$

$$(x-y)S_n = \left[\frac{x^2(x^n-1)}{x-1} - \frac{y^2(y^n-1)}{y-1}\right]$$

SPECIAL SERIES

Key Points

1. Special series are some series whose sum can be expressed by explicit formula.
2. $1^k + 2^k + 3^k + ... + n^k = (x+1)^{k+1} - x^{k+1}$
3. $1 + 2 + 3 + ...+ n = \frac{n(n+1)}{2} = \sum_{k=1}^{n} k$
4. $1 + 3 + 5 + ... + (2n-1) = n^2 = \sum_{k=1}^{n}(2k-1)$
5. $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} = \sum_{k=1}^{n} k^2$
6. $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \sum_{k=1}^{n} k^3$
7. Sum of divisors of one number excluding itself gives the other number. Such numbers are called Amicable Numbers or Friendly Numbers. Example: 220 and 284.
8. The sum of first n natural numbers are called Triangular Numbers because they form triangle shapes.
9. The sum of squares of first n natural numbers are called Square Pyramidal Numbers because they form pyramid shapes with square base.

Worked Examples

2.54 Find the value of (i) $1+2+3+\ldots+50$
(ii) $16+17+18+\ldots+75$

Sol :

(i) $1+2+3+\ldots+50$

Using $1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$

$$1+2+3+\ldots+50 = \frac{50\times(50\times1)}{2} = 1275$$

(ii) $16+17+18+\ldots+75$

$$16+17+18+\ldots+75 = (1+2+3+\ldots+75) - (1+2+3+\ldots+15)$$

$$= \frac{75(75+1)}{2} - \frac{15(15+1)}{2}$$

$$= 2850 - 120 = 2730$$

2.55 Find the sum of (i) $1+3+5+\ldots+$ to 40 terms
(ii) $2+4+6+\ldots+80$
(iii) $1+3+5+\ldots+55$

Sol :

(i) $1+3+5+\ldots$ 40 terms $= 40^2 = 1600$

$[1+3+5+\ldots\ldots+(2n-1)] = n^2$

(ii) $2+4+6+\ldots+80 = 2(1+2+3+\ldots+40)$

$$= 2\times\frac{40\times(40+1)}{2} = 1640$$

(iii) $1+3+5+\ldots+55$

Here the number of terms is not given. Now we have to find the number of terms using the formula, $n = \dfrac{(l-a)}{d}+1 \Rightarrow n = \dfrac{(55-1)}{2}+1 = 28$

Therefore, $1+3+5+\ldots+55 = (28)^2 = 784$

2.56 Find the sum of (i) $1^2+2^2+\ldots+19^2$
(ii) $5^2+10^2+15^2+\ldots+105^2$
(iii) $15^2+16^2+17^2+\ldots+28^2$

Sol :

(i) $1^2+2^2+\ldots+19^2 = \dfrac{19\times(19+1)(2\times19+1)}{6}$

$$= \frac{19\times20\times39}{6} = 2470$$

(ii) $5^2+10^2+15^2+\ldots+105^2 = 5^2(1^2+2^2+3^2+\ldots+21^2)$

$$= 25\times\frac{21\times(21+1)(2\times21+1)}{6}$$

$$= \frac{25\times21\times22\times43}{6} = 82775$$

(iii) $15^2+16^2+17^2+\ldots+28^2$

$= (1^2+2^2+3^2+\ldots+28^2) - (1^2+2^2+3^2+\ldots+14^2)$

$$= \frac{28\times29\times57}{6} - \frac{14\times15\times29}{6}$$

$$= 7714 - 1015 = 6699.$$

2.57 Find the sum of (i) $1^3+2^3+3^3+\ldots+16^3$
(ii) $9^3+10^3+\ldots+21^3$

Sol :

(i) $1^3+2^3+3^3+\ldots+16^3$

$$= \left[\frac{16\times(16+1)}{2}\right]^2 = (136)^2 = 18496$$

(ii) $9^3+10^3+\ldots+21^3 = (1^3+2^3+3^3+\ldots+21^3) - (1^3+2^3+3^3+\ldots+8^3)$

$$= \left[\frac{21\times(21+1)}{2}\right]^2 - \left[\frac{8\times(8+1)}{2}\right]^2$$

$$= (231)^2 - (36)^2 = 53361 - 1296 = 52065.$$

2.58 If $1+2+3+\ldots+n = 666$ then find n.

Since, $1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$,

we have $\dfrac{n(n+1)}{2} = 666$

$n^2+n-1332 = 0 \Rightarrow (n+37)(n-36) = 0$

$\Rightarrow n = -37$ or $n = 36$

But $n \neq -37$ (Since n is a natural number);

Hence $n = 36$.

Progress Check

1. The sum of cubes of first n natural numbers is ______ of the first n natural numbers.

Ans : Square of the sum

2. The average of first 100 natural numbers is ______

Ans :

Sum of first 100 natural numbers $= \dfrac{n(n+1)}{2}$

$$= \frac{100\times101}{2} = 5050$$

$$\text{Average} = \frac{5050}{100} = 50.5$$

Say True or False

1. The sum of first n odd natural numbers is always an odd number.

Ans : False

2. The sum of consecutive even numbers is always an even number.

Ans : True

3. The difference between the sum of squares of first n natural numbers and the sum of first n natural numbers is always divisible by 2.

Ans : False

4. The sum of cubes of first n natural numbers is always a square number.

Ans : True

Thinking Corner

1. How many squares are there in a standard chess board?

Ans : 64

2. How many rectangles are there in a standard chess board?

Ans : 1296

Exercise 2. 9

1. Find the sum of the following series.

(i) $1 + 2 + 3 + \dots + 60$

(ii) $3 + 6 + 9 + \dots + 96$

(iii) $51 + 52 + 53 + \dots + 92$

(iv) $1 + 4 + 9 + 16 + \dots + 225$

(v) $6^2 + 7^2 + 8^2 + \dots + 21^2$

(vi) $10^3 + 11^3 + 12^3 + \dots + 20^3$

(vii) $1 + 3 + 5 + \dots + 71.$

Sol :

(i) $1+2+3+\dots+60$

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$1+2+3+\dots+60 = \frac{60\times(60+1)}{2} = \frac{60\times 61}{2}$$

$$1+2+3+\dots+60 = 1830$$

(ii) $3+6+9+\dots+96.$

$$3+6+9+\dots+96 = 3\,(1+2+3+\dots+32)$$

We know that $1+2+3+\dots+n = \frac{n(n+1)}{2}$

$$\therefore\ 3+6+9+\dots+96 = 3\left[\frac{32\times(32+1)}{2}\right]$$

$$= 3\left[\frac{32\times 33}{2}\right] = 1584$$

$$3+6+9+\dots+96 = 1584$$

(iii) $51+52+53+\dots+92$

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$51+52+53+\dots+92 = (1+2+3+\dots+92) - (1+2+\dots+50)$$

$$= \frac{92\times(92+1)}{2} - \frac{50\times(50+1)}{2}$$

$$= \frac{92\times 93}{2} - \frac{50\times 51}{2}$$

$$= 4278 - 1275 = 3003$$

$$51+52+53+\dots+92 = 3003$$

(iv) $1+4+9+16+\dots+225$

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1+4+9+16+\dots+225 = 1^2+2^2+3^2+\dots+15^2$$

$$= \frac{15(15+1)[2(15)+1]}{6}$$

$$= \frac{15\times 16\times 31}{6}\ 1240$$

$$1+4+9+16+\dots+225 = 1240$$

(v) $6^2+7^2+8^2+\dots+21^2$

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$6^2+7^2+8^2+\dots+21^2 = (1^2+2^2+3^2+\dots+21^2) - (1^2+2^2+\dots+5^2)$$

$$= \frac{21\times(21+1)[2(21)+1]}{6} - \frac{5\times 6\times 11}{6}$$

$$= \frac{21\times 22\times 43}{6} - 55 = 3311 - 55 = 3256$$

$$6^2+7^2+8^2\dots+21^2 = 3256$$

(vi) $10^3+11^3+12^3+\dots+20^3$

$$1^3+2^3+3^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= (1^3+2^3+3^3+\dots+20^3) - (1^3+2^3+\dots+9^3)$$

$$= \left[\frac{20(20+1)}{2}\right]^2 - \left[\frac{9\times(9+1)}{2}\right]^2$$

$$= \left[\frac{20 \times 21}{2}\right]^2 - \left[\frac{9 \times 10}{2}\right]^2 = (210)^2 - (45)^2$$

$$= 44100 - 2025 = 42075$$

$$10^3 + 11^3 + 12^3 + ... + 20^3 = 42075$$

(vii) $1 + 3 + 5 + .. + 71$

$a = 1, d = 3 - 1 = 2$

Number of terms $n = \frac{l - a}{d} + 1$

$$= \frac{71 - 1}{2} + 1 = \frac{70}{2} + 1 = 36$$

Sum of first n odd numbers $= n^2$

$1 + 3 + 5 + ... + 71 = (36)^2 = 1296$

2. If $1 + 2 + 3 + ... + k = 325$, then find $1^3 + 2^3 + 3^3 + ... + k^3$.

Sol :

Sum of first k natural numbers $= \frac{k(k+1)}{2} = 325$

Sum of cube of first k natural numbers

$$= \left[\frac{k(k+1)}{2}\right]^2$$

$$= (325)^2 = 1,05,625$$

$$1^3 + 2^3 + 3^3 + ... + k^3 = 1,05,625$$

3. If $1^3 + 2^3 + 3^3 + ... + k^3 = 44100$, then find $1 + 2 + 3 + ... + k$.

Sol :

$$1^3 + 2^3 + 3^3 + ... + k^3 = \left[\frac{k(k+1)}{2}\right]^2 = 44100 = (210)^2$$

$$\therefore 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2} = 210$$

$$1 + 2 + 3 + ... + k = 210$$

4. How many terms of the series $1^3 + 2^3 + 3^3 + ...$ should be taken to get the sum 14400?

Sol :

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$1^3 + 2^3 + 3^3 + ... + n^3 = (120)^2$$

$$\therefore \frac{n(n+1)}{2} = 120$$

$$n(n+1) = 120 \times 2$$

$$n(n+1) = 240$$

$$n^2 + n - 240 = 0$$

$$(n - 15)(n + 16) = 0$$

$$n = 15 \text{ (or) } (-16)$$

Number of terms cannot be negative

$\therefore$ Number of terms to be taken $= 15$

5. The sum of the squares of the first n natural numbers is 285, while the sum of their cubes is 2025. Find the value of n.

Sol :

$$1^2 + 2^2 + 3^2 + ... + n^2 = 285$$

$$\frac{n(n+1)(2n+1)}{6} = 285 \quad ...(1)$$

$$1^3 + 2^3 + 3^3 + ... + n^3 = 2025$$

$$\left[\frac{n(n+1)}{2}\right]^2 = 2025 = (45)^2 \Rightarrow \frac{n(n+1)}{2} = 45$$

$$(1) \Rightarrow \left[\frac{n(n+1)}{2}\right]\left[\frac{(2n+1)}{3}\right] = 285$$

$$45 \times \left(\frac{2n+1}{3}\right) = 285$$

$$2n + 1 = 285 / 15$$

$$2n + 1 = 19$$

$$2n = 19 - 1 = 18$$

$$n = \frac{18}{2} = 9$$

6. Rekha has 15 square colour papers of sizes 10 cm, 11 cm, 12 cm, ..., 24 cm. How much area can be decorated with these colour papers?

Sol :

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

With the square colour papers are decorated

$$= 10^2 + 11^2 + 12^2 + ... + 24^2$$

$$10^2 + 11^2 + 12^2 + ... + 24^2 = (1^2 + 2^2 + 3^2 + ... + 24^2) - (1^2 + 2^2 + ... + 9^2)$$

$$= \frac{24 \times (24+1)[2(24)+1]}{6} - \frac{9 \times (9+1)[2(9)+1]}{6}$$

$$= (4 \times 25 \times 49) - \frac{9 \times 10 \times 19}{6}$$

$$= 4900 - 285 = 4615$$

4615 cm^2 area can be decorated.

7. Find the sum of the series $(2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + ...$ to (i) n terms (ii) 8 terms

Sol :

(i) $(2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + ...$ n terms

General term of the given series $= (2n)^3 - (2n-1)^3$

$= 8n^3 - [(2n)^3 - 3(2n)^2(1) + 3(2n)(1) - 1^3]$

$= 8n^3 - [8n^3 - 12n^2 + 6n - 1]$

$= 8n^3 - 8n^3 + 12n^2 - 6n + 1$

$= 12n^2 - 6n + 1$

$$= 12\left[\frac{n(n+1)(2n+1)}{6}\right] - 6\left[\frac{n(n+1)}{2}\right] + n$$

$= 2n(n+1)(2n+1) - 3n(n+1) + n$

$= 2n(2n^2 + n + 2n + 1) - 3n^2 - 3n + n$

$= 4n^3 + 6n^2 + 2n - 3n^2 - 3n + n$

$= 4n^3 + 3n^2$

Hence the sum of n terms $= 4n^3 + 3n^2$

(ii) $(2^3 - 1) + (4^3 - 3^3) + (6^3 - 5^3) + ...$ 8 terms

Sum of n term $= 4n^3 + 3n^2$

Here $n = 8$

Sum $= 4(8)^3 + 3(8)^2$

$= 2048 + 192 = 2240$

Sum $= 2240$

Exercise 2.10

Multiple choice question.

1. Euclid's division lemma states that for positive integers a and b, there exist unique integers q and r such that a = bq + r, where r must satisfy

(1) $1 < r < b$ (2) $0 < r < b$

(3) $0 \le r < b$ (4) $0 < r \le b$

[Ans : (3)]

2. Using Euclid's division lemma, if the cube of any positive integer is divided by 9 then the possible remainders are

(1) 0, 1, 8 (2) 1, 4, 8

(3) 0, 1, 3 (4) 1, 3, 5 [Ans : (1)]

3. If the H.C.F. of 65 and 117 is expressible in the form of 65 m – 117, then the value of m is

(1) 4 (2) 2

(3) 1 (4) 3 [Ans : (2)]

Sol :

H.C.F. of 65 and 117 is 13

$65m - 117 = 13$

$65m = 13 + 117 = 130$

$$m = \frac{130}{65} = 2$$

4. The sum of the exponents of the prime factors in the prime factorization of 1729 is

(1) 1 (2) 2

(3) 3 (4) 4 [Ans : (3)]

Sol : $1729 = 7^1 + 13^1 + 19^1$

Sum of powers $= 1 + 1 + 1 = 3$

5. The least number that is divisible by all the numbers from 1 to 10 (both inclusive) is

(1) 2025 (2) 5220

(3) 5025 (4) 2520 [Ans : (4)]

Sol : L.C.M. of 2, 3, 4, 5, 6, 7, 8, 9 is 2520

6. $7^{4k} \equiv$ _____ (mod 100)

(1) 1 (2) 2

(3) 3 (4) 4 [Ans : (1)]

7. Given $F_1 = 1$, $F_2 = 3$ and $F_n = F_{n-1} + F_{n-2}$ then F_5 is

(1) 3 (2) 5

(3) 8 (4) 11 [Ans : (4)]

Sol :

$F_1 = 1, F_2 = 3, F_3 = 3 + 1 = 4, F_4 = 4 + 3 = 7,$
$F_5 = 7 + 4 = 11$

8. The first term of an arithmetic progression is unity and the common difference is 4. Which of the following will be a term of this A.P

(1) 4551 (2) 10091

(3) 7881 (4) 13531 [Ans : (3)]

Sol :

$a = 1, d = 4$

$t_n = a + (n-1)d = 1 + (n-1)4;$

The number should be multiple of 4 + 1 ;

$\therefore$ 7881.

9. If 6 times of 6th term of an A.P. is equal to 7 times the 7th term, then the 13th term of the A.P. is

(1) 0 (2) 6

(3) 7 (4) 13 [Ans : (1)]

Sol :

$$\begin{aligned} 6t_6 &= 7t_7 \\ 6[a+5d] &= 7[a+6d] \\ 6a + 30d &= 7a + 42d \\ 7a - 6a + 42d - 30d &= 0 \\ a + 12d &= 0 \\ a + (13-1)d &= 0 \\ t_{13} &= 0 \end{aligned}$$

10. An A.P. consists of 31 terms. If its 16^{th} term is m, then the sum of all the terms of this A.P. is

(1) 16 m (2) 62 m

(3) 31 m (4) $\frac{31}{2}m$ **[Ans : (3)]**

Sol : $n = 31, t_{16} = m$

$$a + 15d = m$$

$$s_{31} = \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{31}{2}[a+a+30d] = \frac{31}{2}[a+15d+a+15d]$$

$$= \frac{31}{2}[2m] = 31\text{ m}$$

11. In an A.P., the first term is 1 and the common difference is 4. How many terms of the A.P. must be taken for their sum to be equal to 120?

(1) 6 (2) 7

(3) 8 (4) 9 **[Ans : (3)]**

Sol : $a = 1, d = 4$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[2(1)+(n-1)4]$$

$$120 \times 2 = n[2+4(n-1)]$$

$$240 = n[2+4n-4]$$

$$= n[4n-2]$$

$$240 = 2n[2n-1]$$

$$120 = n[2n-1]$$

$$2n^2 - n - 120 = 0$$

$$2n^2 - 16n + 15n - 120 = 0$$

$$2n(n-8) + 15(n-8) = 0$$

$$(n-8)(2n+15) = 0$$

$$n = 8 \text{ or } n = \frac{-15}{2}$$

$$n = \frac{-15}{2} \text{ is not possible}$$

$$\therefore n = 8$$

12. If $A = 2^{65}$ and $B = 2^{64} + 2^{63} + 2^{62} + \ldots + 2^0$ which of the following is true?

(1) B is 2^{64} more than A

(2) A and B are equal

(3) B is larger than A by 1

(4) A is larger than B by 1 **[Ans : (4)]**

Sol :

$$A = 2^{65}$$

$$B = 2^{64} + 2^{63} + 2^{62} + \ldots + 2^0$$

$$\text{Where } a = 2^{64};\ r = \frac{2^{63}}{2^{64}} = \frac{1}{2};\ n = 65$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$B = \frac{2^{64}\left(\left(\frac{1}{2}\right)^{65}-1\right)}{-\frac{1}{2}} = 2^{65}\left(\frac{1}{2^{65}}-1\right)(-1)$$

$$= (1-2^{65})(-1)$$

$$B = 2^{65} - 1$$

$$\text{But } A = 2^{65}$$

$$\therefore B = A - 1$$

$$A = B + 1$$

13. The next term of the sequence $\frac{3}{16}, \frac{1}{8}, \frac{1}{12}, \frac{1}{18}, \ldots$ is

(1) $\frac{1}{24}$ (2) $\frac{1}{27}$

(3) $\frac{2}{3}$ (4) $\frac{1}{81}$ **[Ans : (2)]**

Sol :

$$a = \frac{3}{16};\ r = \frac{\frac{1}{8}}{\frac{3}{16}} = \frac{1}{8} \times \frac{16}{3} = \frac{2}{3}.$$

It is a G.P.

$$\therefore \text{Next term } \frac{1}{18} \times \frac{2}{3} = \frac{1}{27}$$

14. If the sequence $t_1, t_2, t_3, \ldots$ are in A.P. Then the sequence $t_6, t_{12}, t_{18} \ldots$ is

(1) a Geometric progression

(2) an Arithmetic progression

(3) neither an Arithmetic progression nor a Geometric progression

(4) a constant sequence. **[Ans : (2)]**

Sol :

Given $t_1, t_2, t_3, \ldots$ are in A.P.

then $t_6, t_{12}, t_{18}, \ldots$ is an A.P. with common difference 6d.

15. The value of $(1^3 + 2^3 + 3^3 + \dots + 15^3) - (1 + 2 + 3 + \dots + 15)$

(1) 14400 (2) 14200
(3) 14280 (4) 14520 **Ans : (3)**

Sol :

$(1^3 + 2^3 + 3^3 + \dots + 15^3) - (1 + 2 + 3 + \dots + 15)$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \left[\frac{n(n+1)}{2}\right]$$

$$= \left[\frac{15\times 16}{2}\right]^2 - \left[\frac{15\times 16}{2}\right] = (120)^2 - 120$$

$= 14400 - 120 = 14, 280$

UNIT EXERCISE - 2

1. Prove that $n^2 - n$ divisible by 2 for every positive integer n.

Sol :

We know that every positive integers is of the form 2q or 2q + 1 for some integer q.

Case 1:

So Let $n = 2q$

$$\begin{aligned} n^2 - n &= (2q)^2 - (2q) \\ &= 4q^2 - 2q \\ &= 2q(2q-1) \\ n^2 - n &= 2r \quad \text{where} \quad r = q(2q-r) \end{aligned}$$

$\therefore$ $n^2 - n$ is even and divisible by 2

Case 2:

Let $n = 2q + 1$

$$\begin{aligned} n^2 - n &= (2q+1)^2 - (2q+1) \\ &= 4q^2 + 4q + 1 - 2q - 1 \\ &= 4q^2 + 2q \\ &= 2q(2q+1) \\ &= 2r \quad \text{where} \quad r = q(2q+1) \end{aligned}$$

$\therefore n^2 - n$ is even and divisible by 2 . Hence, it is proved that $n^2 - n$ is divisible by 2 for every positive integer n.

2. A milk man has 175 litres of cow's milk and 105 litres of buffalo's milk. He wishes to sell the milk by filling the two types of milk in cans of equal capacity. Calculate the following

(i) Capacity of a can

(ii) Number of cans of cow's milk

(iii) Number of cans of buffalo's milk

Sol :

(i) Cow's milk= 175 litres

Buffalo's = 105 litres

The types of cans are of equal capacity.

$\therefore$ Capacity of a can = H.C.F. of 105, 175

By Euclid's division Algorithm

$$\begin{aligned} 175 &= 105 \times 1 + 70 \\ 105 &= 70 \times 1 + 35 \\ 70 &= 35 \times 2 + 0 \end{aligned}$$

Remainder = 0

$\therefore$ H.C.F. (105, 175) = 35

Capacity of a can = 35 litres.

(ii) Number of cans of Cow's milk

$$= \frac{\textit{Cow's Milk}}{\textit{Capacity of a can}} = \frac{175}{35} = 5$$

5 cans of cow's milk.

(iii) Number of cans of Buffalo's milk

$$= \frac{\text{Buffalo's milk}}{\text{Capacity of a can}} = \frac{105}{35} = 3$$

3 cans buffalo's milk is there.

3. When the positive integers a, b and c are divided by 13 the respective remainders are 9, 7 and 10. Find the remainder when a + 2b + 3c is divided by 13.

Sol :

When a, b, c are divided by 13 leaves the remainder 9, 7, 10 respectively.

$\therefore$ $a = 13q_1 + 9,\ b = 13q_2 + 7,\ c = 13q_3 + 10$

Now $a + 2b + 3c = (13q_1 + 9) + 2(13q_2 + 7) + 3(13q_3 + 10)$

$= 13q_1 + 9 + 26q_2 + 14 + 39q_3 + 30$

$= 13(q_1 + 2q_2 + 3q_3) + 53$

$= 13(q_1 + 2q_2 + 3q_3) + (4 \times 13 + 1)$

$=13(q_1 + 2q_2 + 3q_3 + 4) + 1$

$\therefore$ $a + 2b + 3c$ is divided by 13, the remainder is 1.

4. Show that 107 is of the form 4q + 3 for any integer q.

Sol :

Given the number 107,

It is a positive odd integer.

Let $a = 107$ and $b = 4$

Applying division algorithm we have,

$107 = 4q + r$ where $0 \le r < 4$

$\therefore$ The possible $r = 0, 1, 2, 3$.

But 107 is odd, the remainders cannot be 0 or 2.

i.e, $4q$ or $4q + 2$ is not possible to express 107.

The other possibilities are $4q + 1$ or $4q + 3$

Suppose $4q + 1 = 107$

$$4q = 107 - 1 = 106$$

$$q = \frac{106}{4} \text{ not a natural number.}$$

$\therefore$ Only possibility is $107 = 4q + 3$

$$\therefore 107 = 4q + 3$$
$$4q = 107 - 3 = 104$$
$$q = \frac{104}{4} = 26$$
$$q = 26$$

5. If $(m + 1)^{th}$ term of an A.P. is twice the $(n + 1)^{th}$ term, then prove that $(3m + 1)^{th}$ term is twice the $(m + n + 1)^{th}$ term.

Sol :

Let a-first term and d-common difference.

Given $(m + 1)^{th}$ term $= (n + 1)^{th}$ term $\times 2$

We know that $t_n = a + (n - 1)d$

$$t_{m+1} = a + (m + 1 - 1)d = a + md$$
$$t_{n+1} = a + (n + 1 - 1)d = a + nd$$
$$t_{m+1} = 2t_{n+1}$$
$$a + md = 2(a + nd)$$
$$a + md = 2a + 2nd$$
$$md - 2nd = 2a - a$$
$$(m - 2n)d = a$$

$(m + n + 1)^{th}$ term $= t_{m+n+1} = a + (m + n + 1 - 1)d$

$$= a + (m + n)d$$
$$= (m - 2n)d + (m + n)d$$
$$= md - 2nd + md + nd$$
$$= 2md - nd$$

twice $(m+ n + 1)^{th}$ term $= 2(2md - nd)$

$$= 2(2m - n)d \quad \ldots (1)$$

Taking $(3m + 1)^{th}$ term

$$= t_{3m+1} = a + (3m + 1 - 1)d$$
$$= a + 3md$$
$$= (m - 2n)d + 3md$$
$$= md - 2nd + 3md$$
$$= 4md - 2nd$$
$$= 2(2m - n)d \quad \ldots (2)$$

From (1) and (2) we proved that $(3m + 1)^{th}$ term = twice the $(m + n + 1)^{th}$ term.

6. Find the 12^{th} term from the last term of the A.P. – 2, – 4, – 6 … – 100.

Sol :

We have to find out the 12^{th} term from the end.

So we can assume the last term is the first term.

i.e., $-100, \ldots, -6, -4, -2$ be the A.P

first term $a = -100$

$$d = -4 - (-6) = -4 + 6 = 2$$
$$t_n = a + (n - 1)d$$
$$t_{12} = -100 + (12 - 1)(2)$$
$$= -100 + 22 = -78$$

12^{th} term from the last is -78.

7. Two A.P's have the same common difference. The first term of one A.P. is 2 and that of the other is 7. Show that the difference between their 10^{th} terms is the same as the difference between their 21^{st} terms, which is the same as the difference between any two corresponding terms.

Sol :

Given two A.P.'s have same common difference.

Let it be d.

For first A.P. $a = 2$

10^{th} term $t_{10} = a + 9d$

10^{th} term $t_{10} = 2 + 9d$

For second A.P. $a = 7$

10^{th} term $t_{10} = a + 9d$

$$= 7 + 9d$$

Their Difference $= (7 + 9d) - (2 + 9d)$

$$= 7 + 9d - 2 - 9d$$
$$= 5 \quad \ldots (1)$$

Also 21^{th} term of 1^{st} AP $= a + 20d$

$$= 2 + 20d$$

21^{st} term of 2^{nd} AP $= 7 + 20d$

Difference $= 7 + 20\,d - (2 + 20\,d)$
$= 7 + 20\,d - 2 - 20\,d = 5$...(2)

From (1) and (2) the difference is same.
Also Difference between n^{th} term of two A.P.'s
$= 7 + (n-1)d - [2 + (n-1)\,d]$
$= 7 + nd - d - 2 - nd + d$
$= 7 - 2 = 5$

$\therefore$ Difference between any two corresponding terms is 5 always.

8. A man saved ₹ 16500 in ten years. In each year after the first he saved ₹ 100 more than he did in the preceding year. How much did he save in the first year?

Sol :

Let the amount he saved in the first year be x

Then $x + (x + 100) + (x + 200) + \ldots$ 10 terms $= 16500$

$x + x + 100 + x + 200 + \ldots$ 10 terms $= 16500$

$(x + x + \ldots$ 10 terms$) + (100 + 200 + \ldots$ 9 terms$) = 16500$

$$10\,x + \frac{9}{2}\left[2\,(100) + 8\,(100)\right] = 16500$$

$$10\,x = 16500 - 4500$$

$$10x = 12000$$

$$x = \frac{12000}{10} = 1200$$

His 1^{st} year saving = ₹ 1200

9. Find the G.P. in which the 2^{nd} term is $\sqrt{6}$ and the 6^{th} term is $9\sqrt{6}$.

Sol :

n^{th} term of a G.P $= ar^{n-1}$

Given 2^{nd} term $= \sqrt{6}$

$$ar^{2-1} = \sqrt{6}$$

$$ar = \sqrt{6}$$

6^{th} term $= 9\sqrt{6}$

$$ar^{6-1} = 9\sqrt{6}$$

$$ar^5 = 9\sqrt{6}$$

$$\therefore \frac{ar^5}{ar} = \frac{9\sqrt{6}}{\sqrt{6}}$$

$$r^4 = 9$$

$$r^2 = 3$$

$$r = \sqrt{3}$$

Also $ar = \sqrt{6}$

$$a = \frac{\sqrt{6}}{r} = \frac{\sqrt{2}\sqrt{3}}{\sqrt{3}}$$

$$a = \sqrt{2}$$

$\therefore$ The G.P. is $a, ar, ar^2, \ldots$

$$= \sqrt{2}, \sqrt{6}, \sqrt{2}\left(\sqrt{3}\right)^2 \ldots\ldots$$

$$= \sqrt{2}, \sqrt{6}, 3\sqrt{2} \ldots\ldots$$

10. The value of a motor cycle depreciates at the rate of 15% per year. What will be the value of the motor cycle 3 year hence, which is now purchased for ₹ 45,000?

Sol :

Value of the motor cycle
= ₹ 45000

The value decreases at the rate of 15%

$$= \frac{15}{100}$$

$\therefore$ Value of the machine after a year

$$= 45000 \times \frac{85}{100}$$

Continuing this way the value of the machine

$45000,\ 45000 \times \frac{85}{100},\ 45000 \times \left(\frac{85}{100}\right)^2, \ldots$ form a G.P.

$$a = 45000,\ r = \frac{85}{100}$$

$\therefore$ We need to find the value of the machine after 3 years i.e., $n = 4$.

$$t_n = ar^{n-1}$$

$$t_4 = 45000 \times \left(\frac{85}{100}\right)^{4-1}$$

$$= 45000 \times \frac{85}{100} \times \frac{85}{100} \times \frac{85}{100}$$

$$= 27635.625$$

$\therefore$ Value of the machine after 3 years = ₹ 27636

CREATIVE QUESTIONS

I. Multiple Choice Questions

Euclid's Division Lemma and Algorithm

1. Euclid's division lemma can be used to find the _____ of any two positive integers.

(1) HCF (2) Multiples
(3) Both (4) None of these

[Ans: (1)]

2. Euclid's division lemma is not applicable for which values of b?

(1) Positive integer
(2) Zero
(3) Negative integer
(4) All of these **[Ans: (2)]**

3. Using Euclid's division lemma HCF of 455 and 42 can be expressed as _____

(1) $455 = 42 \times 9 + 77$
(2) $455 = 42 \times 10 + 35$
(3) $455 = 42 \times 11 - 7$
(4) $455 = 42 \times 12 - 49$ **[Ans: (2)]**

Fundamental Theorem of Arithmetic

4. The number 132 is to be written as product of its prime factors. Which of the following is correct?

(1) $132 = 2 \times 6 \times 11$
(2) $132 = 2^2 \times 3 \times 11$
(3) $132 = 2^2 \times 3^2 \times 5$
(4) $132 = 3 \times 4 \times 11$ **[Ans: (2)]**

Sol :

$$\begin{array}{r|l} 2 & 132 \\ \hline 2 & 66 \\ \hline 3 & 33 \\ \hline 11 & 11 \\ \hline & 1 \end{array}$$

$$\begin{aligned} 132 &= 2 \times 2 \times 3 \times 11 \\ &= 2^2 \times 3 \times 11 \end{aligned}$$

5. What is the sum of the prime factors of 240?

(1) 16 (2) 14
(3) 12 (4) 10 **[Ans: (1)]**

Sol :

$$\begin{array}{r|l} 2 & 240 \\ \hline 2 & 120 \\ \hline 2 & 60 \\ \hline 2 & 30 \\ \hline 3 & 15 \\ \hline 5 & 5 \\ \hline & 1 \end{array}$$

Prime factors of 240 $= 2 \times 2 \times 2 \times 2 \times 3 \times 5$

Their sum $= 2 + 2 + 2 + 2 + 3 + 5 = 16$

6. Solve the following $25 + 37 \equiv$ _____ (mod 12)

(1) 2 (2) 3
(3) 1 (4) 62 **[Ans: (1)]**

Sol :

$$\begin{aligned} 25 + 37 &\equiv x \pmod{12} \\ 62 &\equiv x \pmod{12} \\ 62 - x &= 12k \text{ for some } k \\ \frac{62 - x}{12} &= k \text{ for some } k \\ \frac{62 - 2}{12} &= 5 \\ \therefore\ x &= 2. \end{aligned}$$

7. What does 144 reduces to mod 11?

(1) 144 mod 11 (2) 1 mod 11
(3) 2 mod 11 (4) 143 mod 11 **[Ans: (2)]**

Sol : $144 \equiv x \pmod{11}$

$144 - x$ is a multiple of 11

$143 = (144 - 1)$ is a multiple of 11

$\therefore\ x = 1$

$144 \equiv 1 \pmod{11}$.

Sequences

8. First term and common difference in the sequence 7, 10, 13, ...

(1) 1, 7 (2) 7, 10
(3) 7, 3 (4) 13, 10 **[Ans: (3)]**

Sol :

The sequence is 7, 10, 13, ...

$t_2 - t_1 = t_3 - t_2 \Rightarrow 10 - 7 = 13 - 10 = 3$

$\therefore$ Common difference $d = 3$

First term is 7.

Arithmetic Progression

9. If the first term of an A.P. is a and n^{th} term is b, then the common difference is ______.

(1) $\frac{b-a}{n+1}$ (2) $\frac{b-a}{n-1}$

(3) $\frac{b-a}{n}$ (4) $\frac{b+a}{n-1}$ **[Ans: (2)]**

Sol :

First term = a

Let common difference be d

$$n^{th} \text{ term} = b$$
$$a + (n-1)d = b$$
$$(n-1)d = b - a$$
$$d = \frac{b-a}{n-1}$$

10. The common differences of the A.P. $\frac{1}{3}, \frac{1-3b}{3}, \frac{1-6b}{3}, \ldots$ is

(1) $\frac{1}{3}$ (2) $\frac{-1}{3}$

(3) $-b$ (4) b **[Ans: (3)]**

Sol :

Common difference = $t_2 - t_1$

$$= \frac{1-3b}{3} - \frac{1}{3} = \frac{1-3b-1}{3} = -\frac{3b}{3} = -b$$

Series

11. The sum of n terms of an A.P. is $3n^2 + 5n$, then which of its term is 164?

(1) 26^{th} (2) 27^{th}

(3) 28^{th} (4) None of these

[Ans: (2)]

Sol :

$$S_n = 3n^2 + 5n$$
$$S_1 = 3(1)^2 + 5 = 8 = a = t_1$$
$$S_2 = 3(2)^2 + 5(2) = 3(4) + 10$$
$$= 12 + 10 = 22$$
$$t_2 = S_2 - S_1 = 22 - 8 = 14$$
$$d = t_2 - t_1 = 14 - 8 = 6$$
$$a + (n-1)d = 164$$
$$8 + (n-1)6 = 164$$
$$n - 1 = \frac{164-8}{6} = \frac{156}{6} = 26$$
$$n = 26 + 1 = 27$$

12. The first, second and last term of an A.P. are a, b and 2a respectively, its sum is

(1) $\frac{ab}{2(b-a)}$ (2) $\frac{ab}{b-a}$

(3) $\frac{3ab}{2(b-a)}$ (4) None of these

[Ans: (3)]

Sol :

$$n = \frac{l-a}{d} + 1$$
$$= \frac{2a-a}{b-a} + 1 = \frac{a}{b-a} + 1$$
$$= \frac{a+b-a}{b-a} = \frac{b}{b-a}$$
$$\text{Sum} = \frac{n}{2}(l+a)$$
$$= \frac{b}{2(b-a)}(2a+a)$$
$$\text{Sum} = \frac{3ab}{2(b-a)}$$

Geometric Sequences

13. 7^{th} term of a G.P. 2, 6, 18, . . . is

(1) 5832 (2) 2919

(3) 1458 (4) 729 **[Ans: (3)]**

Sol : $a = 2,\ r = \frac{t_2}{t_1} = \frac{6}{2} = 3$

$$t_n = ar^{n-1}$$
$$t_7 = 2(3)^{7-1} = 2(3)^6$$
$$= 2 \times 729 = 1458.$$

14. No term of a geometric sequence be

(1) 3 (2) 1

(3) 2 (4) 0 **[Ans: (4)]**

Sum of G.P.

15. Sum of n terms of a G.P. is

(1) $\frac{n}{2}[2a+(n-1)d]$ (2) $\frac{a(1-r^n)}{1-r}$

(3) $\frac{2ab}{(a+b)}$ (4) $\frac{a+b}{2}$ **[Ans: (2)]**

16. Sum of 7 terms of – 2, 6, – 18, . . . is

(1) 1094 (2) – 1094

(3) 9041 (4) – 9041 **[Ans: (2)]**

Sol : $a = -2,\ r = \frac{6}{-2} = -3$

$$\text{Sum} = \frac{a(r^n - 1)}{r-1}$$

$$S_7 = \frac{(-2)[(-3)^7 - 1]}{-3-1}$$

$$= \frac{(-2)(-2187-1)}{-4} = \frac{(-2)(-2188)}{-4}$$

$$= -1094.$$

Special Series

17. $\dfrac{5+9+13+\ldots \text{ to } n \text{ terms}}{7+9+11+\ldots \text{ to } (n+1) \text{ terms}} = \dfrac{17}{16}$ **then n = ?**

(1) 8 (2) 7
(3) 10 (4) 11 **[Ans: (2)]**

Sol :

Sum of 5 + 9 + 13 + ... to n terms

$$= \frac{n}{2}\{2a + (n-1)d\}$$

d = 9 – 5 = 4; a = 5

$$\text{Sum } = \frac{n}{2}[2 \times 5 + (n-1)4]$$

$$= \frac{n}{2}[10 + 4n - 4]$$

$$= \frac{n}{2}[4n + 6] = n(2n+3)$$

Sum of 7 + 9 + 11 + ... to (n + 1) terms.

$$= \frac{n+1}{2}[2 \times 7 + (n+1-1)2]$$

$$= \frac{n+1}{2}[14 + 2n]$$

$$= \frac{(n+1)}{2} 2(7+n)$$

$$= (n+1)(7+n)$$

$$\frac{5+9+13+\ldots \text{ to } n \text{ terms}}{7+9+11+\ldots \text{ to } (n+1) \text{ terms}} = \frac{17}{16}$$

$$\frac{n(3+2n)}{(n+1)(7+n)} = \frac{17}{16}$$

$16n(3+2n) = 17(n+1)(7+n)$
$48n + 32n^2 = 17n^2 + 136n + 119$
$48n + 32n^2 - 17n^2 - 136n - 119 = 0$
$15n^2 - 88n - 119 = 0$
$15n^2 - 105n + 17n - 119 = 0$
$15n(n-7) + 17(n-7) = 0$
$(n-7)(15n+17) = 0$

$n = 7$ or $n = -\dfrac{17}{15}$

$n = -\dfrac{17}{15}$ is not possible.

$\therefore\ n = 7.$

18. The sum of first n odd natural number is

(1) $2n - 1$ (2) $2n + 1$
(3) n^2 (4) $n^2 - 1$ **[Ans: (3)]**

19. If 1 + 2 + 3 + . . . + 10 = 55, then, $1^3 + 2^3 + 3^3 + \ldots + 10^3 = ?$

(1) 55^2 (2) 10^2
(3) 55^3 (4) 10^3 **[Ans: (1)]**

20. $1^2 + 2^2 + 3^2 \ldots + n^2 = ?$

(1) $\left[\dfrac{n(n+1)}{2}\right]^2$ (2) $\dfrac{n(n+1)}{2}$

(3) n^2 (4) $\dfrac{n(n+1)(2n+1)}{6}$

[Ans: (4)]

II. Very Short Answer Questions

1. State the fundamental theorem of Arithmetic.

Sol :

Every natural number except 1 can be factorized as a product of primes and this factorization is unique except for the order in which the prime factors are written.

2. If 6^n is a number such that n is a natural number. Check whether is any value of $n \in N$ for which 6^n is divisible by 7.

Sol :

We have $6^n = (2 \times 3)^n$
$= 2^n \times 3^n$

Prime factorization of 6^n does not contain the prime number 7. Therefore 6^n is not divisible by 7.

3. Find the H.C.F. and L.C.M. of 100 and 190 by fundamental theorem of arithmetic.

Sol :

By fundamental theorem we have every composite number can be expressed as a product of primes.

2	100
2	50
5	25
	5

2	190
5	95
	19

$\therefore$ Factorizing 100 and 190

$$100 = 2^2 \times 5^2$$
$$190 = 2^1 \times 5^1 \times 19^1$$
$$\therefore \text{H.C.F. of 100 and 190} = 2^1 \times 5^1 = 10$$
$$\text{H.C.F.} \times \text{L.C.M.} = \text{Product of two numbers}$$
$$10 \times \text{L.C.M.} = 100 \times 190$$
$$\text{L.C.M.} = \frac{100 \times 190}{10} = 1900$$

4. Write the H.C.F. of smallest composite number and the smallest prime number.

Sol :

Smallest composite number $= 4 = 2^2$

Smallest prime number $= 2 = 2^1$

$\therefore$ H.C.F. of 4 and 2 $= 2$

5. The traffic lights at three different road crossings change after every 48 sec, 72 sec and 108 sec respectively. If they all change simultaneously at 8.20 am, then at what time will they again change simultaneously?

Sol :

Interval of change = L.C.M. of (48, 72, 108) sec

$$\begin{array}{r|l} 12 & 48, 72, 108 \\ \hline 3 & 4, 6, 9 \\ \hline 2 & 4, 2, 3 \\ \hline & 2, 1, 3 \end{array}$$

$$\text{LCM} = 12 \times 3 \times 2 \times 2 \times 3 = 432$$

So the lights will again change simultaneously after every 432 sec.

$= 432/60 = 7$ min 12 sec.

Hence next change will be at 8 : 27 : 12 am.

6. Does 7 divides $(2^{29} + 3)$?

Sol :

We have

$$2^3 \equiv 1 \pmod 7$$
$$(2^3)^8 \equiv 1^8 \pmod 7$$
$$2^3 2^{24} \equiv 1 \times 1^8 \pmod 7$$
$$2^{27} \equiv 1 \pmod 7$$
$$2^2\, 2^{27} \equiv 4 \times 1\ (1 \bmod 7)$$
$$2^{29} \equiv 4 \pmod 7$$
$$2^{29} + 3 = (4+3) \pmod 7$$
$$2^{29} + 3 \equiv 0 \pmod 7$$

$2^{29} + 3$ is divisible by 7.

7. What is the remainder when $3^{202} + 5^9$ is divided by 8?

Sol :

$$3^2 = 1 \pmod 8$$
$$(3^2)^{101} \equiv 1^{101} \pmod 8$$
$$3^{202} \equiv 1 \pmod 8$$
$$5^2 \equiv 1 \pmod 8$$
$$5 \equiv 5 \pmod 8$$
$$(5^2)^4 \equiv 1^4 \pmod 8$$
$$5^8.5^1 \equiv 5 \pmod 8$$
$$3^{202} + 5^9 = 6 \pmod 8$$

Remainder is 6 when divided by 8.

8. Write the first three terms of the sequence defined by $a_n = (-1)^{n-1}, 2^n$.

Sol :

Given $a_n = (-1)^{n-1}\, 2^n$

$$a_1 = (-1)^{1-1}\, 2^1 = (-1)^0\, 2^1 = 1 \times 2 = 2$$
$$a_2 = (-1)^{2-1} \times 2^2 = (-1)^1 \times 4 = -4$$
$$a_3 = (-1)^{3-1} \times 2^3 = (-1)^2 \times 8 = 8$$

$\therefore$ First three terms are 2, – 4, 8

9. What is the 15th term of the sequence defined by $a_n = \dfrac{n(n-3)}{n+4}$?

Sol :

Given $a_n = \dfrac{n(n-3)}{n+4}$

$$a_{15} = \frac{15(15-3)}{15+4} = \frac{15 \times 12}{19}$$
$$a_{15} = \frac{180}{19}$$

10. Find the n^{th} term of the sequence 5, 8, 11,...

Sol :

Given the sequence 5, 8, 11,... Every term is 3 more than the previous term but the first term is 5.

$\therefore$ The general term may be $a_n = 3n + 2$

11. Write the general term of the sequence $\frac{-1}{2}, 0, \frac{3}{2}, \frac{8}{2}, \ldots$

Sol :

In each term the numerator increases by n (n – 2) and denominator is 2.

$\therefore$ The general term is $a_n = \dfrac{n(n-2)}{2}$

12. Find the first three terms of $a_n = \dfrac{2n-3}{6}$

Sol :

Given $a_n = \dfrac{2n-3}{6}$

$$\therefore a_1 = \frac{2(1)-3}{6} = \frac{2-3}{6} = \frac{-1}{6}$$

$$a_2 = \frac{2(2)-3}{6} = \frac{4-3}{6} = \frac{1}{6}$$

$$a_3 = \frac{2(3)-3}{6} = \frac{6-3}{6} = \frac{3}{6}$$

$\therefore$ First three terms are $\dfrac{-1}{6}, \dfrac{1}{6}, \dfrac{3}{6}$.

13. If 2x, x + 10, 3x + 2 are in A.P. Find x.

Sol :

Given 2x, x + 10, 3x + 2 are in A.P

$$x + 10 - 2x = 3x + 2 - [x + 10]$$
$$-x + 10 = 3x + 2 - x - 10 = 2x - 8$$
$$10 + 8 = 2x + x$$
$$3x = 18$$
$$x = \frac{18}{3} = 6$$

14. Find four terms of an A.P. whose sum is 20 and the sum of whose squares is 120.

Sol :

Let the four terms be (a – 3d),(a – d) (a + d), (a + 3d)

$$\text{Sum} = 20$$
$$(a-3d) + (a-d) + (a+d) + (a+3d) = 20$$
$$a - 3d + a - d + a + d + a + 3d = 20$$
$$4a = 20$$
$$a = \frac{20}{4} = 5$$
$$\text{Given sum of squares} = 120$$
$$(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 120$$
$$a^2 + 9d^2 - 6ad + a^2 + d^2 - 2ad + a^2 + d^2 + 2ad + a^2 + 9d^2 + 6ad = 120$$
$$4a^2 + 20d^2 = 120$$
$$a^2 + 5d^2 = 30$$
$$25 + 5d^2 = 30$$
$$5d^2 = 5$$
$$d = \pm 1$$

If d = 1 then the numbers are 2, 4, 6, 8.

If d = – 1 then the numbers are 8, 6, 4, 2.

$\therefore$ The four numbers are 2, 4, 6, 8.

15. Which term of the A.P. –1, 3, 7, 11, ... is 95?

Sol :

$a = -1, \quad d = t_2 - t_1 = 3 - (-1) = 4$

Let 95 be the n^{th} term of the A.P.

Then
$$t_n = 95$$
$$a + (n-1)d = 95$$
$$-1 + (n-1)4 = 95$$
$$(n-1)4 = 95 + 1 = 96$$
$$n - 1 = \frac{96}{4} = 24$$
$$n = 24 + 1 = 25$$

$\therefore$ 95 is the 25^{th} term of the A.P.

16. How many terms are there in the sequence 3, 6, 9, 12, ..., 111?

Sol :

Here $t_2 - t_1 = 6 - 3 = 3$

$t_3 - t_2 = 9 - 6 = 3$ and so on.

$\therefore$ It is an A.P.

$a = 3, d = 3.$

Let the n^{th} term of the A.P. = 111

$$t_n = 111$$
$$a + (n-1)d = 111$$
$$3 + (n-1)3 = 111$$
$$(n-1)3 = 111 - 3 = 108$$
$$(n-1) = \frac{108}{3}$$
$$n - 1 = 36$$
$$n = 36 + 1 = 37$$

$\therefore$ The A.P. has 37 terms.

17. Is 184 a term of the sequence 3, 7, 11, ...?

Sol :

Clearly 7 – 3 = 4, 11 – 7 = 4 and it is an A.P.

Suppose 184 is the n^{th} term of the A.P.

then
$$t_n = 184$$
$$a + (n-1)d = 184$$
$$3 + (n-1)4 = 184$$
$$(n-1)4 = 184 - 3$$
$$(n-1)4 = 181$$
$$n - 1 = \frac{181}{4}$$
$$n - 1 = 45.25$$
$$n = 45.25 + 1 = 46.25$$

Since n is not a natural number. 184 is not a term of the given A.P.

18. Which term of the sequence $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ is the first negative term?

Sol :

Here $t_2 - t_1 = 19\frac{1}{4} - 20 = -\frac{3}{4}$

$$t_3 - t_2 = 18\frac{1}{2} - 19\frac{1}{4} = -\frac{3}{4}$$

$$t_2 - t_1 = t_3 - t_2$$

$\therefore$ The sequence is an A.P. with $a = 20; d = -\frac{3}{4}$

Let the n^{th} term of A.P. be the first negative term.

i.e., $t_n < 0$

$$a + (n-1)d < 0$$

$$20 + (n-1) \times \left(\frac{-3}{4}\right) < 0.$$

$$(n-1)\left(-\frac{3}{4}\right) < -20$$

$$(n-1)\left(\frac{3}{4}\right) > 20$$

$$n - 1 > \frac{20}{\left(\frac{3}{4}\right)}$$

$$n - 1 > 20 \times \frac{4}{3}$$

$$n > \frac{80}{3} + 1$$

$$n > \frac{80+3}{3} = \frac{83}{3}$$

$$n > 27\frac{2}{3}$$

$$n \geq 28 \quad [\because n \text{ is a natural number}]$$

$\therefore$ If $n \geq 28$, the terms t_n becomes negative.

$\therefore$ 28^{th} term is the first negative term.

19. How many numbers of two digits are divisible by 7?

Sol :

We know that first two digit number divisible by 7 is 14.

Last two digit number divisible by 7 is 98.

It is enough to find the number of terms of the A.P. 14, 21, 28, ..., 98.

$$a = 14, d = 7$$

$$n^{th} \text{ term} = 98$$

$$a + (n-1)d = 98$$

$$14 + (n-1)(7) = 98$$

$$(n-1)(7) = 98 - 14 = 84$$

$$n - 1 = \frac{84}{7} = 12$$

$$n = 12 + 1 = 13$$

$\therefore$ There are 13 numbers of two digits which are divisible by 7.

20. Find the sum of first n natural numbers.

Sol :

First n natural numbers are given by 1, 2, 3, 4,.... n

The first term $a = 1; d = 2 - 1 = 1, l = n.$

$$\text{Sum } S_n = \frac{n}{2}(a + l)$$

$$= \frac{n}{2}(1 + n) = \frac{n(n+1)}{2}$$

$\therefore$ Sum of first n natural numbers is $\frac{n(n+1)}{2}$

21. How many terms of the A.P. 9, 17, 25,... must be taken to give a sum of 636?

Sol :

Here $a = 9, d = 17 - 9 = 8$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$636 = \frac{n}{2}\{2(9) + (n-1)(8)\}$$

$$= \frac{n}{2} \times 2\{9 + (n-1)4\}$$

$$= n\{9 + 4n - 4\} = n\{4n + 5\}$$

$$636 = 4n^2 + 5n$$

$$4n^2 + 5n - 636 = 0$$

$$n = \frac{-5 \pm \sqrt{25 - 4(4)(-636)}}{2(4)}$$

$$= \frac{-5 \pm \sqrt{25 + 10176}}{8}$$

$$= \frac{-5 \pm \sqrt{10201}}{8}$$

$$= \frac{-5 \pm 101}{8}$$

$$= \frac{-5+101}{8}, \frac{-5-101}{8}$$

$$= \frac{96}{8}, \frac{-106}{8} = 12 \text{ or } \frac{-53}{4}$$

$$n = -\frac{53}{4} \text{ is not possible}$$

$\therefore n = 12$

$\therefore$ 12 terms are taken to get the sum 636.

22. If S_n the sum of first n terms of an A.P. is given by $S_n = 5n^2 + 3n$. Then find the n^{th} term.

Sol :

$$\begin{aligned} S_n &= 5n^2 + 3n \\ S_{n-1} &= 5(n-1)^2 + 3(n-1) \\ &= 5(n^2 - 2n + 1) + 3(n-1) \\ &= 5n^2 - 10n + 5 + 3n - 3 \\ &= 5n^2 - 7n + 2 \\ \text{Now } n^{th} \text{ term} &= S_n - S_{n-1} \\ \therefore \text{The required } n^{th} \text{ term} &= [5n^2 + 3n] - [5n^2 - 7n + 2] \\ &= 5n^2 + 3n - 5n^2 + 7n - 2 \\ t_n &= 10n - 2 \end{aligned}$$

23. Find the 9^{th} term and the general term of the G.P. $\frac{1}{4}, \frac{-1}{2}, 1, -2, \ldots$

Sol :

$$\text{Were } a = \frac{1}{4}, \quad r = -2$$

$$\begin{aligned} a_n &= ar^{n-1} \\ a_9 &= \frac{1}{4}(-2)^8 = 64 \\ a_n &= \frac{1}{4}(-2)^{n-1} \\ a_n &= (-1)^{n-1} 2^{n-3} \end{aligned}$$

24. The fourth, seventh and last term of a G.P. are 10, 80 and 2560 respectively. Find the first term and the number of terms in the G.P.

Sol :

Let the first term = a, common ratio = r

$$\begin{aligned} a_4 &= 10, a_7 = 80, a_n = 2560 \\ ar^3 &= 10 \\ ar^6 &= 80 \\ \frac{ar^6}{ar^3} &= \frac{80}{10} \\ r^3 &= 8 \\ r &= 2 \\ ar^3 &= 10 \\ \therefore a(2)^3 &= 10 \\ a(8) &= 10 \\ a &= \frac{10}{8} = \frac{5}{4} \end{aligned}$$

$$\begin{aligned} ar^{n-1} &= 2560 \\ \frac{5}{4}(2)^{n-1} &= 2560 \Rightarrow 2^{n-1} = 2560 \times \frac{4}{5} \\ &= 512 \times 4 \\ &= 2048 \\ 2^{n-1} &= 2^{11} \\ \therefore n - 1 &= 11 \Rightarrow n = 12. \end{aligned}$$

25. The 7^{th} term of G.P. is 8 times the 4^{th} term and 5^{th} term is 48. Find the G.P.

Sol :

$$\begin{aligned} a_7 &= 8a_4 \quad \text{and} \quad a_5 = 48 \\ ar^6 &= 8ar^3 \quad \text{and} \quad ar^4 = 48 \\ r^3 &= 8 \quad \Rightarrow a(2)^4 = 48 \\ r &= 2 \quad\quad a = \frac{48}{16} = 3 \\ a &= 3 \end{aligned}$$

$\therefore$ The required G.P. is 3, 6, 12, 24, 48...

26. What term of the G.P. 5, 10, 20, 40,... is 5120?

Sol : $a = 5, \; r = 2$

$$\begin{aligned} a_n &= 5120 \\ a_n &= ar^{n-1} \\ 5120 &= 5(2)^{n-1} \\ 2^{n-1} &= 1024 \\ 2^{n-1} &= 2^{10} \\ n - 1 &= 10 \\ n &= 10 + 1 = 11 \end{aligned}$$

11^{th} term of the G.P. is 5120.

27. Find the sum upto infinity of the G.P. $1, \frac{1}{3}, \frac{1}{9}, \ldots$

Sol :

$$a = 1, r = \frac{1}{3}$$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1.5$$

28. Find the sum of infinity of the G.P. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, \ldots$

Sol :

$$\text{Here } a = \frac{-3}{4}, r = \frac{-1}{4}$$

$$\text{Sum} = \frac{a}{1-r} = \frac{-\frac{3}{4}}{1-\left(\frac{-1}{4}\right)}$$

$$\text{Sum} = \frac{\frac{-3}{4}}{\frac{5}{4}} = \frac{-3}{5}$$

29. Prove that $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times .... = 3$

Sol :

$$\text{L.H.S} = 3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \times ....$$

$$= 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....}$$

Now take $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...$

It is a G.P. with $a = \frac{1}{2}$, $r = \frac{1/4}{1/2} = \frac{1}{2}$

$$\text{Sum} = \frac{a}{1-r}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$\therefore 3^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...} = 3^1 = 3 = \text{RHS}$$

30. If $y = x + x^2 + x^3 + ...$ prove that $x = \frac{y}{1+y}$

Sol :

$$\text{Given } y = x + x^2 + x^3 + ...$$

$$= x(1 + x + x^2 + ...)$$

$$y = x\left[\frac{1}{1-x}\right] \quad \left[\because S_\infty = \frac{a}{1-r}\right]$$

$$y = \frac{x}{1-x}$$

$$y - yx = x$$

$$x + xy = y \Rightarrow x(1+y) = y$$

$$x = \frac{y}{1+y}$$

31. Using Geometric Series rationalise $0.1\overline{42}$

Sol :

$$0.1\overline{42} = 0.1424242...$$

$$= 0.1 + 0.042 + 0.00042 + 0.0000042 + ...$$

$$= 0.1 + \frac{42}{10^3} + \frac{42}{10^5} + \frac{42}{10^7} + ...$$

$$= \frac{1}{10} + \left[\frac{\frac{42}{10^3}}{1-\frac{1}{10^2}}\right] = \frac{1}{10} + \frac{42}{990}$$

$$\therefore 0.1\overline{42} = \frac{141}{990}$$

32. Find the sum: 1 + 2 + 3 + 4 + ... + 80.

Sol :

$$\text{We have } 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

$$\therefore 1 + 2 + 3 + ... + 80 = \frac{80 \times 81}{2} = 3240$$

33. Find the sum of $1^3 + 2^3 + 3^3 + ... + 70^3$.

Sol :

$$\text{We have } 1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$= \left[\frac{70 \times 71}{2}\right]^2$$

$$= (35 \times 71)^2 = (2485)^2$$

$$1^3 + 2^3 + 3^3 + ... + 70^3 = 61,75,225$$

34. Find the sum of $1^2 + 2^2 + 3^2 + ... + 23^2$

Sol :

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\text{Sum of } 1^2 + 2^2 + 3^2 + ... + 23^2 = \frac{23 \times 24 \times 47}{6}$$

$$= 4324$$

III. Short Answer Questions:

1. What is the greatest possible length which can be used to measure exactly the lengths 7 m; 3 m 85 cm; 12 m 95 cm?

Sol :

Required Length = H.C.F. of (7 m, 3 m 85 cm, 12 m 95 cm)

= H.C.F. of 700 cm, 385 cm, 1295 cm

$$\begin{array}{r|l} 5 & 700 \\ \hline 7 & 140 \\ \hline 2 & 20 \\ \hline 2 & 10 \\ \hline 5 & 5 \\ \hline & 1 \end{array} \qquad \begin{array}{r|l} 5 & 385 \\ \hline 7 & 77 \\ \hline & 11 \end{array} \qquad \begin{array}{r|l} 5 & 1295 \\ \hline 7 & 259 \\ \hline & 37 \end{array}$$

$$\begin{aligned} 700 &= 2^2 \times 5^2 \times 7^1 \\ 385 &= 5^1 \times 7^1 \times 11^1 \\ 1295 &= 5^1 \times 7^1 \times 37^1 \\ \text{H.C.F. is } 5^1 \times 7^1 &= 35 \end{aligned}$$

$\therefore$ The required length is 35 cm.

2. Find the greatest number that will divide 43, 91 and 183 so as to leave the same remainder in each case.

Sol :

Required number = H.C.F. of (91 – 43), (183 – 91) and (183 – 43)

= H.C.F. of 48, 92, and 140

$$\begin{array}{r|l} 2 & 48 \\ \hline 2 & 24 \\ \hline 2 & 12 \\ \hline 2 & 6 \\ \hline 3 & 3 \\ \hline & 1 \end{array} \qquad \begin{array}{r|l} 2 & 92 \\ \hline 2 & 46 \\ \hline & 23 \end{array} \qquad \begin{array}{r|l} 7 & 140 \\ \hline 2 & 20 \\ \hline 2 & 10 \\ \hline & 5 \end{array}$$

$$\begin{aligned} 48 &= 2^4 \times 3^1 \\ 92 &= 2^2 \times 23^1 \\ 140 &= 2^2 \times 5^1 \times 7^1 \end{aligned}$$

$\therefore$ H.C.F. is $2^2 = 4$

$\therefore$ The required number is 4

3. What will be the least number which when doubled will be exactly divisible by 12, 18, 21 and 30?

Sol :

$$\begin{array}{r|l} 2 & 12, 18, 21, 30 \\ \hline 3 & 6, 9, 21, 15 \\ \hline & 2, 3, 7, 5 \end{array}$$

$$\begin{aligned} \text{L.C.M. of } 12, 18, 21, 30 &= 2^2 \times 3^2 \times 5^1 \times 7^1 \\ &= 4 \times 9 \times 35 \\ &= 1260 \\ \text{Required number} &= 1260 \div 2 \\ &= 630 \end{aligned}$$

4. If the exams are over by Monday. The results will be published 29 days after exams. What day the results will be published.

Sol :

Monday stands for 1.

29 days after Monday is $1 + 29 \pmod 7$

$\equiv 30 \pmod 7$

$\equiv 2 \pmod 7$

2 stands for Tuesday.

Results will be published on Tuesday.

5. Let the sequence defined by $a_1 = 3$, $a_n = 3a_{n-1} + 1$ for all $n > 1$. Find the first three terms of the sequence.

Sol :

$$\begin{aligned} \text{Given } a_1 &= 3. \\ a_n &= 3a_{n-1}+1 \quad \text{for all } n > 1 \\ \text{Put } n &= 2; \\ a_2 &= 3a_{2-1}+1 = 3a_1 + 1 \\ &= 3(3) + 1 = 9 + 1 = 10 \\ \text{Put } n &= 3; \\ a_3 &= 3a_{3-1} + 1 = 3a_2 + 1 \\ &= 3(10) + 1 = 30 + 1 = 31 \end{aligned}$$

$\therefore$ First three terms are 3, 10, 31.

6. If $a_n = (-1)^n n$ find a_3, a_5 and a_8

Sol :

Given $a_n = (-1)^n n$

We know that

$(-1)^{\text{odd number}}$ = '–' ve

$(-1)^{\text{even number}}$ = + ve

$$\begin{aligned} a_3 &= (-1)^3\, 3 = -3 \\ a_5 &= (-1)^5\, 5 = -5 \\ a_8 &= (-1)^8\, 8 = 8 \\ a_3 &= -3; a_5 = -5; a_8 = 8 \end{aligned}$$

7. If $a_n = (n-1)(2-n)(3+n)$ find a_1, a_2, a_3.

Sol :

Given $a_n = (n-1)(2-n)(3+n)$

Put $n = 1$; $a_1 = (1-1)(2-1)(3+1)$

$a_1 = 0 \times 1 \times 4 = 0$

Put $n = 2$; $a_2 = (2-1)(2-2)(3+2)$

$a_2 = 1 \times (0) \times 5 = 0$

Put $n = 3$; $a_3 = (3-1)(2-3)(3+3)$

$= 2 \times (-1) \times 6 = -2 \times 6$

$a_3 = -12$

$a_1 = 0; a_2 = 0; a_3 = -12$

8. For what value of n the n^{th} term of the A.P. 69, 68, 67,... and 1, 7, 13, 19 ... are the same

Sol :

Consider the A.P. 69, 68, 67,...

$a = 69; d = 68 - 69 = -1$

$t_n = a + (n-1)d$

$t_n = 69 + (n-1)(-1)$ ...(1)

For the A.P. 1, 7, 13, 19,...

$a = 1; d = 7 - 1 = 6$

$T_n = 1 + (n-1)6$ ...(2)

If the two A.P's has an identical term then $t_n = T_n$ for some n.

$69 + (n-1)(-1) = 1 + (n-1)6$

$$69 - n + 1 = 1 + 6n - 6$$
$$69 + 1 - 1 + 6 = 6n + n$$
$$75 = 7n$$
$$n = \frac{75}{7}$$, which is not a natural number.

$\therefore$ The two A.P.'s do not have an identical term for any n.

9. If the 8^{th} term of an A.P. is 31 and the 15^{th} term is 16 more than the 11^{th} term, find the A.P.

Sol :

Let a be the first term and d be the common difference of the A.P.

Given $a_8 = 31$ and $a_{15} = 16 + a_{11}$

$$a + (n-1)\,d = t_n$$
$$a + 7d = 31 \text{ and}$$
$$a + 14\,d = 16 + (a + 10d)$$
$$a + 14\,d = 16 + a + 10d$$
$$a - a + 14d - 10d = 16$$
$$4d = 16$$
$$d = \frac{16}{4} = 4$$

Taking
$$a + 7d = 31$$
$$a + 7\,(4) = 31$$
$$a + 28 = 31$$
$$a = 31 - 28 = 3.$$

$\therefore$ The A.P. is a, a + d, a + 2d, ...

$\Rightarrow$ 3, 7, 11, 15, 19,...

10. Which term of the A.P. 5, 15, 25,... will be 130 more than its 31^{st} term?

Sol :

We have $a = 5; d = 10$

$$a_{31} = a + 30d = 5 + 30 \times 10 = 305$$

Let the n^{th} term of the Given A.P. is 130 more than the 31^{st} term.

$$a_n = 130 + a_{31}$$
$$a + (n-1)\,d = 130 + 305$$
$$5 + 10\,(n-1) = 435$$
$$10\,(n-1) = 430$$
$$n - 1 = 43$$
$$n = 44$$

$\therefore$ 44^{th} term of the given A.P. is 130 more than its 31^{st} term.

11. If the first term of an A.P. is 17 and last term is 350, common difference is 9, how many terms are there in the A.P.? What is their sum?

Sol :

First term $a = 17$

Last term $l = 350 = t_n$

Common difference $d = 9$

$$t_n = a + (n-1)d$$
$$350 = 17 + (n-1)\,9$$
$$\frac{350-17}{9} = n - 1$$
$$n = \frac{333}{9} + 1$$
$$= 37 + 1 = 38$$

$\therefore$ There are 38 terms in the A.P.

$$S_n = \frac{n}{2}(a + l)$$
$$S_{38} = \frac{38}{2}(17 + 350)$$
$$= 19 \times 367$$

$\therefore$ Required sum $= 6973$

12. Find the sum of all three digit numbers which are divisible by 7.

Sol :

Three digit numbers which are divisible by 7 are 105, 112, 119, ..., 994.

It is in A.P. where $a = 105, d = 7$

$$t_n = 994 = l$$
$$t_n = a + (n-1)d$$
$$994 = 105 + (n-1)\,7$$
$$n - 1 = \frac{994 - 105}{7} = \frac{889}{7} = 127$$
$$n = 127 + 1 = 128$$

Now $S_n = \frac{n}{2}(a + l)$

$$= \frac{128}{2}(105 + 994)$$
$$= 64\,(1099)$$

$\therefore$ The required sum $= 70336$

13. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a > b > c$ and $a + b + c = 3/2$ then find the value of a.

Sol :

Let $b = a + d, c = a + 2d$ ...(1)

a^2, b^2, c^2 are in G.P. $(b^2)^2 = a^2c^2$

$\pm b^2 = ac$ ...(2)

$\therefore$ a, b, c are in A.P.

$$2b = a + c$$

Given $$a+b+c = \frac{3}{2}$$

$$\Rightarrow 3b = \frac{3}{2}$$

$$\Rightarrow b = \frac{1}{2}$$

From (1) $$a = b - d$$

$$\Rightarrow a = \frac{1}{2} - d$$

$$c = a + 2d$$

$$= \left(\frac{1}{2} - d\right) + 2d = \frac{1}{2} + d$$

From (2) $$\pm\frac{1}{4} = \left(\frac{1}{2} - d\right)\left(\frac{1}{2} + d\right)$$

$$\Rightarrow \pm\frac{1}{4} = \frac{1}{4} - d^2$$

Taking negative sign $$\therefore d = \pm\frac{1}{\sqrt{2}}$$

$$\therefore a = \frac{1}{2} - d = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$$

$$\Rightarrow a = \frac{1}{2} + \frac{1}{\sqrt{2}} \quad [\because a > b > c]$$

14. If x, (2x + 2), (3x + 3),.... are in G.P. then find the next term of the G.P.

Sol :

Since x, 2x + 2, 3x + 3,.... are in G.P.

$$\frac{2x+2}{x} = \frac{3x+3}{2x+2} = r \ (x \neq -1)$$

$$r = \frac{3}{2}$$

$$\therefore \frac{2x+2}{x} = \frac{3}{2}$$

$$4x + 4 = 3x$$

$$x = -4$$

$\therefore$ The next term $= r(3x+3)$

$$= \frac{3}{2}(-12+3)$$

$$= \frac{-27}{2} = -13.5$$

15. The 5th, 8th and 11th term of a G.P. are p, q and s respectively. Show that $q^2 = ps$.

Sol :

Let 'a' be the first term and r be the common ratio of the given G.P.

Here $a_5 = p \Rightarrow ar^4 = p$ ...(1)

$a_8 = q \Rightarrow ar^7 = q$ ...(2)

$a_{11} = s \Rightarrow ar^{10} = s$ ...(3)

Squaring both sides of (2), we get

$$q^2 = (ar^7)^2$$

$$q^2 = a^2r^{14}$$

$$q^2 = (ar^4)(ar^{10})$$

$$q^2 = ps \quad [\because p = ar^4 \text{ and } s = ar^{10}]$$

16. Find the sum to infinity of the series

$$\frac{1}{7} + \frac{2}{7^2} + \frac{1}{7^3} + \frac{2}{7^4} + ...$$

Sol : This series may be written as

$$\left(\frac{1}{7} + \frac{1}{7^3} + \frac{1}{7^5} + ...\right) + \left(\frac{2}{7^2} + \frac{2}{7^4} + \frac{2}{7^6} + ...\right)$$

$$= \left(\frac{\frac{1}{7}}{1 - \frac{1}{7^2}}\right) - \left(\frac{\frac{2}{7^2}}{1 - \frac{1}{7^2}}\right) \quad \left[\text{Using sum} = \frac{a}{1-r}\right]$$

$$= \frac{\frac{1}{7}}{\frac{48}{49}} + \frac{\frac{2}{49}}{\frac{48}{49}}$$

$$= \frac{7}{48} + \frac{2}{48} = \frac{9}{48}$$

$$\frac{1}{7} + \frac{2}{7^2} + \frac{1}{7^3} + \frac{2}{7^4} + ... = \frac{9}{48}$$

17. Find the sum to infinity of the G.P.

$$\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + ...$$

Sol :

The given series may be written as

$$\left(\frac{2}{5} + \frac{2}{5^3} + \frac{2}{5^5} + ...\right) + \left(\frac{3}{5^2} + \frac{3}{5^4} + \frac{3}{5^6} + ...\right)$$

$$S_\infty = \frac{a}{1-r}$$

$$\text{Sum} = \frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2} + \frac{\frac{3}{5^2}}{1 - \left(\frac{1}{5}\right)^2}$$

$$= \frac{\frac{2}{5}}{1-\frac{1}{25}} + \frac{\frac{3}{25}}{1-\frac{1}{25}}$$

$$= \frac{5}{12} + \frac{1}{8} = \frac{13}{24}$$

$$\therefore \frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \ldots = \frac{13}{24}$$

18. Find $16^2 + 17^2 + 18^2 + \ldots + 30^2$

Sol :

We have

$1^2 + 2^2 + 3^2 + \ldots + 30^2$

$$= \frac{n(n+1)(2n+1)}{6}$$

$16^2 + 17^2 + 18^2 + \ldots + 30^2$

$$= (1^2 + 2^2 + \ldots + 30^2) - (1^2 + 2^2 + \ldots + 15^2)$$

$$= \frac{30 \times 31 \times 61}{6} - \frac{15 \times 16 \times 31}{6}$$

$$= 9455 - 1240 = 8215$$

19. Find $8^3 + 9^3 + 10^3 + \ldots + 23^3$.

Sol :

We have

$1^3 + 2^3 + 3^3 + \ldots + n^3$

$$= \left[\frac{n(n+1)}{2}\right]^2$$

$8^3 + 9^3 + 10^3 + \ldots + 23^3$

$$= (1^3 + 2^3 + \ldots + 23^3) - (1^3 + 2^3 + \ldots + 7^3)$$

$$= \left[\frac{23 \times 24}{2}\right]^2 - \left[\frac{7 \times 8}{2}\right]^2$$

$$= (23 \times 12)^2 - (7 \times 4)^2$$

$$= (276)^2 - (28)^2$$

$$= 76176 - 784$$

$$= 75,392$$

20. 1 + 3 + 5 + … + 20 terms

Sol :

Sum of first n odd number $= n^2$

Sum of first 20 odd numbers $= 20^2$

$= 400$

IV. Long Answer Questions

1. Show that the square of an odd positive integer is of the form 8q + 1, for some integer q.

Sol :

First we will prove "any odd positive integer n is of the form 4q + 1 or 4q + 3, where q is some integer".

By Euclid's division Lemma,

If 'a' and 'b' are two positive integers then

$a = bq + r$ where $0 \le r < |b|$.

Suppose the positive integer be 'a' and $b = 4$

then $a = 4q + r$ where $0 \le r < |4|$

$a = 4q$, $a = 4q + 1$; $a = 4q + 2$; $a = 4q + 3$

$= 2(2q)$; $a = 4q + 1$ $= 2(2q + 1)$; $=$ odd

$=$ even; $=$ odd; $=$ even;

$\therefore$ Any positive odd integer is of the form 4q + 1 or 4q + 3

Case (1):

If $a = 4q + 1$

$a^2 = (4q + 1)^2$

$= 16q^2 + 8q + 1$

$= 8q(2q + 1) + 1$

$= 8m + 1$ where $m = q(2q + 1)$

Case (2):

If $a = 4q + 3$

then $a^2 = (4q + 3)^2 = 16q^2 + 24q + 9$

$= 8[2q^2 + 3q] + 8 + 1$

$= 8[2q^2 + 3q + 1] + 1$

$= 8m + 1$ where $m = 2q^2 + 3q + 1$

$\therefore$ We conclude that the square of an odd positive integer is of the form 8q + 1, for some integer q.

2. Show that if x and y are both odd positive integers, then $x^2 + y^2$ is even but not divisible by 4.

Sol :

Let m and n be any integers, then

$x = 2m + 1$ and $y = 2n + 1$ since x and y are odd

$x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$

$= 4(m^2 + n^2) + 4(m + n) + (1 + 1)$

$= 4(m^2 + n^2) + 4(m + n) + 2$

$= 4q + 2$, where $q = (m^2 + n^2) + (m + n)$

$= 4q + 2$

$x^2 + y^2 = 2[2q + 1]$ which is an even number.

Thus $4q + 2$ is an even number, which is not divisible by 4.
i.e., It leaves the remainder 2.
Hence $x^2 + y^2$ is even but not divisible by 4.

3. If the H.C.F. of 210 and 55 is expressible in the form $210 \times 5 + 55y$. Find y.

Sol :

Let us find the H.C.F. of 210 and 55, Applying Euclid's Division Algorithm

$$\begin{aligned} 210 &= 55 \times 3 + 45 \\ 55 &= 45 \times 1 + 10 \\ 45 &= 10 \times 4 + 5 \\ 10 &= 5 \times 2 + 0 \\ \text{Remainder} &= 0 \end{aligned}$$

$\therefore$ H.C.F. of 210 and 55 is 5

$$\begin{aligned} \text{Given} \quad 5 &= 210 \times 5 + 55\,y \\ 5 - 210 \times 5 &= 55\,y \\ 55\,y &= 5 - 1050 \\ &= -1045 \\ y &= \frac{-1045}{55} \\ y &= -19 \end{aligned}$$

4. If d is the H.C.F. of 56 and 72, find x and y satisfying $d = 56x + 72y$.

Sol :

Applying Euclid's division Algorithm to find the H.C.F. of 56 and 72, we have.

$$\begin{aligned} 72 &= 56 \times 1 + 16 \quad \text{... (1)} \\ 56 &= 16 \times 3 + 8 \quad \text{... (2)} \\ 16 &= 8 \times 2 + 0 \\ \text{Remainder} &= 0 \end{aligned}$$

$\therefore$ H.C.F. of 56 and 72 $= 8$

From (2) we have

$$\begin{aligned} 8 &= 56 - 16 \times 3 \\ 8 &= 56 - (72 - 56 \times 1) \times 3 \\ & \qquad [\because \textit{from } (1)\ 16 = 72 - 56 \times 1] \\ &= 56 - (3 \times 72) + 3 \times 56 \\ &= 56\,(1 + 3) - (3 \times 72) \\ &= 56\,(4) + 72\,(-3) \end{aligned}$$

Comparing with $d = 56x + 72y$, we have

$$x = 4 \text{ and } y = -3$$

5. In an Interview, the number of participants in Mathematics, Physics and Chemistry are 60, 84 and 108 respectively. Find the minimum number of rooms required if in each room the same number of participants to be seated and all of them being in the same subject.

Sol :

The number of participants in each room is the H.C.F. of 60, 84 and 108.

First we find the H.C.F. of 60 and 84, by applying Euclid's Division Algorithm.

$$\begin{aligned} 84 &= 60 \times 1 + 24 \\ 60 &= 24 \times 2 + 12 \\ 24 &= 12 \times 2 + 0 \\ \text{The remainder} &= 0 \end{aligned}$$

$\therefore$ H.C.F. of 60 and 84 is 12.

Now applying Euclid's Division Algorithm to 12 and 108

$$\begin{aligned} 108 &= 12 \times 9 + 0 \\ \text{Remainder} &= 0 \\ \therefore \text{H.C.F. of 12 and 108} &= 12 \\ \therefore \text{H.C.F. } (60, 84, 108) &= 12. \end{aligned}$$

$\therefore$ In each room, the minimum 12 number of participants can be seated.

$$\begin{aligned} \text{Total number of participants} &= 60 + 84 + 108 \\ &= 252 \\ \text{Number of rooms required} &= \frac{252}{12} = 21 \end{aligned}$$

H.C.F.

If a composite number n divides ab, then n need not divide neither a nor b. For example 6 divides 4×3, but 6 neither divides 4 nor 3.

6. Find the greatest number which can divide 1356, 1868 and 2764 leaving the same remainder 12 in each case

Sol :

Required number = H.C.F. of $(1356 - 12), (1868 - 12), (2764 - 12)$

= H.C.F. of 1344, 1856, 2752

2	2752
2	1376
2	688
2	344
2	172
2	86
	43

2	1344
2	672
3	336
7	112
2	16
2	8
2	4
2	2
	1

2	1856
2	928
2	464
2	232
2	116
2	58
	29

$$2752 = 2^6 \times 43^1$$

$$1344 = 2^6 \times 3^1 \times 7^1$$
$$1856 = 2^6 \times 29^1$$
$$\text{H.C.F. is } 2^6 = 64$$
$\therefore$ The required number $= 64$

7. Find the smallest number of five digits exactly divisible by 16, 24, 36 and 54.

Sol :

Smallest number of five digits is 10000.

Required number must be divisible by 16, 24, 36 and 54

2	16, 24, 36, 54
2	8, 12, 18, 27
3	4, 6, 9, 27
3	4, 2, 3, 9
2	4, 2, 1, 3
	2, 1, 1, 3

$$\text{L.C.M.} = 2^4 \times 3^3$$
$$= 16 \times 27 = 432$$

L.C.M. of 16, 24, 36, 54 $= 432$

$\therefore$ Required number is divisible by 432.

$$\begin{array}{r|l} & 231 \\ \hline 432 & 10000 \\ & 864 \\ \hline & 1360 \\ & 1296 \\ \hline & \ \ 64 \end{array}$$

On dividing 10000 by 432, we get remainder $= 64$

Required number $= 10000 + (432 - 64)$

$= 10368$

8. Six bells commence tolling together and toll at intervals of 2, 4, 6, 8, 10 and 12 seconds respectively. In 30 minutes, how many times do they toll together?

Sol :

2	2, 4, 6, 8, 10, 12
3	1, 2, 3, 4, 5, 6
2	1, 2, 1, 4, 5, 2
	1, 1, 1, 2, 5, 1

L.C.M. $= 12 \times 10 = 120$

L.C.M. of 2,4,6,8,10, 12 is 120

$\therefore$ The bells will toll together after every 120 sec. i.e., 2 min.

In 30 minutes they will toll together $\left(\frac{30}{2}+1\right) = 16$ times.

9. Solve $9x \equiv 5 \pmod{13}$

Sol :

$$9x \equiv 5 \pmod{13}$$
$9x - 5 = 13$ k for some integer k.

$x = \dfrac{13k+5}{9}$ for some integer k.

When we put 1, 10, 19,...... the 13k + 5 is divisible by 13.

$$x = \frac{13(1)+5}{9} = 2$$
$$x = \frac{13(10)+5}{9} = 15$$
$$x = \frac{13(19)+5}{9} = 28$$
$$x = \frac{13(28)+5}{9} = 41$$

$\therefore$ The solutions are 2, 15, 28, 41,...

10. Find the next five terms of the given sequence $a_1 = 1, a_n = a_{n-1} + 2, n \geq 2$

Sol :

Given $a_1 = 1$

$a_n = a_{n-1} + 2$ for $n \geq 2$

Put n = 2, $a_2 = a_{2-1} + 2$

$= a_1 + 2 = 1 + 2$

$a_2 = 3$

Put n = 3, $a_3 = a_{3-1} + 2$

$= a_2 + 2 = 3 + 2$

$a_3 = 5$

Put n = 4, $a_4 = a_{4-1} + 2$

$= a_3 + 2 = 5 + 2$

$a_4 = 7$

Put n = 5, $a_5 = a_{5-1} + 2$

$= a_4 + 2 = 7 + 2$

$a_5 = 9$

Put n = 6, $a_6 = a_{6-1} + 2$

$= a_5 + 2 = 9 + 2$

$a_6 = 11$

The next five terms are $a_2 = 3$; $a_3 = 5$; $a_4 = 7$; $a_5 = 9$ and $a_6 = 11$.

11. Find the next five terms of the sequence given by $a_1 = 4; a_n = 4a_{n-1} + 3, n > 1$

Sol :

Given $a_1 = 4$

$a_n = 4a_{n-1} + 3$ for $n > 1$

Put $n = 2$; $a_2 = 4a_{2-1} + 3 = 4a_1 + 3$
$= 4(4) + 3 = 16 + 3$
$a_2 = 19$

Put $n = 3$; $a_3 = 4a_{3-1} + 3 = 4a_2 + 3$
$= 4(19) + 3 = 76 + 3$
$a_3 = 79$

Put $n = 4$; $a_4 = 4a_{4-1} + 3 = 4a_3 + 3$
$= 4(79) + 3 = 316 + 3$
$a_4 = 319$

Put $n = 5$; $a_5 = 4a_{5-1} + 3 = 4a_4 + 3$
$= 4(319) + 3 = 1276 + 3$
$a_5 = 1279$

Put $n = 6$; $a_6 = 4a_{6-1} + 3 = 4a_5 + 3$
$= 4(1279) + 3 = 5116 + 3$
$a_6 = 5119$

$\therefore$ The next five terms are $a_2 = 19, a_3 = 79, a_4 = 319, a_5 = 1279, a_6 = 5119$.

12. A sequence is defined by $a_n = n^3 - 6n^2 + 11n - 6$ show that the first three terms of the sequence are zero and all other terms are positive.

Sol :

Given $a_n = n^3 - 6n^2 + 11n-6$

Put $n = 1$, $a_1 = 1^3 - 6(1)^2 + 11(1) - 6$
$= 1 - 6 + 11 - 6 = 12 - 12$
$a_1 = 0$

Put $n = 2$, $a_2 = 2^3 - 6(2)^2 + 11(2) - 6$
$= 8 - (6 \times 4) + 22 - 6$
$= 8 - 24 + 22 - 6 = 30 - 30$
$a_2 = 0$

Put $n = 3$, $a_3 = 3^3 - 6(3)^2 + 11(3) - 6$
$= 27 - 54 + 33 - 6 = 60 - 60$
$a_3 = 0$

So we have $a_1 = a_2 = a_3 = 0$.

That is the value of the cubic polynomial $n^3 - 6n^2 + 11n - 6$ becomes zero for $n = 1, 2, 3$

$\therefore$ $(n-1), (n-2)$ and $(n-3)$ are factors of a_n.

$\therefore$ $a_n = (n-1)(n-2)(n-3)$ for which $a_n > 0$ for all $n > 3$.

$\therefore$ Other terms are positive.

13. The 10^{th} term an A.P. is 52 and 16^{th} term is 82. Find the 32^{nd} term and the general term.

Sol :

$t_n = a + (n-1)d$

10^{th} term of an A.P. is 52.

$t_{10} = a + (10-1)d = 52$

$a + 9d = 52$ ...(1)

16^{th} term is 82

$a + 15d = 82$ ...(2)

$a + 9d = 52$ ...(1)

Solving (1) and (2)

(2) – (1), $6d = 30$
$d = 5$
$a + 9d = 52$ from (1)
$a + 9(5) = 52$
$a + 45 = 52$
$a = 52 - 45 = 7$

$\therefore$ $t_{32} = a + 31d$
$= 7 + 31(5) = 7 + 155$
$t_{32} = 162$

The general term $a_n = 5n + 2$

14. The sum of 5^{th} and 9^{th} terms of an A.P. is 72 and the sum of 7^{th} and 12^{th} term is 97. Find the A.P.

Sol :

Given $a_5 + a_9 = 72$ and $a_7 + a_{12} = 97$

$a + 4d + a + 8d = 72$ and $a + 6d + a + 11d = 97$

$2a + 12d = 72$...(1) and $2a + 17d = 97$ ...(2)

Solving (1) and (2)

$2a + 12d = 72$ ...(1)

$2a + 17d = 97$ ...(2)

(2) – (1) $\Rightarrow$ $5d = 25$
$d = 5$
$2a + 12d = 72$
$2a + 12(5) = 72$
$2a + 60 = 72$
$2a = 72 - 60 = 12$
$a = \frac{12}{2} = 6$

$\therefore$ The A.P. is $a, a + d, a + 2d, a + 3d$ ...

$6, 6 + 5, 6 + 2(5), 6 + 3(5)$ ...

6, 11, 16, 21,...

15. If $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P. then prove that a^2, b^2, c^2 are in A.P

Sol :

Given $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$$\therefore t_2 - t_1 = t_3 - t_2$$

$$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$$

$$\frac{(b+c)-(c+a)}{(c+a)(b+c)} = \frac{(c+a)-(a+b)}{(a+b)(c+a)}$$

$$\frac{b+c-c-a}{cb+ab+c^2+ac} = \frac{c+a-a-b}{ca+bc+a^2+ab}$$

$$\frac{b-a}{ab+cb+ac+c^2} = \frac{c-b}{ab+bc+ca+a^2}$$

$$(b-a)(ab+bc+ca+a^2) = (c-b)(ab+cb+ac+c^2)$$

$$ab^2+b^2c+cab+a^2b-a^2b-abc-a^2c-a^3 = abc+c^2b+ac^2+c^3-ab^2-cb^2-abc-bc^2$$

$$ab^2+b^2c-a^2c-a^3 = ac^2+c^3-ab^2-cb^2$$

$$b^2c-a^2c+ab^2-a^3 = c^3-cb^2+ac^2-ab^2$$

$$c(b^2-a^2)+a(b^2-a^2) = c(c^2-b^2)+a(c^2-b^2)$$

$$(b^2-a^2)(c+a) = (c^2-b^2)(c+a)$$

$$b^2-a^2 = c^2-b^2$$

$t_2 - t_1 = t_3 - t_2$ of the sequence a^2, b^2, c^2.

$\therefore$ a^2, b^2, c^2 are in A.P.

16. Find the 6^{th} term of the A.P. $\frac{2m+1}{m}, \frac{2m-1}{m}, \frac{2m-3}{m}, \ldots$

Sol :

Here $t_1 = \frac{2m+1}{m}$, $t_2 = \frac{2m-1}{m}$

$$d = t_2 - t_1 = \frac{2m-1}{m} - \frac{2m+1}{m} = \frac{2m-1-2m-1}{m} = \frac{-2}{m}$$

Now $t_n = a + (n-1)d$

$$t_n = \left[\frac{2m+1}{m}\right] + (n-1)\left[\frac{-2}{m}\right] = \left[\frac{2m+1}{m}\right] + \left[\frac{-2n}{m}\right] - 1\left[\frac{-2}{m}\right] = \frac{2m+1}{m} - \frac{2n}{m} + \frac{2}{m} = \frac{2m+1-2n+2}{m}$$

Thus n^{th} term $= \frac{2m-2n+3}{m}$

$\therefore$ 6^{th} term $t_6 = \frac{2m-2(6)+3}{m}$

6^{th} term $= \frac{2m-9}{m}$

17. If S_n the sum of n terms of an A.P. is given by $3n^2 - 4n$. Find the n^{th} term.

Sol :

We have $S_n = 3n^2 - 4n$

$$S_{n-1} = 3(n-1)^2 - 4(n-1) = 3(n^2-2n+1) - 4n + 4 = 3n^2 - 6n + 3 - 4n + 4 = 3n^2 - 10n + 7$$

$\therefore n^{th}$ term $= S_n - S_{n-1} = [3n^2 - 4n] - [3n^2 - 10n + 7] = 3n^2 - 4n - 3n^2 + 10n - 7$

$\therefore$ n^{th} term is $= 6n - 7$

18. The sum of first six terms of an A.P. is 42. The ratio of 10^{th} term to its 30^{th} term is 1 : 3. Calculate the first term and 13^{th} term of the A.P.

Sol :

$$S_6 = \frac{6}{2}\{2a + (6-1)d\} = 42$$

$\therefore$ $6a + 15d = 42$ ...(1)

Given $t_{10} : t_{30} = 1 : 3$

$$\frac{a+9d}{a+29d} = \frac{1}{3}$$

$$3(a+9d) = a + 29d$$

$$3a + 27d = a + 29d$$

$$3a - a = 29d - 27d$$

$$2a = 2d$$

$$a = d$$

From (1) $6a + 15d = 42$

$6d + 15d = 42$ $[\because a = d]$

$$21d = 42$$

$$d = \frac{42}{21} = 2$$

$$a = d = 2$$

Now $t_{13} = a + 12d = 2 + 12(2) = 2 + 24 = 26$

13^{th} term $= 26$

19. Which term of the A.P. 3, 15, 27, 39,... will be 120 more than its 21^{st} term?

Sol :

Let the first term a and common difference d.

$$a = 3; d = 15 - 3 = 12.$$

$$t_n = a + (n-1)d$$

$$t_{21} = 3 + (21-1) \times 12$$

$$= 3 + 20 \times 12 = 3 + 240$$

$$t_{21} = 243$$

Let the required term be the n^{th} term.

$$n^{th} \text{ term} = 120 + 21^{st} \text{ term}$$

$$= 120 + 243 = 363$$

$$t_n = a + (n-1)d$$

$$363 = 3 + (n-1) \times 12$$

$$363 - 3 = (n-1) \times 12$$

$$n - 1 = \frac{360}{12} = 30$$

$$n = 30 + 1 \; n = 31$$

$\therefore$ The required term is 31^{st} term of the A.P.

20. If p^{th} term of an A.P. . is $\frac{1}{q}$ and q^{th} term is $\frac{1}{p}$, prove that the sum of the first pq terms is $\frac{1}{2}(pq+1)$

Sol :

n^{th} term of an A.P.

$$t_n = a + (n-1)d$$

Given $\quad p^{th} \text{ term} = \frac{1}{q}$

$$a + (p-1)d = \frac{1}{q} \quad ...(1)$$

$$q^{th} \text{ term} = \frac{1}{p} \Rightarrow a + (q-1)d = \frac{1}{p} \quad ...(2)$$

$$(1) - (2) \Rightarrow [(p-1) - (q-1)]d = \frac{1}{q} - \frac{1}{p}$$

$$(p - 1 - q + 1)d = \frac{p-q}{pq}$$

$$(p-q)d = \frac{p-q}{pq}$$

$$d = \frac{p-q}{pq(p-q)}$$

$$d = \frac{1}{pq}$$

Substituting $d = \frac{1}{pq}$ in (1)

$$a + (p-1)\left(\frac{1}{pq}\right) = \frac{1}{q}$$

$$a = \frac{1}{q} - \frac{p-1}{pq}$$

$$= \frac{p-p+1}{pq}$$

$$a = \frac{1}{pq}$$

$$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore S_{pq} = \frac{pq}{2}\left[2\left(\frac{1}{pq}\right) + (pq-1)\left(\frac{1}{pq}\right)\right]$$

$$= \frac{pq}{2}\left[\frac{2}{pq} + \frac{pq}{pq} - \frac{1}{pq}\right]$$

$$= \frac{pq}{2}\left[\frac{1}{pq} + 1\right]$$

$$= \frac{1}{2}\left[\frac{pq}{pq} + pq\right] = \frac{1}{2}(1 + pq)$$

$$\therefore S_{pq} = \frac{1}{2}(pq+1)$$

21. If the sum of first 7 terms of an A.P. is 49 and that of first 17 terms is 289. Find the sum of n terms.

Sol :

Let 'a' be the first term.

'd' be the common difference.

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$S_7 = \frac{7}{2}[2a + 6d] = 49$$

$$\frac{7}{2} \times 2(a + 3d) = 49$$

$$7(a + 3d) = 49$$

$$a + 3d = \frac{49}{7} = 7 \quad ...(1)$$

$$S_{17} = \frac{17}{2}[2a + 16d] = 289$$

$$\Rightarrow \frac{17}{2} \times 2[a+8d] = 289$$

$$17(a+8d) = 289$$

$$a+8d = \frac{289}{17} = 17 \qquad \text{...(2)}$$

$$(2)-(1)\quad a+8d-a-3d = 17-7$$

$$5d = 10$$

$$d = \frac{10}{5} = 2$$

From (1) we have

$$a+3(2) = 7$$

$$a+6 = 7$$

$$a = 7-6 = 1$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$= \frac{n}{2}[2(1)+(n-1)(2)]$$

$$= \frac{n}{2}[2+2n-2]$$

$$= \frac{n}{2}[2n] = n^2$$

$\therefore$ Sum of n terms $= n^2$

22. Three numbers, the third which is being 12 form decreasing G.P. If the last term were 9 instead of 12, the three numbers would have formed an A.P. Find the common ratio of the G.P.

Sol :

The numbers a, b, 12 are in GP.

$$b^2 = 12a \qquad \text{...(1)}$$

a, b, 9 are in A.P

$$2b = a+9$$

$$a = 2b-9 \qquad \text{...(2)}$$

From (1) and (2)

$$b^2 = 12(2b-9)$$

$$b^2 - 24b + 108 = 0$$

$$(b-18)(b-6) = 0$$

$$b = 6, 18$$

From (2) $a = 2b-9 = 2(6)-9 = 3$ and

$$a = 2b-9 = 2(18)-9 = 27$$

$$\therefore \text{Common ratio} = \frac{b}{a} = \frac{6}{3} \text{ and } \frac{18}{27}$$

$$= 2 \text{ and } \frac{2}{3}$$

Since it is a decreasing G.P., Common ratio $= \frac{2}{3}$

23. If each term of a G.P. is positive and each term is the sum of its two succeeding terms find the common ratio of the G.P.

Sol :

If a, ar, ar^2,.... are in GP.

$$a = ar + ar^2 \Rightarrow r^2 + r - 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} \quad \left[\because x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}\right]$$

$$r = \frac{-1+\sqrt{5}}{2}$$

$$r = \frac{\sqrt{5}-1}{2}$$

24. Given $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x \neq 0)$ then show that a, b, c, d are in G.P.

Sol :

Given $$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$$

$$\frac{\frac{a}{b}+x}{\frac{a}{b}-x} = \frac{\frac{b}{c}+x}{\frac{b}{c}-x} = \frac{\frac{c}{d}+x}{\frac{c}{d}-x}$$

From first two relations

$$\frac{2\frac{a}{b}}{2x} = \frac{2\frac{b}{c}}{2x}$$

$$\therefore \frac{a}{b} = \frac{b}{c} \qquad \text{...(1)}$$

From last two relations

$$\frac{2\frac{b}{c}}{2x} = \frac{2\frac{c}{d}}{2x} \Rightarrow \frac{b}{c} = \frac{c}{d} \qquad \text{...(2)}$$

From (1) and (2) $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$

$\therefore$ a, b, c, d are in G.P.

25. Find the sum to infinity of the G.P.
$(\sqrt{2}+1)+1+(\sqrt{2}-1)+...$

Sol :

$$a = \sqrt{2}+1,$$

$$r = \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}$$

$$= \frac{\sqrt{2}-1}{2-1} = \frac{\sqrt{2}-1}{1} = \sqrt{2}-1$$

$$\text{Sum} = \frac{a}{1-r}$$

$$= \frac{\sqrt{2}+1}{1-\left(\sqrt{2}-1\right)} = \frac{\sqrt{2}+1}{2-\sqrt{2}}$$

$$= \frac{\sqrt{2}+1}{\sqrt{2}\left(\sqrt{2}-1\right)}$$

$$= \frac{\left(\sqrt{2}+1\right)\left(\sqrt{2}+1\right)}{\sqrt{2}\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}$$

$$= \frac{3+2\sqrt{2}}{\sqrt{2}}$$

$$\text{Sum} = \frac{4+3\sqrt{2}}{2}$$

26. The fourth term of a G.P. is 4. Find the product of its first seven terms.

Sol :

Let a be the first term and r be the common ratio.

$$t_4 = 4 \Rightarrow ar^{4-1} = 4 \Rightarrow ar^3 = 4$$

Product of seven terms $t_1 \times t_2 \times t_3 \times t_4 \times t_5 \times t_6 \times t_7$

$$= (a)(ar)(ar^2)(ar^3)(ar^4)(ar^5)(ar^6)$$

$$= a^7 r^{21} = (ar^3)^7$$

$$= (4)^7 = 16384$$

Product of first seven terms = 16384

27. Find the 1025^{th} term in the sequence 1, 22, 4444, 88888888,....

Sol :

Number of digits in the given sequence are 1, 2, 4, 8,...

Which are in G.P. Let 1025^{th} term $= 2^n$

Then $1 + 2 + 4 + 8 + \ldots + 2^{n-1} < 1025 \leq 1 + 2 + 4 + \ldots + 2^n$

$$1.\frac{2^n-1}{2-1} < 1025 \leq 1.\frac{2^{n+1}-1}{2-1}$$

$$2^n - 1 < 1025 \leq 2^{n+1} - 1$$

$$2^n < 1026 < 2^{n+1} - 1 \quad \ldots(1)$$

$$2^{n+1} \geq 1026 > 1024$$

$$2^{n+1} > 2^{10}$$

$$n + 1 > 10$$

$$n > 9$$

$\therefore$ 1025^{th} term is 2^{10}

28. Find the sum of 0.6 + 0.66 + 0.666 + ...upto n terms.

Sol :

Let S_n = 0.6 + 0.66 + 0.666 + ...upto n terms.

= 6 [0.1 + 0.11 + 0.111 + ...upto n terms]

$= \frac{6}{9}$ [0.9 + 0.99+ 0.999 + ...upto n terms]

(multiply and divide by 9)

$$= \frac{2}{3}\left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots\text{upto n terms}\right]$$

$$= \frac{2}{3}\left[\left(1-\frac{1}{10}\right)\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots\text{ up to n terms}\right]$$

$$= \frac{2}{3}\left[n - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \ldots\text{ upto n terms}\right)\right]$$

$$= \frac{2}{3}\left[n - \frac{\frac{1}{10}\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}\right]$$

$$= \frac{2}{3}\left[n - \frac{1}{9}\left(1-\frac{1}{10^n}\right)\right]$$

$$0.6 + 0.666 + \ldots \text{ n terms} = \left[\frac{2n}{3} - \frac{2}{27}\left(1-\frac{1}{10^n}\right)\right]$$

29. Find the sum to n terms 1 × 2 + 2 × 3 + 3 × 4 + 4 × 5 + ...

Sol :

$(1 \times 2) + (2 \times 3) + (3 \times 4) + (4 \times 5) + \ldots = 2 + 6 + 12 + 20 + \ldots + \text{n terms}$

$$= (1^2 + 1) + (2^2 + 2) + (3^2 + 3) + \ldots + \text{n terms}$$

$$= (1^2 + 2^2 + 3^2 + \ldots \text{n terms}) + (1 + 2 + 3 + \ldots + \text{n terms})$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right]$$

$$= \frac{n(n+1)}{2} \times \frac{2n+4}{3}$$

$$= \frac{n(n+1)(n+2)}{3}$$

30. Find the sum to n terms $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots +$ n terms

Sol :

$(3 \times 1^2) + (5 \times 2^2) + (7 \times 3^2) + \ldots + \text{n terms} = 3 + 20 + 63 + \ldots + \text{n terms}$

$= (2\times 1^3+1^2)+(2\times 2^3+2^2)+(2\times 3^3+3^2)+\ldots+(2\times n^3+n^2)$

$= 2(1^3+2^3+3^3+\ldots+n^3)+(1^2+2^2+\ldots+n^2)$

$$= 2\left[\frac{n(n+1)}{2}\right]^2+\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2[n(n+1)]^2}{4}+\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1)+\frac{(2n+1)}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3(n^2+n)+(2n+1)}{3}\right]$$

$$= \frac{n(n+1)(3n^2+3n+2n+1)}{6}$$

$$= \frac{n(n+1)(3n^2+5n+1)}{6}$$

31. Find the sum to n terms $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$

Sol :

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots \text{ n terms}$$

$$= \frac{1}{4}[1^2+2.1+1]+\frac{1}{4}[2^2+2.2+1]+\ldots+\frac{1}{4}(n^2+2n+1)$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}+\frac{2n(n+1)}{2}+\frac{n}{1}\right]$$

$$= \frac{n}{4}\left[\frac{(n+1)(2n+1)}{6}+(n+1)+\frac{1}{1}\right]$$

$$= \frac{n}{4}\left[\frac{2n^2+3n+1+6n+6+6}{6}\right]$$

$$= \frac{n}{24}[2n^2+9n+13]$$

32. Find the value of the sum 3 + 5 + 6 + 9 + 10 + 12 + 15 + 18 + 20 + … + 100

Sol :

$3+5+6+9+10+12+15+18+20+\ldots+100$

$= (3+6+9+12+15+18+\ldots 99)+(5+10+15+20+\ldots+100)-(15+30+\ldots 90)$

$= 3(1+2+3+\ldots+33)+5(1+2+3+\ldots+20)-15(1+2+\ldots+6)$

$$= 3\left(\frac{33\times 34}{2}\right)+5\left(\frac{20\times 21}{2}\right)-15\left(\frac{6\times 7}{2}\right)$$

$= (3\times 33\times 17)+(5\times 10\times 21)-(15\times 3\times 7)$

$= 1683+1050-315$

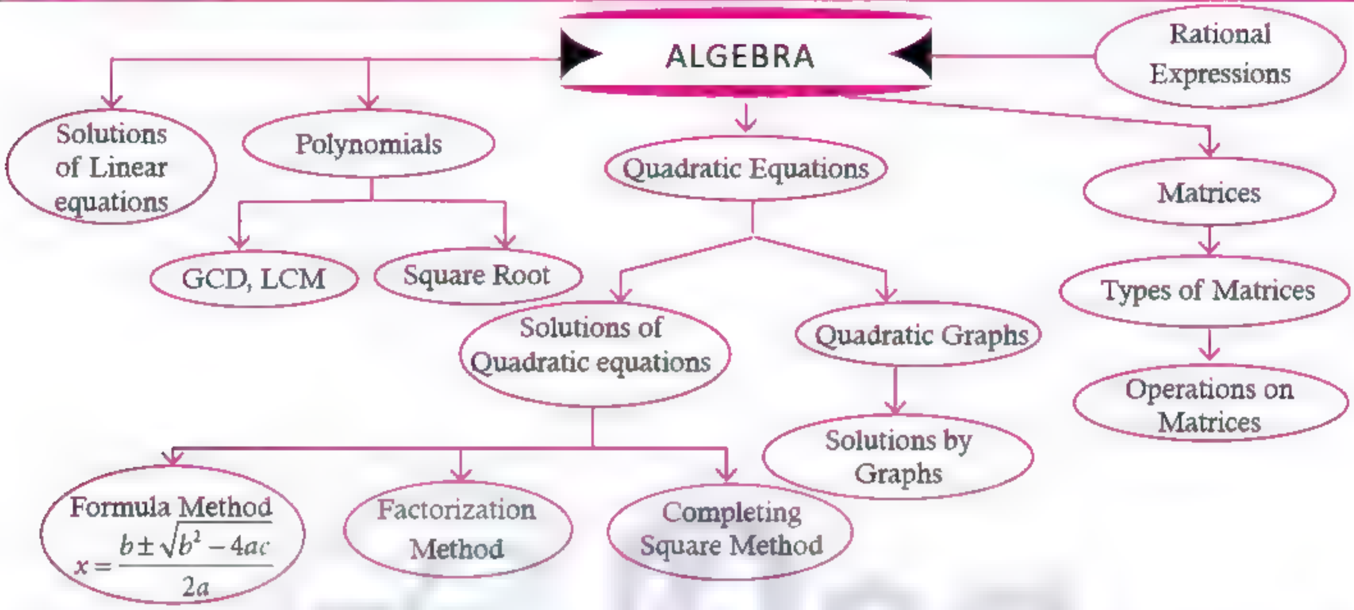
$= 2733-315$

$= 2418.$

UNIT 3

ALGEBRA

MIND MAP



SIMULTANEOUS LINEAR EQUATIONS IN THREE VARIABLES

Key Points

- Any first degree equation containing two variables x and y is called a linear equation in two variables. The general form of linear equation in two variables x and y is $ax + by + c = 0$ where a, b, c are real numbers.
- A linear equation in two variables represents a straight line in xy-plane.
- A linear equation with three variables x, y and z will be of the form $ax + by + cz + d = 0$ where a, b, c, d are real numbers and atleast one a, b, c is non-zero.
- A linear equation in three variables represents a plane.
- A system of equations can have unique solution (or) infinitely many solutions (or) No solutions.
- The system of equations has no solution if any step comes as $0 = 1$ while solving.
- The system of equations has infinitely many solutions if any step comes as $0 = 0$ while solving.

Worked Examples

3.1 The father's age is six times his son's age. Six years hence the age of father will be four times his son's age. Find the present ages (in years) of the son and father respectively.

Sol : Let present age of father be x years and present age of son be y years

Given, $x = 6y$ ... (1)

$x + 6 = 4(y + 6)$ ... (2)

Substituting (1) in (2), $6y + 6 = 4(y + 6)$

$6y + 6 = 4y + 24 \Rightarrow y = 9$

Therefore, son's age = 9 years and

father's age = 54 years.

3.2 Solve $2x - 3y = 6, x + y = 1$

Sol :

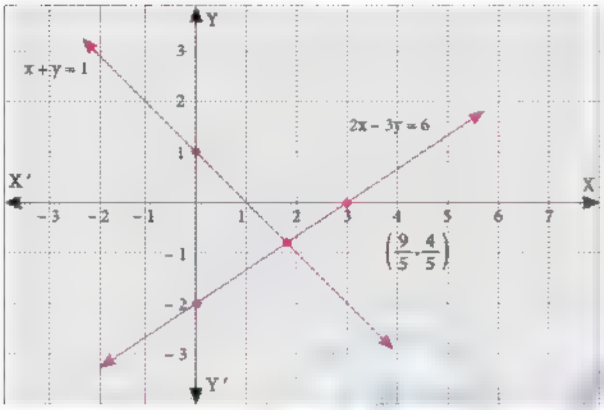
$2x - 3y = 6$ ... (1)

$x + y = 1$ ... (2)

$(1) \times 1 \Rightarrow 2x - 3y = 6$

$(2) \times 2 \Rightarrow 2x + 2y = 2$

$-5y = 4 \Rightarrow y = \frac{-4}{5}$



Substituting $y = \frac{-4}{5}$ in (2),

$x - \frac{4}{5} = 1 \Rightarrow x = \frac{9}{5}$

Therefore, $x = \frac{9}{5}, y = \frac{-4}{5}$

3.3 Solve the following system of linear equations in three variables $3x - 2y + z = 2, 2x + 3y - z = 5, x + y + z = 6$

Sol :

$3x - 2y + z = 2$ ... (1)

$2x + 3y - z = 5$ ... (2)

$x + y + z = 6$ ... (3)

Adding (1) and (2), $3x - 2y + z = 2$

$2x + 3y - z = 5$ (+)

$5x + y = 7$ ... (4)

Adding (2) and (3), $2x + 3y - z = 5$

$x + y + z = 6$ (+)

$3x + 4y = 11$ ... (5)

$4 \times (4) - (5)$ $20x + 4y = 28$

$3x + 4y = 11$ (−)

$17x = 17$

$\Rightarrow x = 1$

Substituting $x = 1$ in (4), $5 + y = 7 \Rightarrow y = 2$

Substituting $x = 1, y = 2$ in (3),

$1 + 2 + z = 6 \Rightarrow z = 3$

Therefore, $x = 1, y = 2, z = 3$.

3.4. In an interschool Atheletic meet, with 24 individual-events, securing a total of 56 points, a first place provides 5 points, a second place provides 3 points and a third place secures 1 point. Having as many third place finishers as first and second place finishers, find how many Athelets finished in each place.

Sol :

Let, the number of I, II and III place finishers be x, y & z respectively.

Total number of events = 24 ; Total number of points = 56.

Hence, the linear equations in three variables are

$x + y + z = 24$ ... (1)

$5x + 3y + z = 56$ ....(2)

$x + y = z$ ...(3)

Substituting (3) in (1) we get,

$z + z = 24 \Rightarrow z = 12$

Therefore, (3) $\Rightarrow$ $x + y = 12$

(2) $\Rightarrow$ $5x + 3y = 44$

$3 \times (3) \Rightarrow$ $3x + 3y = 36$ (−)

$2x = 8 \Rightarrow x = 4$

Substituting $x = 4, z = 12$ in (3) we get,

$y = 12 - 4 = 8$

Therefore, number of first place finishers is $x = 4$;

number of second place finishers is $y = 8$;

number of third place finishers is $z = 12$

3.5 Solve $x + 2y - z = 5; x - y + z = -2; -5x - 4y + z = -11$

Sol :

Let, $x + 2y - z = 5$ ... (1)

$x - y + z = -2$ ... (2)

$-5x - 4y + z = -11$ ...(3)

Adding (1) and (2) $\Rightarrow$

$x + 2y - z = 5$

$x - y + z = -2$ (+)

$2x + y = 3$ ... (4)

Subtracting (2) and (3),

$x - y + z = 2$

$-5x - 4y + z = -11$ (−)

$6x + 3y = 9$

Dividing by 3 $\Rightarrow$ $2x + y = 3$ ... (5)

Subtracting (4) and (5),

$$\begin{array}{r} 2x + y = 3 \\ 2x + y = 3 \\ \hline 0 = 0 \\ \hline \end{array}$$

Here we arrive at an identity 0 = 0.
Hence the system has an infinite number of solutions.

3.6. Solve $3x + y - 3z = 1; -2x - y + 2z = 1; -x - y + z = 2.$

Sol :

$$\text{Let } 3x + y - 3z = 1 \quad \text{...(1)}$$
$$-2x - y + 2z = 1 \quad \text{... (2)}$$
$$-x - y + z = 2 \quad \text{... (3)}$$

Adding (1) and (2),

$$\begin{array}{r} 3x + y - 3z = 1 \\ -2x - y + 2z = 1 \quad (+) \\ \hline x \qquad - z = 2 \\ \hline \end{array} \quad \text{... (4)}$$

Adding (1) and (3),

$$\begin{array}{r} 3x + y - 3z = 1 \\ -x - y + z = 2 \ (+) \\ \hline 2x \qquad - 2z = 3 \\ \hline \end{array} \quad \text{... (5)}$$

$$\begin{array}{rr} & 2x - 2z = 3 \\ \text{Now, } (5) - 2 \times (4) \Rightarrow & 2x - 2z = 4 \quad (-) \\ \hline & 0 = -1 \\ \hline \end{array}$$

Here we arrive at a contradiction as 0 = −1
This means that the system is inconsistent and has no solution.

3.7. Solve $\frac{x}{2} - 1 = \frac{y}{6} + 1 = \frac{z}{7} + 2; \ \frac{y}{3} + \frac{z}{2} = 13$

Sol :

$$\text{Considering, } \frac{x}{2} - 1 = \frac{y}{6} + 1$$

$$\frac{x}{2} - \frac{y}{6} = 1 + 1$$

$$\Rightarrow \frac{6x - 2y}{12} = 2$$

$$\Rightarrow 3x - y = 12 \quad \text{... (1)}$$

$$\text{Considering, } \frac{x}{2} - 1 = \frac{z}{7} + 2$$

$$\frac{x}{2} - \frac{z}{7} = 1 + 2$$

$$\Rightarrow \frac{7x - 2z}{14} = 3$$

$$\Rightarrow 7x - 2z = 42 \quad \text{... (2)}$$

$$\text{Also, from } \frac{y}{3} + \frac{z}{2} = 13$$

$$\Rightarrow \frac{2y + 3z}{6} = 13$$

$$\Rightarrow 2y + 3z = 78 \quad \text{... (3)}$$

Eliminating z from (2) and (3)

$$\begin{array}{rr} (2) \times 3 \Rightarrow & 21x - 6z = 126 \\ (3) \times 2 \Rightarrow & 4y + 6z = 156 \\ \hline & 21x + 4y = 282 \\ (1) \times 4 & 12x - 4y = 48 \\ \hline & 33x \qquad = 330 \Rightarrow x = 10 \\ \hline \end{array}$$

Substituting x = 10 in (1), 30 − y = 12 ⇒ y = 18

Substituting x = 10 in (2), 70 − 2z = 42 ⇒ z = 14

Therefore, x = 10, y = 18, z = 14.

3.8 Solve :

$$\frac{1}{2x} + \frac{1}{4y} - \frac{1}{3z} = \frac{1}{4}; \frac{1}{x} = \frac{1}{3y}; \frac{1}{x} - \frac{1}{5y} + \frac{4}{z} = 2\frac{2}{15}$$

Sol :

$$\text{Let } \frac{1}{x} = p, \frac{1}{y} = q, \frac{1}{z} = r$$

$$\Rightarrow \quad \frac{p}{2} + \frac{q}{4} - \frac{r}{3} = \frac{1}{4}$$

$$\Rightarrow \quad p = \frac{q}{3}$$

$$\Rightarrow \quad p - \frac{q}{5} + 4r = 2\frac{2}{15} = \frac{32}{15}$$

$$6p + 3q - 4r = 3 \quad \text{... (1)}$$
$$3p = q \quad \text{... (2)}$$
$$15p - 3q + 60r = 32 \quad \text{... (3)}$$

Substituting (2) in (1) and (3)

$$\text{we get, } 15p - 4r = 3 \quad \text{... (4)}$$

$$(9) \Rightarrow \quad 6p + 60r = 32$$

$$\text{reduces to } \Rightarrow 3p + 30r = 16 \quad \text{... (5)}$$

Solving (4) and (5)

$$\begin{array}{r} 15p - \quad 4r = 3 \\ 15p + 150r = 80 \ (-) \\ \hline -154r = -77 \\ \hline \end{array} \Rightarrow r = \frac{1}{2}$$

Substituting $r = \frac{1}{2}$ in (4),

we get $15p - 2 = 3$

$\Rightarrow \quad p = \frac{1}{3}$

From (2), $q = 3p \Rightarrow q = 1$

Therefore, $x = \frac{1}{p} = 3, y = \frac{1}{q} = 1, \; z = \frac{1}{r} = 2.$

That is, $x = 3, y = 1, z = 2$

3.9. **The sum of thrice the first number, second number and twice the third number is 5. If thrice the second number is subtracted from the sum of first number and thrice the third we get 2. If the third number is subtracted from the sum of twice the first, thrice the second, we get 1. Find the numbers.**

Sol :

Let the three numbers be x, y, z

From the given data we get the following equations,

$$3x + y + 2z = 5 \quad \ldots (1)$$
$$x + 3z - 3y = 2 \quad \ldots (2)$$
$$2x + 3y - z = 1 \quad \ldots (3)$$

$$(1) \times 1 \Rightarrow 3x + y + 2z = 5$$
$$(2) \times 3 \Rightarrow 3x - 9y + 9z = 6 \quad (-)$$
$$10y - 7z = -1 \quad \ldots (4)$$

$$(1) \times 2 \Rightarrow 6x + 2y + 4z = 10$$
$$(3) \times 3 \Rightarrow 6x + 9y - 3z = 3 \quad (-)$$
$$-7y + 7z = 7 \quad \ldots (5)$$

Adding (4) and (5),
$$10y - 7z = -1$$
$$-7y + 7z = 7$$
$$3y = 6$$
$$\Rightarrow y = 2$$

Substituting $y = 2$ in (5), $-14 + 7z = 7 \Rightarrow z = 3$

Substituting $y = 2$ and $z = 3$ in (1),

$$3x + 2 + 6 = 5$$
$$\Rightarrow x = -1$$

Therefore, $x = -1, y = 2, z = 3$.

Progress Check

1. For a system of linear equations with three variables the minimum number of equations required to get unique solution is _____

Ans : 3

2. A system with _____ will reduce to identity.

Ans : infinitely many solutions.

3. A system with ________ will provide absurd equation.

Ans : No solution.

Thinking Corner

1. The number of possible solutions when solving system of linear equations in three variables are _____

Ans : 3

2. If three planes are parallel then the number of possible point(s) of intersection is / are _____

Ans : zero.

Exercise 3.1

1. Solve the following system of linear equations in three variables

(i) $x + y + z = 5, \; 2x - y + z = 9, \; x - 2y + 3z = 16$

(ii) $\frac{1}{x} - \frac{2}{y} + 4 = 0, \; \frac{1}{y} - \frac{1}{z} + 1 = 0, \; \frac{2}{z} + \frac{3}{x} = 14$

(iii) $x + 20 = \frac{3y}{2} + 10 = 2z + 5 = 110 - (y + z)$

Sol :

(i)
$$x + y + z = 5 \quad \ldots (1)$$
$$2x - y + z = 9 \quad \ldots (2)$$
$$x - 2y + 3z = 16 \quad \ldots (3)$$

Consider (1) and (2)
$$x + y + z = 5 \quad \ldots (1)$$
$$2x - y + z = 9 \; (+) \quad \ldots (2)$$
$$(1) + (2) \Rightarrow 3x + 2z = 14 \quad \ldots (4)$$

Consider (1) and (3)
$$x - 2y + 3z = 16 \quad \ldots (3)$$
$$(1) \times (2) \Rightarrow 2x + 2y + 2z = 10 \; (+) \quad \ldots (5)$$
$$(3) + (5) \Rightarrow 3x + 5z = 26 \quad \ldots (6)$$

Consider (4) and (6)

$$3x + 5z = 26 \quad ...(6)$$
$$3x + 2z = 14 \quad ...(4)$$
$$(6) - (4) \Rightarrow \quad 3z = 12$$
$$z = \frac{12}{3} = 4$$

Substituting z = 4 in (4)

$$3x + 2(4) = 14$$
$$3x + 8 = 14$$
$$3x = 14 - 8 = 6$$
$$3x = 6$$
$$x = \frac{6}{3} = 2$$

Substituting x = 2, z = 4 in (1)

$$2 + y + 4 = 5$$
$$y + 6 = 5$$
$$y = 5 - 6$$
$$y = -1$$

∴ Solution : x = 2, y = −1, z = 4

(ii)
$$\frac{1}{x} - \frac{2}{y} + 4 = 0$$
$$\frac{1}{y} - \frac{1}{z} + 1 = 0$$
$$\frac{2}{z} + \frac{3}{x} = 14$$

Let $\frac{1}{x} = a$, $\frac{1}{y} = b$ and $\frac{1}{z} = c$

∴ The equations become

$$a - 2b = -4 \quad ...(1)$$
$$b - c = -1 \quad ...(2)$$
$$3a + 2c = 14 \quad ...(3)$$
$$(1) + (2) + (3) \Rightarrow \quad 4a - b + c = 9 \quad ...(4)$$

Now, consider (4) and (2)

$$4a - b + c = 9 \quad ...(4)$$
$$b - c = -1 \quad ...(2)$$
$$(4) + (2) \Rightarrow \quad 4a = 8$$
$$a = \frac{8}{4} = 2$$

Substituting the value a = 2 in (1)

$$2 - 2b = -4$$
$$-2b = -4 - 2 = -6$$
$$b = \frac{6}{2} = 3$$

Substituting b = 3 in (2)

$$3 - c = -1$$
$$c = 4$$

Now a = 2, b = 3, c = 4

∴ Solution $x = \frac{1}{a} = \frac{1}{2}$,

$$y = \frac{1}{b} = \frac{1}{3},$$
$$z = \frac{1}{c} = \frac{1}{4}$$

(iii)
$$x + 20 = \frac{3y}{2} + 10$$
$$= 2z + 5$$
$$= 110 - (y + z)$$

Equating
$$x + 20 = \frac{3y}{2} + 10$$
$$2x + 40 = 3y + 20$$

On simplifying $\quad 2x - 3y = -20 \quad ...(1)$

Equating
$$\frac{3y}{2} + 10 = 2z + 5$$
$$3y + 20 = 4z + 10$$

On simplifying $\quad 3y - 4z = -10 \quad ...(2)$

Now equating $\quad 2z + 5 = 110 - (y + z)$

Simplifying, $\quad 3z + y = 105 \quad ...(3)$

Considering (2) and (3)

$$(3) \times 3 \Rightarrow \quad 3y + 9z = 315 \quad ...(4)$$
$$(2) \times 1 \Rightarrow \quad 3y - 4z = -10\ (+) \quad ...(2)$$
$$(4) - (2) \Rightarrow \quad 13z = 325$$
$$z = \frac{325}{13} = 25$$

Substituting in (3)

$$3(25) + y = 105$$
$$y = 105 - 75$$
$$y = 30$$

Substituting y = 30 in (1)

$$2x - 3(30) = -20$$
$$2x = -20 + 90$$
$$2x = 70$$
$$x = \frac{70}{2} = 35$$

∴ Solution : x = 35, y = 30, z = 25.

2. Discuss the nature of solutions of the following system of equations

(i) $x + 2y - z = 6$, $-3x - 2y + 5z = -12$, $x - 2z = 3$

(ii) $2y + z = 3(-x + 1)$, $-x + 3y - z = -4$, $3x + 2y + z = -\frac{1}{2}$

(iii) $\frac{y+z}{4} = \frac{z+x}{3} = \frac{x+y}{2}$, $x + y + z = 27$

Sol :

(i) $\quad x + 2y - z = 6 \quad ...(1)$

$$-3x - 2y + 5z = -12 \quad \text{... (2)}$$
$$x - 2z = 3 \quad \text{... (3)}$$

Consider (1) and (3)

$$x + 2y - z = 6 \quad \text{... (1)}$$
$$x \quad - 2z = 3\,(-) \quad \text{... (3)}$$

(1) – (3) ⇒ $2y + z = 3$ ... (4)

Consider (4) and (2)

$$2y + z = 3 \quad \text{... (4)}$$
$$-3x - 2y + 5z = -12\,(+) \quad \text{... (2)}$$

(4) + (2) ⇒ $-3x + 6z = -9$ ... (5)

Dividing equation (5) by (–3)

We get, $x - 2z = 3$ ... (6)

Now consider (6) and (3)

$$x - 2z = 3 \quad \text{... (6)}$$
$$x - 2z = 3 \quad \text{... (3)}$$

(6) – (3) ⇒ $0 = 0$ which is an Identity.

∴ The system of equations has infinitely many solutions.

(ii) $2y + z = 3\,(-x + 1)$

$$3x + 2y + z = 3 \quad \text{... (1)}$$
$$-x + 3y - z = -4 \quad \text{... (2)}$$
$$3x + 2y + z = -\frac{1}{2} \quad \text{... (3)}$$

Consider (1) and (3)

$$3x + 2y + z = 3 \quad \text{... (1)}$$
$$3x + 2y + z = -\frac{1}{2} \quad \text{... (2)}$$

(1) – (2) ⇒ $0 = \frac{7}{2}$ (or)

$$0 = 7$$

which is a contradiction.

∴ The system of equations has no solution.

(iii) $\frac{y+z}{4} = \frac{z+x}{3} = \frac{x+y}{2}$,

$$x + y + z = 27 \quad \text{... (3)}$$

Equating $\frac{y+z}{4} = \frac{z+x}{3}$

We get, $4x - 3y + z = 0$ ... (1)

Equating $\frac{y+z}{4} = \frac{x+y}{2}$

On simplifying, $2x + y - z = 0$ ... (2)

Consider (1) and (2)

$$4x - 3y + z = 0 \quad \text{...(1)}$$
$$2x + y - z = 0 \quad \text{...(2)}$$

(1) + (2) ⇒ $6x - 2y = 0$ ...(4)

Consider (2) and (3)

$$2x + y - z = 0 \quad \text{...(2)}$$
$$x + y + z = 27 \quad \text{...(3)}$$

(2) + (3) ⇒ $3x + 2y = 27$ ...(5)

Consider (4) and (5)

$$6x - 2y = 0 \quad \text{...(4)}$$
$$3x + 2y = 27 \quad \text{...(5)}$$
$$9x = 27$$
$$x = \frac{27}{9} = 3$$

Substituting the value x = 2 in (4)

$$6(3) - 2y = 0$$
$$-2y = -18$$
$$y = \frac{18}{2} = 9$$

Substituting $x = 3, y = 9$ in (3)

$$3 + 9 + z = 27$$
$$z = 27 - 12$$
$$z = 15$$

Solution: $x = 3, y = 9, z = 15$

3. Vani, her father and her grand father have an average age of 53. One-half of her grand father's age plus one-third of her father's age plus one fourth of Vani's age is 65. If 4 years ago Vani's grandfather was four times as old as Vani then how old are they all now?

Sol :

Let the present ages of Vani, her father and her grand father be x, y, z respectively.

∴ Given $\frac{x+y+z}{3} = 53$

$$\therefore x + y + z = 159 \quad \text{... (1)}$$
$$\frac{z}{2} + \frac{y}{3} + \frac{x}{4} = 65$$
$$6z + 4y + 3x = 780 \quad \text{... (2)}$$
$$(z - 4) = 4(x - 4)$$
$$4x - z = 12 \quad \text{... (3)}$$

Consider (1) and (3)

$$x + y + z = 159 \quad \text{... (1)}$$
$$4x \quad - z = 12 \quad \text{... (3)}$$

(1) + (3) ⇒ $5x \quad + y = 171$ ... (4)

Consider (2) and (3)

$$3x + 4y + 6z = 780 \quad \text{... (2)}$$

(3) × (6) ⇒ $24x \quad - 6z = 72$ ... (5)

(2) + (5) ⇒ $27x + 4y = 852$ ... (6)

Consider (4) and (6)

$$(4) \times 4 \Rightarrow \quad 20x + 4y = 684 \quad \ldots (7)$$
$$(6) \times 1 \Rightarrow \quad 27x + 4y = 852 \quad \ldots (6)$$
$$(7) - (6) \Rightarrow \quad -7x = -168$$
$$x = \frac{168}{7} = 24$$

Substituting $x = 24$ in (4)

$$5(24) + y = 171$$
$$y = 171 - 120 = 51$$

Substituting $x = 24$, $y = 51$ in (1)

$$24 + 51 + z = 159$$
$$z = 159 - 75 = 84$$

$\therefore$ Present age of Vani = 24 years

Present age of her father = 51 years

Present age of her grandfather = 84 years.

4. The sum of the digits of a three-digit number is 11. If the digits are reversed, the new number is 46 more than five times the old number. If the hundreds digit plus twice the tens digit is equal to the units digit, then find the original three digit number?

Sol :

Let the 100's digit be 'x', 10's be 'y' and Unit digit be 'z'

$\therefore$ The three digit number is $100x + 10y + z$

Now given,
$$x + y + z = 11 \quad \ldots (1)$$
$$100z + 10y + x = 5(100x + 10y + z) + 46$$

Simplifying
$$499x + 40y - 95z = -46 \quad \ldots (2)$$
$$x + 2y = z$$
$$\Rightarrow \quad x + 2y - z = 0 \quad \ldots (3)$$

Consider (1) and (3)
$$x + y + z = 11 \quad \ldots (1)$$
$$x + 2y - z = 0 \quad \ldots (3)$$
$$(1) + (3) \Rightarrow \quad 2x + 3y = 11 \quad \ldots (4)$$

Consider (1)
$$499x + 40y - 95z = -46 \quad \ldots (2)$$
$(1) \times 95 \Rightarrow$
$$95x + 95y + 95z = 1045 \quad \ldots (5)$$
$$594x + 135y = 999 \quad \ldots (6)$$

Consider (6) and (4)
$$594x + 135y = 999 \quad \ldots (6)$$
$$(4) \times 45 \Rightarrow \quad 90x + 135y = 495 \quad \ldots (7)$$
$$(6) - (7) \Rightarrow \quad 504x = 504$$
$$x = \frac{504}{504} = 1$$

Substituting the value $x = 1$ in (4)
$$2(1) + 3y = 11$$
$$3y = 11 - 2 = 9$$
$$y = \frac{9}{3} = 3$$

Substituting $x = 1$, $y = 3$ in (1)
$$1 + 3 + z = 11$$
$$z = 11 - 4 = 7$$

$\therefore x = 1, \ y = 3, z = 7$

$\therefore$ The original three digit number is 137

i.e., $100(1) + 10(3) + 1(7) = 100 + 30 + 7 = 137$

5. There are 12 pieces of five, ten and twenty rupee currencies whose total value is ₹ 105. But when first 2 sorts are interchanged in their numbers its value will be increased by ₹ 20. Find the number of currencies in each sort.

Sol :

Let the number of five, ten and twenty rupee currencies be x, y and z respectively.

$\therefore$ Given
$$x + y + z = 12 \quad \ldots (1)$$
$$5x + 10y + 20z = 105 \quad \ldots (2)$$
$$10x + 5y + 20z = 125 \quad \ldots (3)$$

Consider (2) and (3)
$$5x + 10y + 20z = 105 \quad \ldots (2)$$
$$10x + 5y + 20z = 125 \quad \ldots (3)$$
$$(2) - (3) \Rightarrow -5x + 5y = -20 \quad \ldots (4)$$

Consider (1) and (2)
$$(1) \times 20 \Rightarrow 20x + 20y + 20z = 240 \quad \ldots (5)$$
$$(2) \times 1 \Rightarrow 5x + 10y + 20z = 105 \quad \ldots (2)$$
$$(5) - (2) \Rightarrow 15x + 10y = 135 \quad \ldots (6)$$

Consider (4) and (6)
$$(4) \times 3 \Rightarrow \quad -15x + 15y = -60 \quad \ldots (7)$$
$$15x + 10y = 135 \quad \ldots (6)$$
$$25y = 75$$
$$y = \frac{75}{25} = 3$$

Substituting $y = 3$ in (4)
$$-5x + 5(3) = -20$$
$$-5x = -20 - 15$$
$$-5x = -35$$
$$x = \frac{35}{5} = 7$$

Substituting $x = 7$, $y = 3$ in (1)
$$7 + 3 + z = 12$$
$$z = 12 - 10 = 2$$

$\therefore$ The number of five rupee currencies 7

The number of ten rupee currencies 3

The number of twenty rupee currencies 2

GCD AND LCM OF POLYNOMIALS

Key Points

- GCD can be found out by using long division method.
- If f(x) and g(x) are both polynomials of same degree then the polynomial carrying the highest co efficient will be the dividend.
- Least Common Multiple: The Least Common Multiple of two or more algebraic expressions is the expression of lowest degree (or power) such that the expressions exactly divide it.
- Relationship between LCM and GCD: For two polynomials f(x) and g(x),
  f(x) × g(x) = [LCM of f(x), g(x)] × [GCD of f(x), g(x)]

Worked Examples

3.10 **Find the GCD of the polynomials $x^3 + x^2 - x + 2$ and $2x^3 - 5x^2 + 5x - 3$.**

Sol :

Let $f(x) = 2x^3 - 5x^2 + 5x - 3$ and
$g(x) = x^3 + x^2 - x + 2$

Divisor: $x^3 + x^2 - x + 2$, quotient: 2

$2x^3 - 5x^2 + 5x - 3$
$2x^3 + 2x^2 - 2x + 4$ (−)
$-7x^2 + 7x - 7$

$= -7(x^2 - x + 1)$

$-7(x^2 - x + 1) \neq 0$, note that -7 is not a divisor of g(x)

Now dividing $g(x) = x^3 + x^2 - x + 2$ by the new remainder (leaving the constant factor), we get

Divisor: $x^2 - x + 1$, quotient: x

$x^3 + x^2 - x + 2$
$x^3 - x^2 + x$ (−)
$2x^2 - 2x + 2$
$2x^2 - 2x + 2$ (−)
0

$= 2(x^2 - x + 1)$

Here, we get zero as remainder.
Therefore,
GCD $(2x^3 - 5x^2 + 5x - 3, x^3 + x^2 - x + 2) = x^2 - x + 1$

3.11. **Find the GCD of $6x^3 - 30x^2 + 60x - 48$ and $3x^3 - 12x^2 + 21x - 18$.**

Sol :

Let, $f(x) = 6x^3 - 30x^2 + 60x - 48$
$= 6(x^3 - 5x^2 + 10x - 8)$ and

$g(x) = 3x^3 - 12x^2 + 21x - 18$
$= 3(x^3 - 4x^2 + 7x - 6)$

Now, we shall find the GCD of $x^3 - 5x^2 + 10x - 8$ and $x^3 - 4x^2 + 7x - 6$

Divisor: $x^3 - 5x^2 + 10x - 8$, quotient: 1

$x^3 - 4x^2 + 7x - 6$
$x^3 - 5x^2 + 10x - 8$ (−)
$x^2 - 3x + 2$

Divisor: $x^2 - 3x + 2$, quotient: $x - 2$

$x^3 - 5x^2 + 10x - 8$
$x^3 - 3x^2 + 2x$ (−)
$-2x^2 + 8x - 8$
$-2x^2 + 6x - 4$ (−)
$2x - 4 = 2(x - 2)$

Divisor: $x - 2$, quotient: $x - 1$

$x^2 - 3x + 2$
$x^2 - 2x$ (−)
$-x + 2$
$-x + 2$ (−)
0

Here, we get zero as remainder.
GCD of leading coefficients 3 and 6 is 3.
Thus, GCD $(6x^3 - 30x^2 + 60x - 48, 3x^3 - 12x^2 + 21x - 18) = 3(x - 2)$.

3.12. **Find the LCM of the following**

(i) $8x^4 y^2, 48x^2 y^4$ **(ii)** $(5x - 10), (5x^2 - 20)$
(iii) $(x^4 - 1), x^2 - 2x + 1$
(iv) $x^3 - 27, (x - 3)^2, x^2 - 9$

Sol :

(i) $8x^4y^2$, $48x^2y^4$

First let us find the LCM of the numerical coefficients.

That is, LCM (8, 48) $= 2 \times 2 \times 2 \times 6 = 48$

Then find the LCM of the terms involving variables.

That is, LCM $(x^4y^2, x^2y^4) = x^4y^4$

Finally find the LCM of the given expression.

We conclude that the LCM of the given expression is the product of the LCM of the numerical coefficient and the LCM of the terms with variables.

Therefore, LCM $(8x^4y^2, 48x^2y^4) = 48x^4y^4$

(ii) $(5x - 10), (5x^2 - 20)$

$$5x - 10 = 5(x - 2)$$
$$5x^2 - 20 = 5(x^2 - 4) = 5(x + 2)(x - 2)$$

Therefore, LCM

$[(5x - 10), (5x^2 - 20)] = 5(x + 2)(x - 2)$

(iii) $(x^4 - 1), x^2 - 2x + 1$

$$x^4 - 1 = (x^2)^2 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$$
$$x^2 - 2x + 1 = (x - 1)^2$$

Therefore,

LCM $[(x^4 - 1), (x^2 - 2x + 1)] = (x^2 + 1)(x + 1)(x - 1)^2$

(iv) $x^3 - 27, (x - 3)^2, x^2 - 9$

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9);$$
$$(x - 3)^2 = (x - 3)^2;$$
$$(x^2 - 9) = (x + 3)(x - 3)$$

Therefore, LCM $[(x^3 - 27), (x - 3)^2, (x^2 - 9)]$

$= (x - 3)^2(x + 3)(x^2 + 3x + 9)$

Progress Check

1. **When two polynomials of same degree has to be divided, _____ should be considered to fix the dividend and divisor.**

   Ans : Highest coefficient

2. **If r (x) = 0 when f (x) is divided by g (x) then g (x) is called _____ of the polynomials.**

   Ans : Factor

3. **If f (x) = g (x) q(x) + r(x), _____ must be added to f (x) to make f(x) completely divisible by g(x).**

   Ans : q (x)

4. **If f(x) = g(x) q(x) + r(x), _____ must be subtracted to f(x) to make f(x) completely divisible by g(x).**

   Ans : r (x)

Thinking Corner

1. **Complete the factor tree for the given polynomials f(x) and g(x), Hence find their GCD and LCM.**

$f(x) = 2x^3 - 9x^2 - 32x - 21$

2x + 3		x + 1

$g(x) = 2x^3 - 7x^2 - 43x - 42$

		x + 2

GCD [f(x) and g(x)] = _____ ?

LCM [f(x) and g(x)] = _____ ?

Sol :

$f(x) = 2x^3 - 9x^2 - 32x - 21$

2x + 3	x – 7	x + 1

$g(x) = 2x^3 - 7x^2 - 43x - 42$

2x + 3	x – 7	x + 2

GCD [f(x) and g(x)] = $(2x + 3)(x - 7)$

LCM [f(x) and g(x)] = $(2x + 3)(x - 7)(x + 2)(x + 1)$

Exercise 3.2

1. **Find the GCD of the given polynomials**

   (i) $x^4 + 3x^3 - x - 3,\ x^3 + x^2 - 5x + 3$

   (ii) $x^4 - 1,\ x^3 - 11x^2 + x - 11$

   (iii) $3x^4 + 6x^3 - 12x^2 - 24x,\ 4x^4 + 14x^3 + 8x^2 - 8x$

   (iv) $3x^3 + 3x^2 + 3x + 3,\ 6x^3 + 12x^2 + 6x + 12$

Sol :

(i) $x^4 + 3x^3 - x - 3,\ x^3 + x^2 - 5x + 3$

Let us divide the highest degree polynomial by least degree polynomial and

Let $f(x) = x^4 + 3x^3 - x - 3$

$g(x) = x^3 + x^2 - 5x + 3$

Now, dividing f (x) by g (x).

$$\begin{array}{r|l} & \quad x+2 \\ \cline{2-2} x^3+x^2-5x+3 & x^4+3x^3+0x^2-x-3 \\ & x^4+x^3-5x^2+3x \qquad (-) \\ \cline{2-2} & \quad 2x^3+5x^2-4x-3 \\ & \quad 2x^3+2x^2-10x+6 \qquad (-) \\ \cline{2-2} & \qquad 3x^2+6x-9 \end{array}$$

$= 3\,(x^2+2x-3) \neq 0$

Since '3' is not the divisor of g (x), let us divide g (x) by x^2+2x-3

$$\begin{array}{r|l} & \quad x-1 \\ \cline{2-2} x^2+2x-3 & x^3+x^2-5x+3 \\ & x^3+2x^2-3x \qquad (-) \\ \cline{2-2} & \quad -x^2-2x+3 \\ & \quad -x^2-2x+3 \qquad (-) \\ \cline{2-2} & \qquad 0 \end{array}$$

Since, the Remainder is zero, the GCD is x^2+2x-3.

(ii) Let $f(x) = x^4-1$, $g(x) = x^3-11x^2+x-11$

Dividing f (x) by g (x)

$$\begin{array}{r|l} & \quad x+11 \\ \cline{2-2} x^3-11x^2+x-11 & x^4+0x^3+0x^2+0x-1 \\ & x^4-11x^3+x^2-11x \qquad (-) \\ \cline{2-2} & \quad 11x^3-x^2+11x-1 \\ & \quad 11x^3-121x^2+11x-121 \; (-) \\ \cline{2-2} & \qquad 120x^2+120 \end{array}$$

$= 120\,(x^2+1) \neq 0$

'120' is not the divisor of g (x).

So, dividing g (x) by x^2+1

$$\begin{array}{r|l} & \quad x-11 \\ \cline{2-2} x^2+1 & x^3-11x^2+x-11 \\ & x^3 \qquad\quad +x \qquad (-) \\ \cline{2-2} & \quad -11x^2-11 \\ & \quad -11x^2-11 \qquad (-) \\ \cline{2-2} & \qquad 0 \end{array}$$

Since, the Remainder is zero, the GCD is x^2+1.

(iii) Let $f(x) = 4x^4+14x^3+8x^2-8x$

$= 2\,(2x^4+7x^3+4x^2-4x)$

$g(x) = 3x^4+6x^3-12x^2-24x$

$= 3\,(x^4+2x^3-4x^2-8x)$

Dividing $2x^4+7x^3+4x^2-4x$ by $x^4+2x^3-4x^2-8x$

$$\begin{array}{r|l} & \quad 2 \\ \cline{2-2} x^4+2x^3-4x^2-8x & 2x^4+7x^3+4x^2-4x \\ & 2x^4+4x^3-8x^2-16x \qquad (-) \\ \cline{2-2} & \quad 3x^3+12x^2+12x \end{array}$$

$= 3\,(x^3+4x^2+4x)$

Now, dividing $x^4+2x^3-4x^2-8x$ by x^3+4x^2+4x

$$\begin{array}{r|l} & \quad x-2 \\ \cline{2-2} x^3+4x^2+4x & x^4+2x^3-4x^2-8x \\ & x^4+4x^3+4x^2 \qquad (-) \\ \cline{2-2} & \quad -2x^3-8x^2-8x \\ & \quad -2x^3-8x^2-8x \qquad (-) \\ \cline{2-2} & \qquad 0 \end{array}$$

Since, the remainder is zero, the GCD is x^3+4x^2+4x

i.e., $x\,(x^2+4x+4)$

$= x\,(x+2)^2$

(iv) Let $f(x) = 6x^3+12x^2+6x+12$

$= 6\,(x^3+2x^2+x+2)$

$g(x) = 3x^3+3x^2+3x+3$

$= 3\,(x^3+x^2+x+1)$

For '6' and '3' the GCD is 3

Now, dividing x^3+2x^2+x+2 by x^3+x^2+x+1

$$\begin{array}{r|l} & \quad 1 \\ \cline{2-2} x^3+x^2+x+1 & x^3+2x^2+x+2 \\ & x^3+x^2+x+1 \qquad (-) \\ \cline{2-2} & \quad x^2+1 \end{array}$$

Now, dividing (x^3+x^2+x+1) by (x^2+1)

$$\begin{array}{r|l} & \quad x+1 \\ \cline{2-2} x^2+1 & x^3+x^2+x+1 \\ & x^3 \qquad\quad +x \qquad (-) \\ \cline{2-2} & \quad x^2 \qquad +1 \\ & \quad x^2 \qquad +1 \qquad (-) \\ \cline{2-2} & \qquad 0 \end{array}$$

Since, the Remainder is zero, the GCD is $3\,(x^2+1)$

2. Find the LCM of the given expressions

(i) $4x^2y, 8x^3y^2$

(ii) $-9a^3b^2, 12a^2b^2c$

(iii) $16m, -12m^2n^2, 8n^2$

(iv) p^2-3p+2, p^2-4

(v) $2x^2-5x-3, 4x^2-36$

(vi) $(2x^2-3xy)^2, (4x-6y)^3, (8x^3-27y^3)$

Sol :

(i) $4x^2y, 8x^3y^2$

$$\text{LCM of } (4, 8) = 8$$
$$\text{LCM of } (x^2y, x^3y^2) = x^3y^2$$
$$\therefore \text{LCM of } (4x^2y, 8x^3y^2) = 8x^3y^2$$

(ii) $-9a^3b^2, 12a^2b^2c$

$$\text{LCM of } (-9, 12) = -36$$
$$\text{LCM of } (a^3b^2, a^2b^2c) = a^3b^2c$$
$$\therefore \text{LCM of } (-9a^3b^2, 12a^2b^2c) = -36a^3b^2c$$

(iii) $16m, -12m^2n^2, 8n^2$

$$\text{LCM of } (16, -12, 8) = -48$$
$$\text{LCM of } (m, m^2n^2, n^2) = m^2n^2$$
$$\therefore \text{LCM of } 16m, -12m^2n^2, 8n^2 = -48m^2n^2$$

(iv)
$$p^2-3p+2 = (p-1)(p-2)$$
$$p^2-4 = (p+2)(p-2)$$
$$\text{LCM} = (p-1)(p-2)(p+2)$$

(v)
$$2x^2-5x-3 = 2x^2-6x+x-3$$
$$= 2x(x-3)+1(x-3)$$
$$= (2x+1)(x-3)$$
$$4x^2-36 = 4(x^2-9)$$
$$= 4(x+3)(x-3)$$
$$\therefore \text{LCM} = 4(x+3)(x-3)(2x+1)$$

(vi)
$$(2x^2-3xy)^2 = [x(2x-3y)]^2$$
$$= x^2(2x-3y)^2$$
$$(4x-6y)^3 = [2(2x-3y)]^3$$
$$= 2^3(2x-3y)^3$$
$$8x^3-27y^3 = (2x)^3-(3y)^3$$
$$= (2x-3y)(4x^2+6xy+9y^2)$$
$$\therefore \text{LCM} = 2^3x^2(2x-3y)^3(4x^2+6xy+9y^2)$$
$$= 2^3x^2(2x-3y)^3(4x^2+6xy+9y^2)$$

RELATIONSHIP BETWEEN LCM AND GCD

Thinking Corner

1. Is $f(x) \times g(x) \times r(x) =$ LCM $[f(x), g(x), r(x)] \times$ GCD $[f(x), g(x), r(x)]$?

Sol :

$f(x) \times g(x) \times r(x) \neq$

LCM of $[f(x), g(x), r(x)] \times$ GCD of $[f(x), g(x), r(x)]$

Exercise 3.3

1. Find the LCM and GCD for the following and verify that $f(x) \times g(x) = \text{LCM} \times \text{GCD}$

(i) $21x^2y, 35xy^2$

(ii) $(x^3-1)(x+1), (x^3+1)$

(iii) $(x^2y+xy^2), (x^2+xy)$

Sol :

(i)
$$\text{Let } f(x) = 21x^2y$$
$$g(x) = 35xy^2$$
$$\text{GCD} = 7xy$$
$$\text{LCM of } 21, 35 = 105$$
$$\text{LCM of } x^2y, xy^2 = x^2y^2$$
$$\therefore \text{LCM} = 105x^2y^2$$
$$\text{Now, } f(x) \times g(x) = (21x^2y)(35xy^2)$$
$$= 735x^3y^3$$
$$\text{LCM} \times \text{GCD} = (105x^2y^2)(7xy)$$
$$= 735x^3y^3$$
$$\therefore f(x) \times g(x) = \text{LCM} \times \text{GCD}$$

Hence verified.

(ii)
$$f(x) = (x^3-1)(x+1)$$
$$= (x-1)(x^2+x+1)(x+1)$$
$$g(x) = x^3+1 = (x+1)(x^2-x+1)$$
$$\text{GCD} = x+1$$
$$\text{LCM} = (x+1)(x-1)(x^2+x+1)(x^2-x+1)$$
$$f(x) \times g(x) = (x^3-1)(x+1)(x^3+1)$$
$$= (x+1)((x^3)^2-(1)^2)$$
$$= (x+1)(x^6-1)$$
$$\text{LCM} \times \text{GCD} = (x+1)(x-1)(x^2+x+1)(x^2-x+1)(x+1)$$
$$= (x+1)(x^2-x+1)(x-1)(x^2+x+1)(x+1)$$
$$= (x^3+1)(x^3-1)(x+1)$$
$$= (x^6-1)(x+1)$$
$$\therefore f(x) \times g(x) = \text{LCM} \times \text{GCD}$$

Hence verified.

(iii)
$$\text{Let } f(x) = x^2y+xy^2 = xy(x+y)$$
$$g(x) = x^2+xy = x(x+y)$$
$$\text{GCD} = x(x+y)$$
$$\text{LCM} = xy(x+y)$$
$$f(x) \times g(x) = (x^2y+xy^2)(x^2+xy)$$
$$= xy(x+y)x(x+y)$$

$$= x^2y(x+y)^2$$
$$LCM \times GCD = xy(x+y)\,x(x+y)$$
$$= x^2y(x+y)^2$$
$$\therefore\ f(x)\times g(x) = LCM \times GCD$$

Hence verified.

2. Find the LCM of each pair of the following polynomials

(i) $a^2 + 4a - 12$, $a^2 - 5a + 6$ whose GCD is $a - 2$

(ii) $x^4 - 27a^3x$, $(x - 3a)^2$ whose GCD is $(x - 3a)$

Sol :

(i) Let $f(x) = a^2 + 4a - 12$

$g(x) = a^2 - 5a + 6$

Given GCD $= a - 2$

We know $f(x) \times g(x) = LCM \times GCD$

$$\therefore\ LCM = \frac{f(x)\times g(x)}{GCD}$$
$$= \frac{(a^2+4a-12)(a^2-5a+6)}{a-2}$$
$$= \frac{(a+6)(a-2)(a-3)(a-2)}{a-2}$$
$$\therefore\ LCM = (a-2)(a-3)(a+6)$$

(ii) Let $f(x) = x^4 - 27a^3x$

$$= x(x^3 - 27a^3)$$
$$= x(x^3 - (3a)^3)$$
$$= x(x-3a)(x^2+3ax+9a^2)$$
$$g(x) = (x-3a)^2 = (x-3a)(x-3a)$$

Given GCD $= (x - 3a)$

We know $f(x)\times g(x) = LCM \times GCD$

$$\therefore\ LCM = \frac{f(x)\times g(x)}{GCD}$$
$$= \frac{x(x-3a)(x^2+3ax+9a^2)(x-3a)^2}{x-3a}$$
$$\therefore\ LCM = x(x^2+3ax+9a^2)(x-3a)^2$$

3. Find the GCD of each pair of the following polynomials

(i) $12(x^4 - x^3)$, $8(x^4 - 3x^3 + 2x^2)$ whose LCM is $24x^3(x-1)(x-2)$

(ii) $(x^3 + y^3)$, $(x^4 + x^2y^2 + y^4)$ whose LCM is $(x^3 + y^3)(x^2 + xy + y^2)$

Sol :

(i) Let $f(x) = 12(x^4 - x^3)$

$$= 12x^3(x-1)$$
$$g(x) = 8(x^4 - 3x^3 + 2x^2)$$
$$= 8x^2(x^2 - 3x + 2)$$
$$= 8x^2(x-1)(x-2)$$

Given LCM $= 24x^3(x-1)(x-2)$

We know $f(x)\times g(x) = LCM \times GCD$

$$\therefore\ GCD = \frac{f(x)\times g(x)}{LCM}$$
$$= \frac{12x^3(x-1)\times 8x^2(x-1)(x-2)}{24x^3(x-1)(x-2)}$$
$$GCD = 4x^2(x-1)$$

(ii) Let $f(x) = x^3 + y^3$

$$= (x+y)(x^2 - xy + y^2)$$
$$g(x) = x^4 + x^2y^2 + y^4$$

Given LCM $= (x^3+y^3)(x^2+xy+y^2)$

$$\therefore\ GCD = \frac{f(x)\times g(x)}{LCM}$$
$$= \frac{(x^3+y^3)(x^4+x^2y^2+y^4)}{(x^3+y^3)(x^2+xy+y^2)}$$
$$= \frac{(x^2-xy+y^2)(x^2+xy+y^2)}{x^2+xy+y^2}$$
$$= x^2 - xy + y^2$$

4. Given the LCM and GCD of the two polynomials, find p (x), q (x) find the unknown polynomial in the following table

	LCM	GCD	p (x)	q (x)
(i)	$a^3 - 10a^2 + 11a + 70$	$a - 7$	$a^2 - 12a + 35$	
(ii)	$(x^2+y^2)(x^4+x^2y^2+y^4)$	$(x^2 - y^2)$		$(x^4 - y^4)(x^2 + y^2 - xy)$

Sol :

(i) Given LCM $= a^3 - 10a^2 + 11a + 70$

GCD $= a - 7$

$p(x) = a^2 - 12a + 35$

Now $p(x)\times q(x) = LCM \times GCD$

$$\therefore\ q(x) = \frac{LCM \times GCD}{p(x)}$$
$$\therefore\ q(x) = \frac{(a^3-10a^2+11a+70)(a-7)}{a^2-12a+35}$$
$$= \frac{(a-7)(a-5)(a+2)(a-7)}{(a-7)(a-5)}$$
$$= (a+2)(a-7)$$

(ii) Given LCM = $(x^2 + y^2)(x^4 + x^2y^2 + y^4)$

GCD = $x^2 - y^2$

$q(x) = (x^4 - y^4)(x^2 + y^2 - xy)$

Now $p(x) \times q(x) = \text{LCM} \times \text{GCD}$

$$\therefore p(x) = \frac{LCM \times GCD}{q(x)}$$

$$= \frac{(x^2+y^2)(x^4+x^2y^2+y^4)(x^2-y^2)}{(x^4-y^4)(x^2+y^2-xy)}$$

$$= \frac{(x^4-y^4)(x^2+xy+y^2)(x^2-xy+y^2)}{(x^4-y^4)(x^2+y^2-xy)}$$

$$= x^2 + xy + y$$

RATIONAL EXPRESSIONS

Key Points

- A polynomial is called a rational expression if it can be written in the form $\frac{p(x)}{q(x)}, q(x) \neq 0$.
- A rational expression is the ratio of two polynomials.
- A rational expression $\frac{p(x)}{q(x)}$ is said to be in its lowest form if GCD [p(x), q(x)] = 1.
- A number that makes a rational expression (in its lowest form) undefined is called an Excluded value.

Worked Examples

3.13. **Reduce the rational expressions to its lowest form.**

(i) $\frac{x-3}{x^2-9}$ (ii) $\frac{x^2-16}{x^2+8x+16}$

Sol :

(i) $\frac{x-3}{x^2-9} = \frac{x-3}{(x+3)(x-3)} = \frac{1}{x+3}$

(ii) $\frac{x^2-16}{x^2+8x+16} = \frac{(x+4)(x-4)}{(x+4)^2} = \frac{x-4}{x+4}$

3.14. **Find the excluded values of the following expressions (if any)**

(i) $\frac{x+10}{8x}$ (ii) $\frac{7p+2}{8p^2+13p+5}$ (iii) $\frac{x}{x^2+1}$

Sol :

(i) $\frac{x+10}{8x}$

The expression $\frac{x+10}{8x}$ is undefined when $8x = 0$ or $x = 0$. Hence the excluded value is 0.

(ii) $\frac{7p+2}{8p^2+13p+5}$

The expression $\frac{7p+2}{8p^2+13p+5}$ is undefined when

$8p^2 + 13p + 5 = 0$

that is, $(8p + 5)(p + 1) = 0$

$p = \frac{-5}{8}$, $p = -1$. The excluded values are $\frac{-5}{8}$ and -1.

(iii) $\frac{x}{x^2+1}$

Here $x^2 \geq 0$ for all x. Therefore, $x^2 + 1 \geq 0 + 1 = 1$.

Hence, $x^2 + 1 \neq 0$ for any x. Therefore, there can be no real excluded values for the given rational expression $\frac{x}{x^2+1}$.

Thinking Corner

1. **Are $x^2 - 1$ and $\tan x = \frac{\sin x}{\cos x}$ rational expressions?**

   Ans : No

2. **The number of excluded values of $\frac{x^3+x^2-10x+8}{x^4+8x^2-9}$ is _____.**

   Ans : 2.

Don

Exercise 3.4

1. Reduce each of the following rational expressions to its lowest form

(i) $\frac{x^2-1}{x^2+x}$ (ii) $\frac{x^2-11x+18}{x^2-4x+4}$

(iii) $\frac{9x^2+81x}{x^3+8x^2-9x}$ (iv) $\frac{p^2-3p-40}{2p^3-24p^2+64p}$

Sol :

(i) $$\frac{x^2-1}{x^2+x} = \frac{(x+1)(x-1)}{x(x+1)} = \frac{x-1}{x}$$

(ii) $$\frac{x^2-11x+18}{x^2-4x+4} = \frac{(x-9)(x-2)}{(x-2)^2} = \frac{x-9}{x-2}$$

(iii) $$\frac{9x^2+81x}{x^3+8x^2-9x} = \frac{9x(x+9)}{x(x^2+8x-9)} = \frac{9x(x+9)}{x(x+9)(x-1)} = \frac{9}{x-1}$$

(iv) $$\frac{p^2-3p-40}{2p^3-24p^2+64p} = \frac{(p-8)(p+5)}{2p(p^2-12p+32)} = \frac{(p-8)(p+5)}{2p(p-8)(p-4)} = \frac{p+5}{2p(p-4)}$$

2. Find the excluded values, if any of the following expressions.

(i) $\frac{y}{y^2-25}$ (ii) $\frac{t}{t^2-5t+6}$

(iii) $\frac{x^2+6x+8}{x^2+x-2}$ (iv) $\frac{x^3-27}{x^3+x^2-6x}$

Sol :

(i) $\frac{y}{y^2-25} = \frac{y}{(y+5)(y-5)}$ Which is undefined when y = – 5 and y = 5

∴ Excluded values = – 5, 5

(ii) $\frac{t}{t^2-5t+6} = \frac{t}{(t-2)(t-3)}$ is undefined when t = 2 and t = 3

∴ Excluded values are 2 and 3.

(iii) $$\frac{x^2+6x+8}{x^2+x-2} = \frac{(x+2)(x+4)}{(x+2)(x-1)}$$

$= \frac{x+4}{x-1}$ is undefined when x = 1

∴ Excluded value is 1.

(iv) $$\frac{x^3-27}{x^3+x^2-6x} = \frac{x^3-3^3}{x(x^2+x-6)}$$

$= \frac{(x-3)(x^2+3x+9)}{x(x+3)(x-2)}$ is undefined when x = 0, x = – 3 and x = 2

∴ Excluded values are 0, – 3, 2.

OPERATIONS OF RATIONAL EXPRESSIONS

Key Points

- If $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$ are two rational expressions, their product is $\frac{p(x)}{q(x)} \times \frac{r(x)}{s(x)} = \frac{p(x)\,r(x)}{q(x)\,s(x)}$

- If $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$ are two rational expressions, their quotient is $\frac{p(x)}{q(x)} \div \frac{r(x)}{s(x)} = \frac{p(x).s(x)}{q(x).r(x)}$

- For addition and subtraction of two rational expressions, by taking the LCM of denominators, we get the simplest form.

Worked Examples

3.15 (i) Multiply $\frac{x^3}{9y^2}$ by $\frac{27y}{x^5}$

(ii) Multiply $\frac{x^4b^2}{x-1}$ by $\frac{x^2-1}{a^4b^3}$

Sol :

(i) $$\frac{x^3}{9y^2}\times\frac{27y}{x^5}=\frac{3}{x^2y}$$

(ii) $$\frac{x^4b^2}{x-1}\times\frac{x^2-1}{a^4b^3}=\frac{x^4\times b^2}{x-1}\times\frac{(x+1)(x-1)}{a^4\times b^3}$$

$$=\frac{x^4(x+1)}{a^4b}$$

3.16. Divide:

(i) $\frac{14x^4}{y}\div\frac{7x}{3y^4}$

(ii) $\frac{x^2-16}{x+4}\div\frac{x-4}{x+4}$

(iii) $\frac{16x^2-2x-3}{3x^2-2x-1}\div\frac{8x^2+11x+3}{3x^2-11x-4}$

Sol :

(i) $$\frac{14x^4}{y}\div\frac{7x}{3y^4}=\frac{14x^4}{y}\times\frac{3y^4}{7x}=6x^3y^3$$

(ii) $$\frac{x^2-16}{x+4}\div\frac{x-4}{x+4}=\frac{(x+4)(x-4)}{(x+4)}\times\left(\frac{x+4}{x-4}\right)$$

$$=x+4$$

(iii) $$\frac{16x^2-2x-3}{3x^2-2x-1}\div\frac{8x^2+11x+3}{3x^2-11x-4}$$

$$=\frac{16x^2-2x-3}{3x^2-2x-1}\times\frac{3x^2-11x-4}{8x^2+11x+3}$$

$$=\frac{(8x+3)(2x-1)}{(3x+1)(x-1)}\times\frac{(3x+1)(x-4)}{(8x+3)(x+1)}$$

$$=\frac{(2x-1)(x-4)}{(x-1)(x+1)}=\frac{2x^2-9x+4}{x^2-1}$$

Progress Check

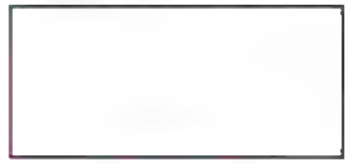
1. Find the unknown expression in the following figures.

Area $=\frac{(x-4)(x+3)}{3x-12}km^2$ Breadth = ?

Length $=\frac{x-3}{3}km$

Ans :

$$\text{Breadth}=\frac{Area}{length}$$



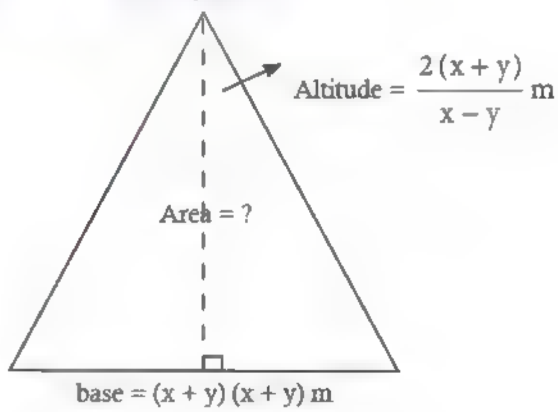
Rectangle

$$=\frac{\frac{(x-4)(x+3)}{(3x-12)}}{\frac{x-3}{3}}$$

$$=\frac{(x-4)(x+3)}{3(x-4)}\times\frac{3}{(x-3)}$$

$$=\frac{x+3}{x-3}$$

2. If x > y and if



Ans :

Triangle

Altitude h $=\frac{2(x+y)}{x-y}m$

Base b = (x + y) (x − y) m

Area = 1/2 bh

$$= \frac{1}{2}(x+y)(x-y)\frac{2(x+y)}{x-y}$$

$$= (x+y)^2 \text{ m}^2$$

Exercise 3.5

1. Simplify

(i) $\frac{4x^2y}{2z^2} \times \frac{6xz^3}{20y^4}$

(ii) $\frac{p^2-10p+21}{p-7} \times \frac{p^2+p-12}{(p-3)^2}$

(iii) $\frac{5t^3}{4t-8} \times \frac{6t-12}{10t}$

Sol :

(i) $\frac{4x^2y}{2z^2} \times \frac{6xz^3}{20y^4} = \frac{3x^3z}{5y^3}$

(ii) $\frac{p^2-10p+21}{p-7} \times \frac{p^2+p-12}{(p-3)^2}$

$$= \frac{(p-7)(p-3)}{(p-7)} \times \frac{(p+4)(p-3)}{(p-3)^2}$$

$$= p+4$$

(iii) $\frac{5t^3}{4t-8} \times \frac{6t-12}{10t} = \frac{5t^3}{4(t-2)} \times \frac{6(t-2)}{10t} = \frac{3}{4}t^2$

2. Simplify

(i) $\frac{x+4}{3x+4y} \times \frac{9x^2-16y^2}{2x^2+3x-20}$

(ii) $\frac{x^3-y^3}{3x^2+9xy+6y^2} \times \frac{x^2+2xy+y^2}{x^2-y^2}$

Sol :

(i) $\frac{x+4}{3x+4y} \times \frac{9x^2-16y^2}{2x^2+3x-20}$

$$= \frac{x+4}{3x+4y} \times \frac{(3x)^2-(4y)^2}{2x^2+8x-5x-20}$$

$$= \frac{x+4}{3x+4y} \times \frac{(3x+4y)(3x-4y)}{(2x-5)(x+4)}$$

$$= \frac{3x-4y}{2x-5}$$

(ii) $\frac{x^3-y^3}{3x^2+9xy+6y^2} \times \frac{x^2+2xy+y^2}{x^2-y^2}$

$$= \frac{(x-y)(x^2+xy+y^2)}{(3x+6y)(x+y)} \times \frac{(x+y)^2}{(x+y)(x-y)}$$

$$= \frac{x^2+xy+y^2}{3x+6y}$$

$$= \frac{x^2+xy+y^2}{3(x+2y)}$$

3. Simplify

(i) $\frac{2a^2+5a+3}{2a^2+7a+6} \div \frac{a^2+6a+5}{-5a^2-35a-50}$

(ii) $\frac{b^2+3b-28}{b^2+4b+4} \div \frac{b^2-49}{b^2-5b-14}$

(iii) $\frac{x+2}{4y} \div \frac{x^2-x-6}{12y^2}$

(iv) $\frac{12t^2-22t+8}{3t} \div \frac{3t^2+2t-8}{2t^2+4t}$

Sol :

(i) $\frac{2a^2+5a+3}{2a^2+7a+6} \div \frac{a^2+6a+5}{-5a^2-35a-50}$

$$= \frac{2a^2+5a+3}{2a^2+7a+6} \times \frac{-5(a^2+7a+10)}{a^2+6a+5}$$

$$= \frac{(2a+3)(a+1)}{(2a+3)(a+2)} \times \frac{-5(a+5)(a+2)}{(a+5)(a+1)}$$

$$= -5$$

(ii) $\frac{b^2+3b-28}{b^2+4b+4} \div \frac{b^2-49}{b^2-5b-14}$

$$= \frac{b^2+3b-28}{b^2+4b+4} \times \frac{b^2-5b-14}{b^2-49}$$

$$= \frac{(b+7)(b-4)}{(b+2)^2} \times \frac{(b-7)(b+2)}{(b-7)(b+7)}$$

$$= \frac{b-4}{b+2}$$

(iii) $\frac{x+2}{4y} \div \frac{x^2-x-6}{12y^2}$

$$= \frac{x+2}{4y} \times \frac{12y^2}{x^2-x-6}$$

$$= \frac{x+2}{4y} \times \frac{12y^2}{(x-3)(x+2)}$$

$$= \frac{3y}{x-3}$$

(iv) $\frac{12t^2-22t+8}{3t} \div \frac{3t^2+2t-8}{2t^2+4t}$

$$= \frac{2(6t^2-11t+4)}{3t} \times \frac{2t(t+2)}{3t^2+2t-8}$$

$$= \frac{2(3t-4)(2t-1)}{3t} \times \frac{2t(t+2)}{(3t-4)(t+2)}$$

$$= \frac{4(2t-1)}{3}$$

4. If x = $\frac{a^2+3a-4}{3a^2-3}$ and y = $\frac{a^2+2a-8}{2a^2-2a-4}$ find the value of x^2y^{-2}.

Sol :

$$\text{Given } x = \frac{a^2+3a-4}{3a^2-3}$$

$$= \frac{(a+4)(a-1)}{3(a^2-1)}$$

$$= \frac{(a+4)(a-1)}{3(a+1)(a-1)} = \frac{a+4}{3(a+1)}$$

$$y = \frac{a^2+2a-8}{2a^2-2a-4} = \frac{(a+4)(a-2)}{2(a^2-a-2)}$$

$$= \frac{(a+4)(a-2)}{2(a-2)(a+1)}$$

$$= \frac{(a+4)}{2(a+1)}$$

$$\text{Now } x^2y^{-2} = \left(\frac{(a+4)}{3(a+1)}\right)^2 \times \left(\frac{a+4}{2(a+1)}\right)^{-2}$$

$$= \left(\frac{a+4}{3(a+1)}\right)^2 \times \left(\frac{2(a+1)}{(a+4)}\right)^2$$

$$= \frac{(a+4)^2}{9(a+1)^2} \times \frac{4(a+1)^2}{(a+4)^2} = \frac{4}{9}.$$

5. If a polynomial p (x) = $x^2 - 5x - 14$ when divided by another polynomial q(x) gets reduced to $\frac{x-7}{x+2}$, find q(x).

Sol :

$$p(x) = x^2 - 5x - 14 \text{ and given}$$

$$p(x) \div q(x) = \frac{x-7}{x+2}$$

$$(x^2-5x-14) \div q(x) = \frac{x-7}{x+2}$$

$$\therefore q(x) = (x^2-5x-14) \times \frac{(x+2)}{(x-7)}$$

$$= (x-7)(x+2) \times \frac{(x+2)}{(x-7)}$$

$$= (x+2)^2$$

$$q(x) = x^2 + 4x + 4$$

ADDING AND SUBTRACTING RATIONAL EXPRESSIONS

Worked Examples

3.17 Find $\frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$

Sol :

$$\frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$$

$$= \frac{(x^2+20x+36)-(x^2+12x+4)}{x^2-3x-28}$$

$$= \frac{8x+32}{x^2-3x-28} = \frac{8(x+4)}{(x-7)(x+4)} = \frac{8}{x-7}$$

3.18. Simplify $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$

Sol :

$$\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$$

$$= \frac{1}{(x-2)(x-3)} + \frac{1}{(x-2)(x-1)} - \frac{1}{(x-5)(x-3)}$$

$$= \frac{(x-1)(x-5)+(x-3)(x-5)-(x-1)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x^2-6x+5)+(x^2-8x+15)-(x^2-3x+2)}{(x-1)(x-2)(x-3)(x-5)}$$

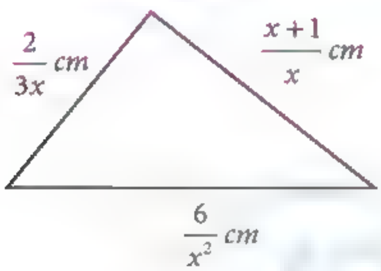
$$= \frac{x^2-11x+18}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x-9)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x-9)}{(x-1)(x-3)(x-5)}$$

Progress Check

1. Write an expression that represents the perimeter of the figure and simplify.



Ans :

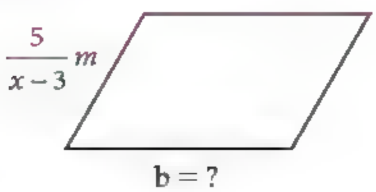
$$\text{Perimeter} = \frac{2}{3x}+\frac{x+1}{x}+\frac{6}{x^2}$$

$$= \frac{2(x)+(x+1)3x+6(3)}{3x^2}$$

$$= \frac{2x+3x^2+3x+18}{3x^2}$$

$$= \frac{3x^2+5x+18}{3x^2}$$

2. Find the base of the given parallelogram whose perimeter is $= \frac{4x^2+10x-50}{(x-3)(x+5)}$



Ans :

$$\text{Perimeter} = \frac{4x^2+10x-50}{(x-3)(x+5)}$$

$$2(l+b) = \frac{4x^2+10x-50}{(x-3)(x+5)}$$

$$2\left(\frac{5}{x-3}+b\right) = \frac{2(2x-5)(x+5)}{(x-3)(x+5)}$$

$$b = \frac{2x-5}{x-3}-\frac{5}{x-3}$$

$$= \frac{2x-5-5}{x-3} = \frac{2x-10}{x-3}$$

Thinking Corner

True or False

1. The sum of two rational expressions is always a rational expression.

Ans : True

2. The product of two rational expressions is always a rational expression.

Ans : True

Exercise 3.6

1. Simplify

(i) $\frac{x(x+1)}{x-2}+\frac{x(1-x)}{x-2}$

(ii) $\frac{x+2}{x+3}+\frac{x-1}{x-2}$

(iii) $\frac{x^3}{x-y}+\frac{y^3}{y-x}$

Sol :

(i) $\frac{x(x+1)}{x-2}+\frac{x(1-x)}{x-2}$

$$= \frac{x(x+1)+x(1-x)}{x-2}$$

$$= \frac{x[x+1+1-x]}{x-2}$$

$$= \frac{x(2)}{x-2} = \frac{2x}{x-2}$$

(ii) $\frac{x+2}{x+3}+\frac{x-1}{x-2}$

$$= \frac{(x+2)(x-2)+(x-1)(x+3)}{(x+3)(x-2)}$$

$$= \frac{x^2-4+x^2+2x-3}{(x+3)(x-2)}$$

$$= \frac{2x^2+2x-7}{(x+3)(x-2)}$$

(iii) $\frac{x^3}{x-y}+\frac{y^3}{y-x}$

$$= \frac{x^3}{x-y}-\frac{y^3}{x-y}$$

$$= \frac{x^3-y^3}{x-y}$$

$$= \frac{(x-y)(x^2+xy+y^2)}{x-y}$$

$$= x^2+xy+y^2$$

2. Simplify

(i) $\frac{(2x+1)(x-2)}{x-4}-\frac{(2x^2-5x+2)}{x-4}$

(ii) $\frac{4x}{x^2-1}-\frac{x+1}{x-1}$

Sol :

(i) $\frac{(2x+1)(x-2)}{x-4}-\frac{(2x^2-5x+2)}{x-4}$

$$= \frac{(2x+1)(x-2)-(2x-1)(x-2)}{x-4}$$

$$= \frac{(x-2)(2x+1-2x+1)}{x-4}$$

$$= \frac{(x-2)(2)}{x-4}$$

$$= \frac{2(x-2)}{x-4}$$

(ii) $\frac{4x}{x^2-1}-\frac{x+1}{x-1}$

$$= \frac{4x}{(x+1)(x-1)}-\frac{x+1}{x-1}$$

$$= \frac{4x-(x+1)(x+1)}{(x+1)(x-1)}$$

$$= \frac{4x-(x^2+2x+1)}{(x+1)(x-1)}$$

$$= \frac{4x-x^2-2x-1}{(x+1)(x-1)}$$

$$= \frac{-x^2+2x-1}{(x+1)(x-1)}$$

$$= -\frac{(x^2-2x+1)}{(x+1)(x-1)}$$

$$= -\frac{(x-1)^2}{(x+1)(x-1)}$$

$$= -\frac{(x-1)}{x+1} = \frac{1-x}{1+x}$$

3. Subtract $\frac{1}{x^2+2}$ from $\frac{2x^3+x^2+3}{(x^2+2)^2}$

Sol :

$$\frac{2x^3+x^2+3}{(x^2+2)^2}-\frac{1}{x^2+2} = \frac{2x^3+x^2+3-(x^2+2)}{(x^2+2)^2}$$

$$= \frac{2x^3+x^2+3-x^2-2}{(x^2+2)^2}$$

$$= \frac{2x^3+1}{(x^2+2)^2}$$

4. Which rational expression should be subtracted from $\frac{x^2+6x+8}{x^3+8}$ to get $\frac{3}{x^2-2x+4}$.

Sol :

$$\frac{x^2+6x+8}{x^3+8}-f(x) = \frac{3}{x^2-2x+4}$$

$$\therefore\ f(x) = \frac{x^2+6x+8}{x^3+2^3}-\frac{3}{x^2-2x+4}$$

$$= \frac{(x+4)(x+2)}{(x+2)(x^2-2x+4)}-\frac{3}{(x^2-2x+4)}$$

$$= \frac{x+4-3}{(x^2-2x+4)} = \frac{x+1}{x^2-2x+4}$$

5. If A = $\frac{2x+1}{2x-1}$, B = $\frac{2x-1}{2x+1}$ find $\frac{1}{A-B}-\frac{2B}{A^2-B^2}$

Sol :

Given A = $\frac{2x+1}{2x-1}$, B = $\frac{2x-1}{2x+1}$

We need to find $\frac{1}{A-B}-\frac{2B}{A^2-B^2}$

Let us simplify : $\frac{1}{A-B}-\frac{2B}{(A+B)(A-B)}$

$$= \frac{A+B-2B}{(A+B)(A-B)}$$

$$= \frac{A-B}{(A+B)(A-B)} = \frac{1}{A+B}$$

Now, $$A + B = \frac{2x+1}{2x-1} + \frac{2x-1}{2x+1}$$

$$= \frac{(2x+1)^2 + (2x-1)^2}{(2x-1)(2x+1)}$$

$$= \frac{4x^2+4x+1+4x^2-4x+1}{4x^2-1}$$

$$A + B = \frac{8x^2+2}{4x^2-1} = \frac{2(4x^2+1)}{4x^2-1}$$

$$\therefore \frac{1}{A-B} - \frac{2B}{A^2-B^2} = \frac{1}{A+B} = \frac{4x^2-1}{2(4x^2+1)}$$

6. If $A = \frac{x}{x+1}$, $B = \frac{1}{x+1}$, prove that

$$\frac{(A+B)^2+(A-B)^2}{\frac{A}{B}} = \frac{2(x^2+1)}{x(x+1)^2}$$

Sol :

Given, $A = \frac{x}{x+1}$, $B = \frac{1}{x+1}$

To prove : $\frac{(A+B)^2+(A-B)^2}{A/B} = \frac{2(x^2+1)}{x(x+1)^2}$

Simplifying $\frac{(A+B)^2+(A-B)^2}{A/B}$

$$= \frac{A^2+2AB+B^2+A^2-2AB+B^2}{A/B}$$

$$= \frac{2A^2+2B^2}{A/B}$$

$$= 2\frac{B}{A}(A^2+B^2)$$

Now $A^2 + B^2 = \left(\frac{x}{x+1}\right)^2 + \left(\frac{1}{x+1}\right)^2$

$$= \frac{x^2+1}{(x+1)^2}$$

$$\therefore 2\frac{B}{A}(A^2+B^2) = 2\frac{\left(\frac{1}{x+1}\right)}{\left(\frac{x}{x+1}\right)}\left[\frac{x^2+1}{(x+1)^2}\right]$$

$$= 2\frac{(x^2+1)}{x(x+1)^2}$$ Hence proved.

7. Pari needs 4 hours to complete a work. His friend Yuvan needs 6 hours to do the same work. How long will the job take if they work together?

Sol :

Let the work done be 'x'

Pari needs 4 hours and Yuvan needs 6 hours

Work done in 1 hr by Pari = $\frac{1}{4}$ of x = $\frac{x}{4}$

Work done in 1 hr by Yuvan = $\frac{1}{6}$ of x = $\frac{x}{6}$

$\therefore$ Work done by both = $\frac{x}{4} + \frac{x}{6}$

$$= \frac{3x+2x}{12} = \frac{5x}{12} = \frac{5}{12} \text{ of } x$$

$\therefore$ Time needed to complete the work together

$$= \frac{12}{5} hrs = \frac{12}{5} \times 60 = 2 \text{ hrs } 24 \text{ minutes.}$$

8. Iniya bought 50 kg of fruits consisting of apples and bananas. She paid twice as much per kg for the apple as she did for the banana. If Iniya bought ₹ 1800 worth of apples and ₹ 600 worth bananas, then how many kg of each fruit did she buy?

Sol :

Let the weight of Apples be 'x' kg.

Let the weight of bananas be 'y' kg.

Given $x + y = 50$ ... (1)

$$\frac{1800}{x} = 2\left(\frac{600}{y}\right) \Rightarrow \frac{3}{x} = \frac{2}{y}$$

$$3y = 2x$$

$$\Rightarrow x = \frac{3}{2}y \quad \text{... (2)}$$

Substituting in (1)

$$\frac{3}{2}y + y = 50$$

$$5y = 100$$

$$y = \frac{100}{5} = 20$$

Substituting in (2)

$$x = \frac{3}{2}(20) = 30$$

$\therefore$ Iniya bought 30 kg of Apples and 20 kg of bananas.

SQUARE ROOT OF POLYNOMIALS

Key Points

- The square root of a given positive number is another number which when multiplied with itself is the given number.
- The square root of a given expression P(x) is another expression Q(x) which when multiplied by itself gives P(x), that is Q(x).Q(x) = P(x)
- Square root of a given expression can be found out by using Factorization method and Division method.

Worked Examples

3.19. **Find the square root of the following expressions.**

(i) $256\,(x-a)^8\,(x-b)^4\,(x-c)^{16}\,(x-d)^{20}$

(ii) $\dfrac{144a^8b^{12}c^{16}}{81f^{12}g^4h^{14}}$

Sol :

(i) $\sqrt{256\,(x-a)^8(x-b)^4(x-c)^{16}(x-d)^{20}}$

$= 16\left|(x-a)^4\,(x-b)^2\,(x-c)^8\,(x-d)^{10}\right|$

(ii) $\sqrt{\dfrac{144a^8b^{12}c^{16}}{81f^{12}g^4h^{14}}} = \dfrac{4}{3}\left|\dfrac{a^4b^6c^8}{f^6g^2h^7}\right|$

3.20. **Find the square root of the following expressions**

(i) $16x^2 + 9y^2 - 24xy + 24x - 18y + 9$

(ii) $(6x^2 + x - 1)\,(3x^2 + 2x - 1)\,(2x^2 + 3x + 1)$

(iii) $\left[\sqrt{15}x^2+(\sqrt{3}+\sqrt{10})x+\sqrt{2}\right]\left[\sqrt{5}x^2+(2\sqrt{5}+1)x+2\right]\left[\sqrt{3}x^2+(\sqrt{2}+2\sqrt{3})x+2\sqrt{2}\right]$

Sol :

(i) $\sqrt{16x^2+9y^2-24xy+24x-18y+9}$

$= \sqrt{(4x)^2+(-3y)^2+(3)^2+2(4x)(-3y)+2(-3y)(3)+2(4x)(3)}$

$= \sqrt{(4x-3y+3)^2}$

$= |4x - 3y + 3|$

(ii) $\sqrt{(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)}$

$= \sqrt{(3x-1)(2x+1)(3x-1)(x+1)(2x+1)(x+1)}$

$= |(3x-1)\,(2x+1)\,(x+1)|$

(iii) First let us factorize the polynomials

$\sqrt{15}x^2+\left(\sqrt{3}+\sqrt{10}\right)x+\sqrt{2}$

$= \sqrt{15}x^2+\sqrt{3}x+\sqrt{10}x+\sqrt{2}$

$= \sqrt{3}x\left(\sqrt{5}x+1\right)+\sqrt{2}\left(\sqrt{5}x+1\right)$

$= \left(\sqrt{5}x+1\right)\times\left(\sqrt{3}x+\sqrt{2}\right)$

$\sqrt{5}x^2+\left(2\sqrt{5}+1\right)x+2 = \sqrt{5}x^2+2\sqrt{5}x+x+2$

$= \sqrt{5}x(x+2)+1(x+2)$

$= \left(\sqrt{5}x+1\right)(x+2)$

$\left[\sqrt{3}x^2+\left(\sqrt{2}+2\sqrt{3}\right)x+2\sqrt{2}\right]$

$= \sqrt{3}x^2+\sqrt{2}x+2\sqrt{3}x+2\sqrt{2}$

$= x\left(\sqrt{3}x+\sqrt{2}\right)+2\left(\sqrt{3}x+\sqrt{2}\right)$

$= (x+2)\left(\sqrt{3}x+\sqrt{2}\right)$

Therefore,

$\sqrt{\left[\sqrt{15}x^2+\left(\sqrt{3}+\sqrt{10}\right)x+\sqrt{2}\right]\left[\sqrt{5}x^2+\left(2\sqrt{5}+1\right)x+2\right]\left[\sqrt{3}x^2+\left(\sqrt{2}+2\sqrt{3}\right)x+2\sqrt{2}\right]}$

$= \sqrt{\left(\sqrt{5}x+1\right)\left(\sqrt{3}x+\sqrt{2}\right)\left(\sqrt{5}x+1\right)(x+2)\left(\sqrt{3}x+\sqrt{2}\right)(x+2)}$

$= \left|\left(\sqrt{5}x+1\right)\left(\sqrt{3}x+\sqrt{2}\right)(x+2)\right|$

Progress Check

1. Is $x^2 + 4x + 4$ a perfect square?

Ans : $x^2 + 4x + 4 = (x+2)^2$ it is a perfect square.

2. What is the value of x in $3\sqrt{x} = 9$?

Ans : $3\sqrt{x} = 9$; $\sqrt{x} = 9/3 = 3$; $x = 9$

3. The square root of $361x^4y^2$ is ______.

Ans : Square root of $361\,x^4y^2 = |19\,x^2y|$

4. $\sqrt{a^2x^2+2abx+b^2}$ = ______.

Ans : $\sqrt{a^2x^2+2abx+b^2} = (ax+b)^2 = |ax+b|$

5. If a polynomial is a perfect square then, its factors will be repeated ______ number of times (odd/even).

Ans : Even

Exercise 3.7

1. Find the square root of the following rational expressions

(i) $\dfrac{400x^4y^{12}z^{16}}{100x^8y^4z^4}$

(ii) $\dfrac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}$

(iii) $\dfrac{121(a+b)^8(x+y)^8(b-c)^8}{81(b-c)^4(a-b)^{12}(b-c)^4}$

Sol :

(i) Square root of $\dfrac{400x^4y^{12}z^{16}}{100x^8y^4z^4}$

$$= \sqrt{\frac{400x^4y^{12}z^{16}}{100x^8y^4z^4}} = \sqrt{\frac{4y^8z^{12}}{x^4}}$$

$$= 2\left|\frac{y^4z^6}{x^2}\right|$$

(ii) Square root of $\dfrac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}$

$$= \sqrt{\frac{7x^2+2\sqrt{14}x+2}{x^2-\frac{1}{2}x+\frac{1}{16}}}$$

$7x^2+2\sqrt{14}x+2$

$$= 7x^2+\sqrt{14}x+\sqrt{14}x+2$$

$$= \sqrt{7}\sqrt{7}x^2+\sqrt{2}\sqrt{7}x+\sqrt{2}\sqrt{7}x+\sqrt{2}\sqrt{2}$$

$$= \sqrt{7}x(\sqrt{7}x+\sqrt{2})+\sqrt{2}(\sqrt{7}x+\sqrt{2})$$

$$= \left(\sqrt{7}x+\sqrt{2}\right)\left(\sqrt{7}x+\sqrt{2}\right)$$

$$= \left(\sqrt{7}x+\sqrt{2}\right)^2$$

$x^2-\frac{1}{2}x+\frac{1}{16}$

$$= x^2-\frac{1}{4}x-\frac{1}{4}x+\frac{1}{16}$$

$$= x\left(x-\frac{1}{4}\right)-1/4\left(x-\frac{1}{4}\right)$$

$$= \left(x-\frac{1}{4}\right)^2$$

$$\therefore \sqrt{\frac{7x^2+2\sqrt{14}x+2}{x^2-1/2\,x+\frac{1}{16}}}$$

$$= \sqrt{\frac{\left(\sqrt{7}x+\sqrt{2}\right)^2}{(x-1/4)^2}}$$

$$= \sqrt{16\frac{\left(\sqrt{7}x+\sqrt{2}\right)^2}{(4x-1)^2}}$$

$$= 4\left|\frac{\sqrt{7}x+\sqrt{2}}{4x-1}\right|$$

(iii) Square root of $\dfrac{121(a+b)^8(x+y)^8(b-c)^8}{81(b-c)^4(a-b)^{12}(b-c)^4}$

$$= \sqrt{\frac{121(a+b)^8(x+y)^8(b-c)^8}{81(b-c)^4(a-b)^{12}(b-c)^4}}$$

$$= \sqrt{\frac{121(a+b)^8(x+y)^8}{81(a-b)^{12}}}$$

$$= \frac{11}{9}\left|\frac{(a+b)^4(x+y)^4}{(a-b)^6}\right|$$

2. Find the square root of the following

(i) $4x^2+20x+25$

(ii) $9x^2-24xy+30xz-40yz+25z^2+16y^2$

(iii) $1+\frac{1}{x^6}+\frac{2}{x^3}$

(iv) $(4x^2-9x+2)(7x^2-13x-2)(28x^2-3x-1)$

(v) $\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)$

Sol :

(i) Square root of $4x^2 + 20x + 25$

$$\sqrt{4x^2 + 20x + 25} = \sqrt{4x^2 + 10x + 10x + 25}$$
$$= \sqrt{2x(2x+5) + 5(2x+5)}$$
$$= \sqrt{(2x+5)(2x+5)}$$
$$= \sqrt{(2x+5)^2}$$
$$\therefore \sqrt{4x^2+20x+25} = |2x+5|$$

(ii) Square root of

$9x^2 - 24xy + 30xz - 40yz + 25z^2 + 16y^2$

Rearranging the terms

$9x^2 + 16y^2 + 25z^2 - 24xy - 40yz + 30xz$

$= (3x)^2 + (-4y)^2 + (5z)^2 + 2(3x)(-4y) + 2(-4y)(5z) + 2(3x)(5z)$

$= (3x - 4y + 5z)^2$

$$\therefore \sqrt{9x^2-24xy+30xz-40yz+25z^2+16y^2}$$
$$= \sqrt{(3x-4y+5z)^2}$$
$$= |3x - 4y + 5z|$$

(iii) Square root of $1+\dfrac{1}{x^6}+\dfrac{2}{x^3}$

$$\text{Now } \sqrt{1+\frac{2}{x^3}+\frac{1}{x^6}} = \sqrt{(1)^2+2\left(\frac{1}{x^3}\right)+\left(\frac{1}{x^3}\right)^2}$$
$$\therefore \sqrt{1+\frac{1}{x^6}+\frac{2}{x^3}} = \sqrt{\left(1+\frac{1}{x^3}\right)^2}$$
$$= \left|\left(1+\frac{1}{x^3}\right)\right|$$

(iv) Square Root of

$(4x^2 - 9x + 2)(7x^2 - 13x - 2)(28x^2 - 3x - 1)$

$$\begin{aligned}
4x^2 - 9x + 2 &= 4x^2 - 8x - x + 2 \\
&= 4x(x-2) - 1(x-2) \\
&= (4x-1)(x-2) \\
7x^2 - 13x - 2 &= 7x^2 - 14x + x - 2 \\
&= 7x(x-2) + 1(x-2) \\
&= (7x+1)(x-2) \\
28x^2 - 3x - 1 &= 28x^2 - 7x + 4x - 1 \\
&= 7x(4x-1) + 1(4x-1) \\
&= (7x+1)(4x-1)
\end{aligned}$$

$$\therefore \sqrt{(4x^2-9x+2)(7x^2-13x-2)(28x^2-3x-1)}$$
$$= \sqrt{(4x-1)(x-2)(7x+1)(x-2)(7x+1)(4x-1)}$$
$$= \sqrt{(4x-1)^2(x-2)^2(7x+1)^2}$$
$$= |(4x-1)(x-2)(7x+1)|$$

(v) Square root of

$$\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)$$

$$\begin{aligned}
2x^2+\frac{17}{6}x+1 &= \frac{12x^2+17x+6}{6} \\
&= \frac{12x^2+9x+8x+6}{6} \\
&= \frac{3x(4x+3)+2(4x+3)}{6} \\
&= \frac{(4x+3)(3x+2)}{6} \\
\frac{3}{2}x^2+4x+2 &= \frac{3x^2+8x+4}{2} \\
&= \frac{3x^2+6x+2x+4}{2} \\
&= \frac{3x(x+2)+2(x+2)}{2} \\
&= \frac{(x+2)(3x+2)}{2} \\
\frac{4}{3}x^2+\frac{11}{3}x+2 &= \frac{4x^2+11x+6}{3} \\
&= \frac{4x^2+8x+3x+6}{3} \\
&= \frac{4x(x+2)+3(x+2)}{3} \\
&= \frac{(x+2)(4x+3)}{3}
\end{aligned}$$

$$\therefore \sqrt{\left(2x^2+\frac{17}{6}x+1\right)\left(\frac{3}{2}x^2+4x+2\right)\left(\frac{4}{3}x^2+\frac{11}{3}x+2\right)}$$
$$= \sqrt{\frac{(4x+3)(3x+2)}{6}\,\frac{(x+2)(3x+2)}{2}\,\frac{(x+2)(4x+3)}{3}}$$
$$= \sqrt{\frac{(4x+3)^2(x+2)^2(3x+2)^2}{36}}$$
$$= \frac{1}{6}\,|(4x+3)(x+2)(3x+2)|$$

FINDING THE SQUARE ROOT OF POLYNOMIAL BY DIVISION METHOD

Worked Examples

3.21. **Find the square root of $64x^4 - 16x^3 + 17x^2 - 2x + 1$**

Sol :

$$\begin{array}{r|l}
 & 8x^2 - x + 1 \\ \hline
8x^2 & 64x^4 - 16x^3 + 17x^2 - 2x + 1 \\
 & 64x^4 \qquad (-) \\ \hline
16x^2 - x & -16x^3 + 17x^2 \\
 & -16x^3 + x^2 \qquad (-) \\ \hline
16x^2 - 2x + 1 & 16x^2 - 2x + 1 \\
 & 16x^2 - 2x + 1 \ (-) \\ \hline
 & 0
\end{array}$$

Therefore, $\sqrt{64x^4 - 16x^3 + 17x^2 - 2x + 1}$

$= |8x^2 - x + 1|$

3.22. **Find the square root of the expression**

$4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2}$

Sol :

$$\begin{array}{r|l}
 & 2\frac{x}{y} + 5 - 3\frac{y}{x} \\ \hline
2\frac{x}{y} & 4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \\
 & \frac{4x^2}{y^2} \qquad (-) \\ \hline
4\frac{x}{y} + 5 & 20\frac{x}{y} + 13 \\
 & 20\frac{x}{y} + 25 \qquad (-) \\ \hline
4\frac{x}{y} + 10 - \frac{3y}{x} & -12 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \\
 & -12 - 30\frac{y}{x} + 9\frac{y^2}{x^2} \ (-) \\ \hline
 & 0
\end{array}$$

Hence, $\sqrt{4\frac{x^2}{y^2} + 20\frac{x}{y} + 13 - 30\frac{y}{x} + 9\frac{y^2}{x^2}}$

$= \left|2\frac{x}{y} + 5 - 3\frac{y}{x}\right|$

3.23. **If $9x^4 + 12x^3 + 28x^2 + ax + b$ is a perfect square, find the values of a and b.**

Sol :

$$\begin{array}{r|l}
 & 3x^2 + 2x + 4 \\ \hline
3x^2 & 9x^4 + 12x^3 + 28x^2 + ax + b \\
 & 9x^4 \qquad (-) \\ \hline
6x^2 + 2x & 12x^3 + 28x^2 \\
 & 12x^3 + 4x^2 \qquad (-) \\ \hline
6x^2 + 4x + 4 & 24x^2 + ax + b \\
 & 24x^2 + 16x + 16 \ (-) \\ \hline
 & 0
\end{array}$$

Because the given polynomial is a perfect square

$a - 16 = 0,\ b - 16 = 0$

Therefore, $a = 16,\ b = 16$.

Exercise 3.8

1. **Find the square root of the following polynomials by division method**

   (i) $x^4 - 12x^3 + 42x^2 - 36x + 9$

   (ii) $4x^4 - 28x^3 + 37x^2 + 42x + 9$

   (iii) $16x^4 + 8x^2 + 1$

   (iv) $121x^4 - 198x^3 - 183x^2 + 216x + 144$

Sol :

(i) $x^4 - 12x^3 + 42x^2 - 36x + 9$

$$\begin{array}{r|l}
 & x^2 - 6x + 3 \\ \hline
x^2 & x^4 - 12x^3 + 42x^2 - 36x + 9 \\
 & x^4 \qquad (-) \\ \hline
2x^2 - 6x & -12x^3 + 42x^2 \\
 & -12x^3 + 36x^2 \qquad (-) \\ \hline
2x^2 - 12x + 3 & 6x^2 - 36x + 9 \\
 & 6x^2 - 36x + 9 \ (-) \\ \hline
 & 0
\end{array}$$

$\therefore \sqrt{x^4 - 12x^3 + 42x^2 - 36x + 9} = |x^2 - 6x + 3|$

(ii) $4x^4 - 28x^3 + 37x^2 + 42x + 9$

$$\begin{array}{r|l}
 & 2x^2-7x-3 \\ \hline
2x^2 & 4x^4-28x^3+37x^2+42x+9 \\
 & 4x^4 \qquad (-) \\ \hline
4x^2-7x & -28x^3+37x^2 \\
 & -28x^3+49x^2 \qquad (-) \\ \hline
4x^2-14x-3 & -12x^2+42x+9 \\
 & -12x^2+42x+9 \qquad (-) \\ \hline
 & 0
\end{array}$$

$\therefore \sqrt{4x^4-28x^3+37x^2+42x+9} = |2x^2-7x-3|$

(iii) $16x^4+8x^2+1$

$$\begin{array}{r|l}
 & 4x^2+1 \\ \hline
4x^2 & 16x^4+8x^2+1 \\
 & 16x^4 \qquad (-) \\ \hline
8x^2+1 & 8x^2+1 \\
 & 8x^2+1 \qquad (-) \\ \hline
 & 0 \\ \hline
\end{array}$$

$\therefore \sqrt{16x^4+8x^2+1} = |4x^2+1|$

(iv) $121x^4-198x^3-183x^2+216x+144$

$$\begin{array}{r|l}
 & 11x^2-9x-12 \\ \hline
11x^2 & 121x^4-198x^3-183x^2+216x+144 \\
 & 121x^4 \qquad (-) \\ \hline
22x^2-9x & -198x^3-183x^2 \\
 & -198x^3+81x^2 \qquad (-) \\ \hline
22x^2-18x-12 & -264x^2+216x+144 \\
 & -264x^2+216x+144 \; (-) \\ \hline
 & 0
\end{array}$$

$\therefore \sqrt{121x^4-198x^3-183x^2+216x+144} = |11x^2-9x-12|$

2. Find the square root of the expression $\frac{x^2}{y^2}-10\frac{x}{y}+27-10\frac{y}{x}+\frac{y^2}{x^2}$

Sol :

$$\begin{array}{r|l}
 & \frac{x}{y}-5+\frac{y}{x} \\ \hline
\frac{x}{y} & \frac{x^2}{y^2}-10\frac{x}{y}+27-10\frac{y}{x}+\frac{y^2}{x^2} \\
 & \frac{4x^2}{y^2} \qquad (-) \\ \hline
2\frac{x}{y}-5 & -10\frac{x}{y}+27 \\
 & -10\frac{x}{y}+25 \qquad (-) \\ \hline
2\frac{x}{y}-10+\frac{y}{x} & 2-10\frac{y}{x}+\frac{y^2}{x^2} \\
 & 2-10\frac{y}{x}+\frac{y^2}{x^2} \qquad (-) \\ \hline
 & 0
\end{array}$$

$\therefore \sqrt{\frac{x^2}{y^2}-10\frac{x}{y}+27-10\frac{y}{x}+\frac{y^2}{x^2}} = \left|\frac{x}{y}-5+\frac{y}{x}\right|$

3. Find the values of a and b if the following polynomials are perfect squares.

(i) $4x^4-12x^3+37x^2+bx+a$

(ii) $ax^4+bx^3+361x^2+220x+100$

Sol :

(i)

$$\begin{array}{r|l}
 & 2x^2-3x+7 \\ \hline
2x^2 & 4x^4-12x^3+37x^2+bx+a \\
 & 4x^4 \qquad (-) \\ \hline
4x^2-3x & -12x^3+37x^2 \\
 & -12x^3+9x^2 \qquad (-) \\ \hline
4x^2-6x+7 & 28x^2+bx+a \\
 & 28x^2-42x+49 \; (-) \\ \hline
 & 0
\end{array}$$

Since, the given polynomial is a perfect square

$b+42=0$, $a-49=0$

$\therefore$ $a=49$, $b=-42$.

(ii) Let us write $ax^4+bx^3+361x^2+220x+100$ in the reverse order as

$100+220x+361x^2+bx^3+ax^4$

Now, finding square root

$$
\begin{array}{r|ll}
 & 10 + 11x + 12x^2 & \\
\hline
10 & 100 + 220x + 361x^2 + bx^3 + ax^4 & \\
 & 100 & (-) \\
\hline
20 + 11x & 220x + 361x^2 & \\
 & 220x + 121x^2 & (-) \\
\hline
20 + 22x + 12x^2 & 240x^2 + bx^3 + ax^4 & \\
 & 240x^2 + 264x^3 + 144x^4 & (-) \\
\hline
 & (b - 264)\,x^3 + (a - 144)x^4 &
\end{array}
$$

Since, the given polynomial is a perfect square,

$b - 264 = 0,\ a - 144 = 0$

$\therefore\ a = 144,\ b = 264$

4. Find the values of m and n if the following expressions are perfect squares.

(i) $\dfrac{1}{x^4} - \dfrac{6}{x^3} + \dfrac{13}{x^2} + \dfrac{m}{x} + n$

(ii) $x^4 - 8x^3 + mx^2 + nx + 16$

Sol :

(i)

$$
\begin{array}{r|ll}
 & \frac{1}{x^2} - \frac{3}{x} + 2 & \\
\hline
\frac{1}{x^2} & \frac{1}{x^4} - \frac{6}{x^3} + \frac{13}{x^2} + \frac{m}{x} + n & \\
 & \frac{1}{x^4} & (-) \\
\hline
\frac{2}{x^2} - \frac{3}{x} & -\frac{6}{x^3} + \frac{13}{x^2} & \\
 & -\frac{6}{x^3} + \frac{9}{x^2} & (-) \\
\hline
\frac{2}{x^2} - \frac{6}{x} + 2 & \frac{4}{x^2} + \frac{m}{x} + n & \\
 & \frac{4}{x^2} - \frac{12}{x} + 4 & (-) \\
\hline
 & (m + 12)\frac{1}{x} + (n - 4) &
\end{array}
$$

Since, the polynomial is a perfect square

$m + 12 = 0 \quad n - 4 = 0$

$\therefore\ m = -12 \quad n = 4$

(ii)

$$
\begin{array}{r|ll}
 & x^2 - 4x + \left(\frac{m-16}{2}\right) & \\
\hline
x^2 & x^4 - 8x^3 + mx^2 + nx + 16 & \\
 & x^4 & (-) \\
\hline
2x^2 - 4x & -8x^3 + mx^2 & \\
 & -8x^3 + 16x^2 & (-) \\
\hline
2x^2 - 8x + \left(\frac{m-16}{2}\right) & (m - 16)x^2 + nx + 16 & \\
 & (m - 16)x^2 - 4(m - 16)x + \left(\frac{m-16}{2}\right) & (-) \\
\hline
 & [n + 4\,(m - 16)]x + 16 - \left(\frac{m-16}{2}\right)^2 &
\end{array}
$$

Since the polynomial is a perfect square,

$n + 4\,(m - 16) = 0$

and $16 - \dfrac{(m-16)^2}{4} = 0$

$\therefore\ 64 - (m - 16)^2 = 0$

$(m - 16)^2 = 64$

$m - 16 = 8$

$m = 8 + 16 = 24$

$m = 24$

$\therefore\ n + 4\,(24 - 16) = 0$

$n + 4\,(8) = 0$

$n + 32 = 0$

$n = -32$

$\therefore\ m = 24,\ n = -32$

QUADRATIC EQUATIONS

Key Points

- An expression of degree 2 is called a Quadratic expression which is expressed as $p(x) = ax^2 + bx + c$, $a \neq 0$ and a, b, c are real numbers.
- The values of 'x' such that the expression $ax^2 + bx + c$ becomes zero are called roots of quadratic equation $ax^2 + bx + c = 0$.
- If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$, then $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$
- x^2 – (sum of the roots) x + Product of the roots = 0 is the general form of the quadratic equation when the roots are given.
- $ax^2 + bx + c = 0$ can equivalently be expressed as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Worked Examples

3.24. Find the zeroes of the quadratic expression $x^2 + 8x + 12$.

Sol :

Let $p(x) = x^2 + 8x + 12 = (x + 2)(x + 6)$

$p(-2) = 4 - 16 + 12 = 0$

$p(-6) = 36 - 48 + 12 = 0$

Therefore,

-2 and -6 are zeros of $p(x) = x^2 + 8x + 12$.

3.25. Write down the quadratic equation in general form for which sum and product of the roots are given below.

(i) 9, 14 **(ii)** $-\frac{7}{2}, \frac{5}{2}$

(iii) $-\frac{3}{5}, -\frac{1}{2}$

Sol :

(i) General form of the quadratic equation when the roots are given is

x^2 – (sum of the roots) x + product of the roots = 0

$$x^2 - 9x + 14 = 0$$

(ii) $$x^2 - \left(-\frac{7}{2}\right)x + \frac{5}{2} = 0 \Rightarrow 2x^2 + 7x + 5 = 0$$

(iii) $$x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{1}{2}\right) = 0$$

$$\Rightarrow \frac{10x^2 + 6x - 5}{10} = 0$$

Therefore, $10x^2 + 6x - 5 = 0$

3.26. Find the sum and product of the roots for each of the following quadratic equations:

(i) $x^2 + 8x - 65 = 0$ **(ii)** $2x^2 + 5x + 7 = 0$

(iii) $kx^2 - k^2x - 2k^3 = 0$

Sol :

(i) $x^2 + 8x - 65 = 0$

$\alpha + \beta = -8;$

$\alpha\beta = -65$

(ii) $2x^2 + 5x + 7 = 0$

$\alpha + \beta = -\frac{5}{2};$

$\alpha\beta = \frac{7}{2}$

(iii) $kx^2 - k^2x - 2k^3 = 0$

$a = k, \ b = -k^2 \ c = -2k^3$

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-k^2)}{k} = k$$

and $$\alpha\beta = \frac{c}{a}$$

$$= \frac{-2k^3}{k} = -2k^2$$

Exercise 3.9

1. Determine the quadratic equations, whose sum and product of roots are

(i) $-9, 20$ **(ii)** $\frac{5}{3}, 4$

(iii) $\frac{-3}{2}, -1$ **(iv)** $-(2-a)^2, (a+5)^2$

Sol :

(i) Given sum of the roots = – 9
product of the roots = 20
General form of the Quadratic equation is
x^2 – (sum of the Roots) x + product of the Roots = 0

$$x^2 + 9x + 20 = 0$$

(ii) Given sum of the roots = 5/3
Product of the roots = 4
General form of the Quadratic equation is
x^2 – (sum of the roots) x + product of the roots = 0

$\therefore\ x^2 - 5/3\,x + 4 = 0$

Simplifying $3x^2 - 5x + 12 = 0$

(iii) Given sum of the roots = – 3/2
product of the roots = – 1
General form of the Quadratic equation is
x^2 – (sum of the roots) x + product of the roots = 0

$x^2 + 3/2\,x - 1 = 0$

$2x^2 + 3x - 2 = 0$

(iv) Given sum of the roots = $-(2 - a)^2$
product of the roots = $(a + 5)^2$
General form of the Quadratic equation is
x^2 – (sum of the roots) x + product of the roots = 0

$\therefore\ x^2 + (2 - a)^2 x + (a + 5)^2 = 0$

2. Find the sum and product of the roots for each of the following quadratic equations

(i) $x^2 + 3x - 28 = 0$ **(ii)** $x^2 + 3x = 0$

(iii) $3 + \frac{1}{a} = \frac{10}{a^2}$ **(iv)** $3y^2 - y - 4 = 0$

Sol :

(i) $x^2 + 3x - 28 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 1, b = 3, c = -28$

$\therefore$ Sum of the roots $= -\frac{b}{a} = -\frac{3}{1} = -3$

product of the roots $= \frac{c}{a} = -\frac{28}{1} = -28$

(ii) $x^2 + 3x = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 1, b = 3, c = 0$

$\therefore$ Sum of the roots $= -\frac{b}{a} = -\frac{3}{1} = -3$

Product of the roots $= \frac{c}{a} = \frac{0}{1} = 0$

(iii) $3 + \frac{1}{a} = \frac{10}{a^2}$

$3 + \frac{1}{a} - \frac{10}{a^2} = 0$

$3a^2 + a - 10 = 0$ is a quadratic equation in 'a'

$A = 3, B = 1, C = -10$

$\therefore$ Sum of the roots $= -\frac{B}{A} = -\frac{1}{3}$

product of the roots $= \frac{C}{A} = -\frac{10}{3}$

(iv) $3y^2 - y - 4 = 0$ is a quadratic equation in 'y'

Comparing with $ax^2 + bx + c = 0$

$\therefore\ a = 3, b = -1, c = -4$

Sum of the roots $= -\frac{b}{a} = -\frac{(-1)}{3} = \frac{1}{3}$

product of the roots $= \frac{c}{a} = -\frac{4}{3}$

SOLVING A QUADRATIC EQUATION

Key Points

- For solving a quadratic equation, we are using different methods, namely
  - (i) Factorization method
  - (ii) Completing the square method and
  - (iii) Formula method.
- Formula for finding roots of a quadratic equation $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Worked Examples

3.27. Solve $2x^2 - 2\sqrt{6}x + 3 = 0$

Sol :

$$2x^2 - 2\sqrt{6}x + 3 = 2x^2 - \sqrt{6}x - \sqrt{6}x + 3$$

(by spliting the middle term)

Now, equating the factors to zero we get,

$$(\sqrt{2}x - \sqrt{3})(\sqrt{2}x - \sqrt{3}) = 0$$

$$\sqrt{2}x - \sqrt{3} = 0 \text{ or } \sqrt{2}x - \sqrt{3} = 0$$

$$\sqrt{2}x = \sqrt{3} \text{ or } \sqrt{2}x = \sqrt{3}$$

Therefore the solution is $x = \dfrac{\sqrt{3}}{\sqrt{2}}$.

3.28. Solve $2m^2 + 19m + 30 = 0$

Sol :

$$\begin{aligned} 2m^2 + 19m + 30 &= 2m^2 + 4m + 15m + 30 \\ &= 2m(m+2) + 15(m+2) \\ &= (m+2)(2m+15) \end{aligned}$$

Now, equating the factors to zero, we get,

$$m + 2 = 0$$

$$\Rightarrow m = -2 \text{ or } 2m + 15 = 0$$

We get, $m = \dfrac{-15}{2}$

Therefore the roots are $-2, \dfrac{-15}{2}$.

Some equations which are not quadratic can be solved by reducing them to quadratic equations by suitable substitutions. Such examples are illustrated below.

3.29. Solve $x^4 - 13x^2 + 42 = 0$

Sol :

Let $x^2 = a$.

Then, $(x^2)^2 - 13x^2 + 42 = a^2 - 13a + 42$
$= (a-7)(a-6)$

Given, $(a-7)(a-6) = 0 \Rightarrow a = 7$ or 6.

Since $a = x^2$, $x^2 = 7 \Rightarrow x = \pm\sqrt{7}$ or

$x^2 = 6 \Rightarrow x = \pm\sqrt{6}$

Therefore the roots are $x = \pm\sqrt{7}, \pm\sqrt{6}$

3.30. Solve $\dfrac{x}{x-1} + \dfrac{x-1}{x} = 2\dfrac{1}{2}$

Sol :

Let $y = \dfrac{x}{x-1}$ then $\dfrac{1}{y} = \dfrac{x-1}{x}$.

Therefore, $\dfrac{x}{x-1} + \dfrac{x-1}{x} = 2\dfrac{1}{2}$

becomes $y + \dfrac{1}{y} = \dfrac{5}{2}$

$$\Rightarrow 2y^2 - 5y + 2 = 0 \Rightarrow y = \frac{1}{2}, 2$$

$$\frac{x}{x-1} = \frac{1}{2} \Rightarrow 2x = x - 1 \Rightarrow x = -1$$

$$\frac{x}{x-1} = 2 \Rightarrow x = 2x - 2 \Rightarrow x = 2$$

Therefore, the roots are $x = -1, 2$.

Exercise 3.10

1. Solve the following quadratic equations by factorization method

(i) $4x^2 - 7x - 2 = 0$ **(ii)** $3(p^2 - 6) = p(p+5)$

(iii) $\sqrt{a(a-7)} = 3\sqrt{2}$ **(iv)** $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

(v) $2x^2 - x + \dfrac{1}{8} = 0$

Sol :

(i)

$$\begin{aligned} 4x^2 - 7x - 2 &= 0 \\ 4x^2 - 8x + x - 2 &= 0 \\ 4x(x-2) + 1(x-2) &= 0 \\ (x-2)(4x+1) &= 0 \end{aligned}$$

$x - 2 = 0, \quad 4x + 1 = 0$

$x = 2, \quad 4x = -1$

$x = -\dfrac{1}{4}$

$\therefore$ Solution : $x = -\dfrac{1}{4}, 2$

(ii)

$$\begin{aligned} 3(p^2 - 6) &= p(p+5) \\ 3p^2 - 18 &= p^2 + 5p \\ 3p^2 - 18 - p^2 - 5p &= 0 \\ 2p^2 - 5p - 18 &= 0 \\ 2p^2 + 4p - 9p - 18 &= 0 \\ 2p(p+2) - 9(p+2) &= 0 \\ (2p-9)(p+2) &= 0 \end{aligned}$$

$2p - 9 = 0, \quad p + 2 = 0$

$2p = 9, \quad p = -2$

$\therefore$ Solution $p = -2, 9/2$.

(iii) $\sqrt{a(a-7)} = 3\sqrt{2}$

Squaring on both sides

$$\begin{aligned} a(a-7) &= 9(2) \\ a^2 - 7a - 18 &= 0 \\ a^2 - 9a + 2a - 18 &= 0 \\ a(a-9) + 2(a-9) &= 0 \\ (a-9)(a+2) &= 0 \end{aligned}$$

Solution is a = 9, – 2.

(iv) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

$$\begin{aligned} \sqrt{2}x^2 + 2x + 5x + 5\sqrt{2} &= 0 \\ \sqrt{2}x^2 + \sqrt{2}\,\sqrt{2}x + 5x + 5\sqrt{2} &= 0 \\ \sqrt{2}x(x+\sqrt{2}) + 5(x+\sqrt{2}) &= 0 \\ (x+\sqrt{2})(\sqrt{2}x+5) &= 0 \end{aligned}$$

$x + \sqrt{2} = 0,$ $\quad x = -\sqrt{2}$

$\sqrt{2}x + 5 = 0$, $\quad \sqrt{2}x = -5$, $\quad x = -\frac{5}{\sqrt{2}}$

$\therefore$ Solution is $x = -\sqrt{2}, -\frac{5}{\sqrt{2}}$

(v) $2x^2 - x + \frac{1}{8} = 0$

$$\begin{aligned} 16x^2 - 8x + 1 &= 0 \\ 16x^2 - 4x - 4x + 1 &= 0 \\ 4x(4x-1) - 1(4x-1) &= 0 \\ (4x-1)(4x-1) &= 0 \\ 4x - 1 &= 0, \; 4x - 1 = 0 \\ x &= \frac{1}{4}, \quad x = \frac{1}{4} \end{aligned}$$

$\therefore$ Solution is $x = \frac{1}{4}$ (twice)

2. The number of volleyball games that must be scheduled in a league with n teams is given by $G(n) = \frac{n^2 - n}{2}$ where each team plays with every other team exactly once. A league schedules 15 games. How many teams are in the league?

Sol :

Given $G(n) = \frac{n^2 - n}{2}$

No. of league schedules = 15

$$\begin{aligned} \therefore \frac{n^2 - n}{2} &= 15 \\ n^2 - n &= 30 \\ n^2 - n - 30 &= 0 \\ (n-6)(n+5) &= 0 \\ n &= 6, -5 \end{aligned}$$

Number of teams can't be negative

$\therefore n = 6.$

SOLUTION OF QUADRATIC EQUATION BY COMPLETING THE SQUARE METHOD

Worked Examples

3.31. Solve $x^2 - 3x - 2 = 0$

Sol :

$$x^2 - 3x - 2 = 0$$

$x^2 - 3x = 2$ (Shifting the Constant to RHS)

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = 2 + \left(\frac{3}{2}\right)^2$$

(Add $\left[\frac{1}{2}(co-efficient\ of\ x)\right]^2$ to both sides)

$$\left(x - \frac{3}{2}\right)^2 = \frac{17}{4}$$

(writing the LHS as complete square)

$$x - \frac{3}{2} = \pm\frac{\sqrt{17}}{2}$$

(Taking the square root on both sides)

$x = \frac{3}{2} + \frac{\sqrt{17}}{2}$ or

$x = \frac{3}{2} - \frac{\sqrt{17}}{2}$ (Solve for x)

Therefore, $x = \frac{3+\sqrt{17}}{2}, \frac{3-\sqrt{17}}{2}$

3.32. Solve $2x^2 - x - 1 = 0$

Sol :

$$2x^2 - x - 1 = 0$$

$$x^2 - \frac{x}{2} - \frac{1}{2} = 0$$

($\div$ 2 make co-efficient of x^2 as 1)

$$x^2 - \frac{x}{2} = \frac{1}{2}$$

$$x^2-\frac{x}{2}+\left(\frac{1}{4}\right)^2 = \frac{1}{2}+\left(\frac{1}{4}\right)^2$$

$$\left(x-\frac{1}{4}\right)^2 = \frac{9}{16}=\left(\frac{3}{4}\right)^2$$

$$x-\frac{1}{4}=\pm\frac{3}{4} \Rightarrow x=1,-\frac{1}{2}$$

3.33. Solve $x^2 + 2x - 2 = 0$ by formula method

Sol :

Compare $x^2 + 2x - 2 = 0$ with the standard form $ax^2 + bx + c = 0$ to get

a, b, c $\Rightarrow$ $a = 1, b = 2, c = -2$

Put a, b and c in the formula

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Substituting the values of a, b and c in the formula

we get, $$x = \frac{-2\pm\sqrt{(2)^2-4(1)(-2)}}{2(1)}$$

$$= \frac{-2\pm\sqrt{12}}{2} = -1\pm\sqrt{3}$$

Therefore, $x = -1+\sqrt{3}, -1-\sqrt{3}$

3.34. Solve $2x^2 - 3x - 3 = 0$ by formula method.

Sol :

Compare $2x^2 - 3x - 3 = 0$ with the standard form $ax^2 + bx + c = 0$ to get

a, b, c $\Rightarrow$ $a = 2, b = -3, c = -3$

Put a, b and c in the formula

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Substituting the values of a, b and c in the formula

we get, $$x = \frac{-(-3)\pm\sqrt{(-3)^2-4(2)(-3)}}{2(2)}$$

$$= \frac{3\pm\sqrt{33}}{4}$$

Therefore, $x = \frac{3+\sqrt{33}}{4}, x = \frac{3-\sqrt{33}}{4}$

3.35. Solve $3p^2 + 2\sqrt{5}p-5 = 0$ by formula method.

Sol :

Compare $3p^2 + 2\sqrt{5}p-5 = 0$ with the standard form $ax^2+bx+c=0$

$$a = 3, b = 2\sqrt{5}, c = -5.$$

$$p = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Substituting the values of a, b and c in the formula

we get, $$p = \frac{-2\sqrt{5}\pm\sqrt{(2\sqrt{5})^2-4(3)(-5)}}{2(3)}$$

$$= \frac{-2\sqrt{5}\pm\sqrt{80}}{6} = \frac{-\sqrt{5}\pm2\sqrt{5}}{3}$$

Therefore, $x = \frac{\sqrt{5}}{3}, -\sqrt{5}$

3.36. Solve $pqx^2 - (p + q)^2 x + (p + q)^2 = 0$

Sol :

Compare the coefficients of the given equation with the standard form $ax^2 + bx + c = 0$ to get

$$a = pq, b = -(p + q)^2, c = (p + q)^2$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Substituting the values of a, b and c in the formula we get,

$$x = \frac{-[-(p+q)^2]\pm\sqrt{[-(p+q)^2]^2-4(pq)(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2\pm\sqrt{(p+q)^4-4(pq)(p+q)^2}}{2pq}$$

$$= \frac{(p+q)^2\pm\sqrt{(p+q)^2[(p+q)^2-4pq]}}{2pq}$$

$$= \frac{(p+q)^2\pm\sqrt{(p+q)^2(p^2+q^2+2pq-4pq)}}{2pq}$$

$$= \frac{(p+q)^2\pm\sqrt{(p+q)^2(p-q)^2}}{2pq}$$

$$= \frac{(p+q)^2\pm(p+q)(p-q)}{2pq}$$

$$= \frac{(p+q)\{(p+q)\pm(p-q)\}}{2pq}$$

$$x = \frac{p+q}{2pq}\times2p \text{ (or) } \frac{p+q}{2pq}\times2q \Rightarrow$$

$$x = \frac{p+q}{q}, \frac{p+q}{p}$$

Exercise 3.11

1. Solve the following quadratic equations by completing the square method

(i) $9x^2 - 12x + 4 = 0$

(ii) $\frac{5x+7}{x-1} = 3x + 2$

Sol :

(i) $9x^2 - 12x + 4 = 0$

$$9x^2 - 12x = -4 \text{ [Dividing throughout by '9']}$$

$$x^2 - \frac{4}{3}x = -\frac{4}{9}$$

$$x^2 - \frac{4}{3}x + \frac{4}{9} = -\frac{4}{9} + \frac{4}{9}$$

$$\left[\because Adding \left[\frac{co\text{-}efficient\ of\ x}{2}\right]^2\right] \text{ on both sides.}$$

$$\left(x - \frac{2}{3}\right)^2 = 0$$

$$x = \frac{2}{3} \text{ (twice)}$$

Solution $x = \frac{2}{3}, \frac{2}{3}$

(ii) $\frac{5x+7}{x-1} = 3x + 2$

$$5x + 7 = (3x + 2)(x - 1)$$
$$5x + 7 = 3x^2 - 3x + 2x - 2$$
$$3x^2 - 6x - 9 = 0$$

Dividing by 3

$$x^2 - 2x - 3 = 0$$
$$x^2 - 2x = 3$$
$$x^2 - 2x + 1 = 3 + 1$$
$$(x - 1)^2 = 4$$
$$\therefore x - 1 = \pm 2$$

$x - 1 = 2,$	$x - 1 = -2$
$x = 2 + 1$	$x = -2 + 1$
$x = 3$	$x = -1$

$\therefore$ Solution $x = -1, 3$

2. Solve the following quadratic equations by formula method

(i) $2x^2 - 5x + 2 = 0$

(ii) $\sqrt{2}f^2 - 6f + 3\sqrt{2} = 0$

(iii) $3y^2 - 20y - 23 = 0$

(iv) $36y^2 - 12ay + (a^2 - b^2) = 0$

Sol :

(i) $2x^2 - 5x + 2 = 0$

Comparing with $ax^2 + bx + c = 0$

$$a = 2, b = -5, c = 2$$

Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{5 \pm \sqrt{(-5)^2 - 4(2)(2)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25 - 16}}{4}$$

$$= \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

$$= \frac{5+3}{4}, \frac{5-3}{4}$$

$$= \frac{8}{4}, \frac{2}{4}$$

$\therefore$ Solution $x = 2, \frac{1}{2}$

(ii) $\sqrt{2}f^2 - 6f + 3\sqrt{2} = 0$

$$a = \sqrt{2}, b = -6, c = 3\sqrt{2}$$

$$\therefore f = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{(-6)^2 - 4(\sqrt{2})(3\sqrt{2})}}{2\sqrt{2}}$$

$$= \frac{6 \pm \sqrt{36 - 24}}{2\sqrt{2}}$$

$$= \frac{6 \pm \sqrt{12}}{2\sqrt{2}} = \frac{6 \pm 2\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{6 + 2\sqrt{3}}{2\sqrt{2}}, \frac{6 - 2\sqrt{3}}{2\sqrt{2}}$$

$\therefore$ Solution $f = \frac{3+\sqrt{3}}{\sqrt{2}}, \frac{3-\sqrt{3}}{\sqrt{2}}$

(iii) $3y^2 - 20y - 23 = 0$

$$a = 3, b = -20, c = -23$$

$\therefore$ Solution $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{20 \pm \sqrt{(-20)^2 - 4(3)(-23)}}{2(3)}$$

$$= \frac{20 \pm \sqrt{400+276}}{6}$$

$$= \frac{20 \pm \sqrt{676}}{6}$$

$$= \frac{20 \pm 26}{6}$$

$$= \frac{20+26}{6}, \frac{20-26}{6}$$

$$\therefore y = \frac{46}{6}, -\frac{6}{6}$$

$$= \frac{23}{3}, -1$$

(iv) $36y^2 - 12ay + (a^2 - b^2) = 0$

$A = 36, B = -12a, C = a^2 - b^2$

$$\therefore y = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \frac{12a \pm \sqrt{(-12a)^2 - 4(36)(a^2 - b^2)}}{2(36)}$$

$$= \frac{12a \pm \sqrt{144a^2 - 144(a^2 - b^2)}}{72}$$

$$= \frac{12a \pm \sqrt{144a^2 - 144a^2 + 144b^2}}{72}$$

$$= \frac{12a \pm 12b}{72}$$

$$= \frac{12(a+b)}{72}, \frac{12(a-b)}{72}$$

$$\therefore \text{Solution } y = \frac{a+b}{6}, \frac{a-b}{6}$$

3. A ball rolls down a slope and travels a distance $d = t^2 - 0.75\,t$ feet in t seconds. Find the time when the distance travelled by the ball is 11.25 feet.

Sol :

Given distance d $= t^2 - 0.75\,t$ and

$t^2 - 0.75\,t = 11.25$

$\therefore t^2 - 0.75\,t - 11.25 = 0$

$a = 1, b = -0.75, c = -11.25$

Using the formula.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{0.75 \pm \sqrt{(-0.75)^2 - 4(1)(-11.25)}}{2(1)}$$

$$= \frac{0.75 \pm \sqrt{0.5625 + 45}}{2}$$

$$= \frac{0.75 \pm \sqrt{45.5625}}{2}$$

$$= \frac{0.75 \pm 6.75}{2}$$

$$= \frac{0.75 + 6.75}{2}, \frac{0.75 - 6.75}{2}$$

$$= \frac{7.5}{2}, -\frac{6}{2}$$

$$= 3.75, -3$$

$\therefore t = 3.75$ seconds.

[$\because$ time cannot be negative $t \neq -3$]

WORD PROBLEMS RELATED TO DAY-TO-DAY LIFE ACTIVITIES

Worked Examples

3.37. The product of Kumaran's age (in years) two years ago and his age four years from now is one more than twice his present age. What is his present age?

Sol :

Let the present age of Kumaran be x years.

Two years ago, his age = (x – 2) years.

Four years from now, his age = (x + 4) years.

Given, $(x-2)(x+4) = 1 + 2x$

$x^2 + 2x - 8 = 1 + 2x$

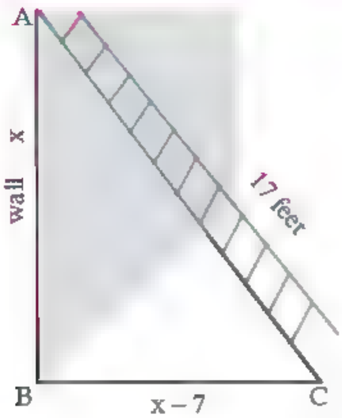
$\Rightarrow (x-3)(x+3) = 0$ then $x = \pm 3$

Therefore, x = 3 (Rejecting – 3 as age cannot be negative)

Kumaran's present age is 3 years.

3.38. A ladder 17 feet long is leaning against a wall. If the ladder, vertical wall and the floor from the bottom of the wall to the ladder form a right triangle, find the height of the wall where the tip of the ladder meets if the distance between bottom of the wall to bottom of the ladder is 7 feet less than the height of the wall?

Sol :



Let the height of the wall AB = x feet.
As per the given data BC = (x – 7) feet.
In ΔABC, AC = 17 feet, BC = (x – 7) feet.
By Pythagoras theorem, $AC^2 = AB^2 + BC^2$

$(17)^2 = x^2 + (x-7)^2;$

$289 = x^2 + x^2 - 14x + 49$

$x^2 - 7x - 120 = 0$

$\Rightarrow (x-15)(x+8) = 0 \Rightarrow x = 15$ (or) -8

Therefore, height of the wall AB = 15 feet
(Rejecting – 8 as height cannot be negative)

3.39. A flock of swans contained x^2 members. As the clouds gathered, 10x went to a lake and one-eigth of the members flew away to a garden. The remaining three pairs played about in the water. How many swans were there in total?

Sol : As given there are x^2 swans.

As per the given data

$$x^2 - 10x - \frac{1}{8}x^2 = 6 \Rightarrow 7x^2 - 80x - 48 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{80 \pm \sqrt{6400 - 4(7)(-48)}}{14}$$

$$= \frac{80 \pm 88}{14}$$

Therefore, $x = 12, -\frac{4}{7}$.

Here, $x = -\frac{4}{7}$ is not possible as the number of swans cannot be negative.
Hence, x = 12. Therefore total number of swans is $x^2 = 144$.

3.40. A passenger train takes 1 hr more than an express train to travel a distance of 240 km from Chennai to Virudhachalam. The speed of passenger train is less than that of an express train by 20 km per hour. Find the average speed of both the trains.

Sol :
Let the average speed of passenger train be x km/hr.
Then the average speed of express train will be (x + 20) km/hr
Time taken by the passenger train to cover distance of 240 km = $\frac{240}{x}$ hr
Time taken by express train to cover distance of 240 km = $\frac{240}{x+20}$ hr

Given, $\frac{240}{x} = \frac{240}{x+20} + 1$

$$240\left[\frac{1}{x} - \frac{1}{x+20}\right] = 1 \Rightarrow$$

$$240\left[\frac{x+20-x}{x(x+20)}\right] = 1 \Rightarrow \quad 4800 = (x^2 + 20x)$$

$x^2 + 20x - 4800 = 0$

$\Rightarrow (x+80)(x-60) = 0 \Rightarrow x = -80$ or 60.

Therefore x = 60 (Rejecting – 80 as speed cannot be negative)
Average speed of the passenger train is 60 km/hr
Average speed of the express train is 80 km/hr.

Exercise 3.12

1. If the difference between a number and its reciprocal is $\frac{24}{5}$, find the number.

Sol :
Let the number be 'x'
$\therefore$ its reciprocal is $\frac{1}{x}$

$$\text{Given } x - \frac{1}{x} = \frac{24}{5}$$

$$\frac{x^2-1}{x} = \frac{24}{5}$$

$$5x^2 - 5 = 24x$$

$$5x^2 - 24x - 5 = 0$$

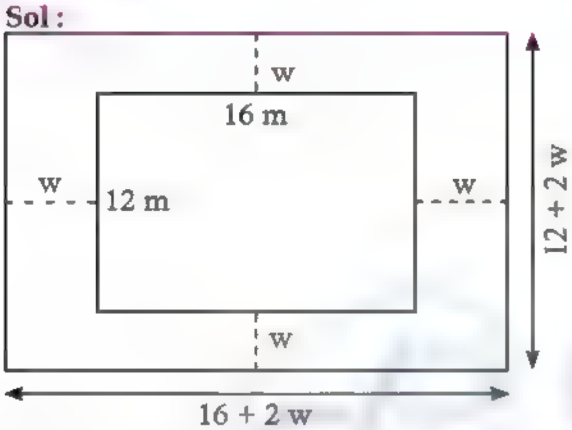
$$5x^2 - 25x + x - 5 = 0$$

$$5x(x-5)+1(x-5) = 0$$
$$(x-5)(5x+1) = 0$$
$$x = 5, -\frac{1}{5}$$

$\therefore$ The number is '5' or '– 1/5'

2. A garden measuring 12 m by 16 m is to have a pedestrian pathway that is '*w*' meters wide installed all the way around so that it increases the total area to 285 m². What is the width of the pathway?

Sol :



Given length of the Garden = 16 m

Breadth of the Garden = 12 m

Width of the path = '*w*' m

$\therefore$ From the figure

$$(16+2w)(12+2w) = 285 \text{ (Given)}$$
$$192+24w+32w+4w^2-285 = 0$$
$$4w^2+56w-93 = 0$$

Using Quadratic formula

$$w = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
$$= \frac{-56 \pm \sqrt{(56)^2-4(4)(-93)}}{2(4)}$$
$$= \frac{-56 \pm \sqrt{4624}}{8}$$
$$= \frac{-56 \pm 68}{8}$$
$$= \frac{-56+68}{8}, \frac{-56-68}{8}$$

[negative value is not possible]

$$= \frac{12}{8} = 1.5$$

$\therefore$ The width of the pathway is 1.5 m

3. A bus covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hour more it would have taken 30 minutes less for the journey. Find the original speed of the journey.

Sol :

Let the original speed of the bus be 'x' km/h then increased speed (x + 15) km/h

$$\text{Usual time} = \frac{90}{x} \text{ hours,}$$
$$\text{New time} = \frac{90}{x+15} \text{ hours}$$
$$\left[\because \text{ time} = \frac{Distance}{Speed}\right.$$
$$\therefore \text{ Given } \frac{90}{x} - \frac{90}{x+15} = \frac{1}{2}$$
$$\left[\because 30 \text{ min} = \frac{1}{2} \text{ hr}\right]$$
$$90\left[\frac{1}{x} - \frac{1}{x+15}\right] = \frac{1}{2}$$
$$90\left[\frac{x+15-x}{x(x+15)}\right] = \frac{1}{2}$$
$$\frac{90 \times 15}{x^2+15x} = \frac{1}{2}$$
$$x^2+15x = 2700$$
$$x^2+15x-2700 = 0$$

Factorizing it $(x+60)(x-45) = 0$

$$x = -60 \text{ and } x = 45$$
$$x = -60 \text{ is not possible}$$

$\therefore$ Original speed of the Bus is 45 km/hour.

4. A girl is twice as old as her sister. Five years hence, the product of their ages (in years) will be 375. Find their present ages.

Sol :

Let the present age of her sister be 'x' years.

$\therefore$ Girl's age is 2x

Five years hence their ages will be (x + 5) and (2x + 5)

Given $(x+5)(2x+5) = 375$

Simplifying $2x^2+15x-350 = 0$

$$2x^2-20x+35x-350 = 0$$
$$2x(x-10)+35(x-10) = 0$$
$$(x-10)(2x+35) = 0$$
$$x = 10, -\frac{35}{2}$$
$$\therefore x = 10$$

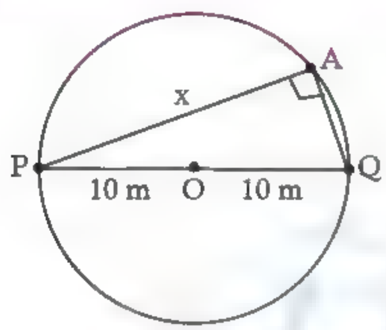
$x = \frac{-35}{2}$ is not possible.

$\therefore$ Present age of a girl is 20 years

Present age of her sister is 10 years.

5. A pole has to be erected at a point on the boundary of a circular ground of diameter 20 m in such a way that the difference of its distance from two diametrically opposite fixed gates P and Q on the boundary is 4 m. Is it possible to do so? If answer is yes at what distance from the two gates should the pole be erected?

Sol :



Let 'A' be the point where the pole is erected.

P and Q are two points which are diametrically opposite.

$$\text{Diameter } PQ = 20 \text{ m}$$

$$\therefore PO = OQ = 10 \text{ m}$$

$$\text{Given } PA - QA = 4$$

Let 'PA' be 'x'

$$\therefore QA = x - 4$$

From the figure, $PA^2 + QA^2 = PQ^2$

[ΔPAQ is a right triangle as $\angle A = 90^\circ$ angle in a semi-circle is a right angle.]

$$x^2 + (x-4)^2 = (20)^2$$

$$x^2 + x^2 - 8x + 16 - 400 = 0$$

$$2x^2 - 8x - 384 = 0$$

Divided by 2

$$x^2 - 4x - 192 = 0$$

$$(x-16)(x+12) = 0$$

$x = 16, -12$ [$x = -12$ is not possible]

$$\therefore x = 16$$

$$\therefore \text{ If } PA = 16, QA = 16 - 4 = 12$$

$$\therefore PA = 16 \text{ m}, QA = 12 \text{ m}$$

and it is very much possible from the given data.

6. From a group of black bees $2x^2$, square root of half of the group went to a tree. Again eight-ninth of the bees went to the same tree. The remaining two got caught up in a fragrant lotus. How many bees were there in total?

Sol :

Given, total no. of Bees $= 2x^2$

Square root of half of the group $= \left(\frac{2x^2}{2}\right)^{1/2}$

eight-ninth of the bees $= \frac{8}{9}(2x^2)$

From the given data,

$$2x^2 - x - \frac{8}{9}(2x^2) = 2$$

$$18x^2 - 9x - 16x^2 = 18$$

$$2x^2 - 9x - 18 = 0$$

$$2x^2 - 12x + 3x - 18 = 0$$

$$2x(x-6) + 3(x-6) = 0$$

$$(x-6)(2x+3) = 0$$

$x = 6$, $x = -\frac{3}{2}$ which is not possible.

$$\therefore x = 6.$$

$\therefore$ No. of Bees in the group $= 2x^2 = 2(6)^2 = 72$

7. Music is been played in two opposite galleries with certain group of people. In the first gallery a group of 4 singers were singing and in the second gallery 9 singers were singing. The two galleries are separated by the distance of 70 m. Where should a person stand for hearing the same intensity of the singers voice? (Hint: The ratio of the sound intensity is equal to the square of the ratio of their corresponding distance).

Sol :

Given

No. of singers in the first gallery = 4

No. of singers in the second gallery = 9

Let the person is standing 'x' m apart from the first gallery.

$\therefore$ $(70 - x)$ m from the second gallery.

From the given data

$$\frac{4k}{9k} = \frac{x^2}{(70-x)^2}$$

[$\because$ k is the constant]

$$4(70-x)^2 = 9x^2$$
$$4(4900-140x+x^2) = 9x^2$$
$$19600-560x+4x^2 = 9x^2$$
$$5x^2+560x-19600 = 0$$

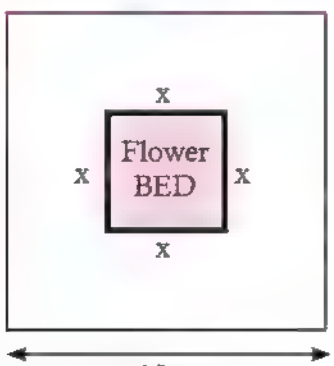
Divided by 5

$$x^2+112x-3920 = 0$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-112\pm\sqrt{(112)^2-4(1)(-3920)}}{2(1)}$$
$$= \frac{-112\pm\sqrt{28224}}{2}$$
$$= \frac{-112\pm168}{2}$$
$$= \frac{-112+168}{2}, \frac{-112-168}{2}$$
$$= \frac{56}{2}, -\frac{280}{2}$$
$$= 28, -140 \quad (-140 \text{ is not possible}).$$
$$\therefore x = 28 \text{ m}$$
$$70-x = 70-28 = 42 \text{ m}.$$

8. There is a square field whose side is 10 m. A square flower bed is prepared in its centre leaving a gravel path all round the flower bed. The total cost of laying the flower bed and gravelling the path at ₹ 3 and ₹ 4 per square metre respectively is ₹ 364. Find the width of the gravel path.

Sol :



10 m

Given side of the field $= 10$ m

$\therefore$ Area of the field $= (10)^2$

$= 100$ sq. m

Let the side of the flower bed be 'x' m

Area of the flower bed $= x^2$

$\therefore$ Area of the path $= 100-x^2$

Given total cost for laying the flower bed and gravelling the path at ₹ 3 and ₹ 4 respectively is ₹ 364.

$$\therefore\ 3x^2+4(100-x^2) = 364$$
$$3x^2+400-4x^2 = 364$$
$$x^2-36 = 0$$
$$x^2 = 36$$
$$x = \pm 6$$
$$[\because x = -6 \text{ is not possible,}]$$
$$\therefore x = 6$$

$$\therefore \text{ Width of the gravel path } = \frac{10-6}{2}$$
$$= \frac{4}{2} = 2 \text{ m.}$$

9. Two women together took 100 eggs to a market, one had more than the other. Both sold them for the same sum of money. The first then said to the second: "If I had your eggs, I would have earned ₹ 15", to which the second replied: "If I had your eggs, I would have earned ₹ $6\frac{2}{3}$". How many eggs did each had in the beginning?

Sol :

Let the no. of eggs with the first woman be 'x'

$\therefore$ Number of eggs with the second woman "(100 – x)"

Now, let 'a' be the cost of an egg sold by the first woman and 'b' be the cost of an egg sold by the second woman.

Given, $x > 100-x$

and $\quad a(100-x) = 15, \qquad bx = 6\frac{2}{3}$

$$a = \frac{15}{100-x}, \qquad b = \frac{20}{3x}$$

Given that eggs been sold for same sum of money

$$\therefore\ ax = (100-x)\,b$$
$$\Rightarrow x = \frac{100b}{a+b}$$

Substituting the values of 'a' and 'b'

$$x = \frac{100\left(\frac{20}{3x}\right)}{\frac{15}{100-x}+\frac{20}{3x}}$$
$$x = \frac{2000(100-x)}{2000+25x}$$
$$25(x+80)x = 25(80)(100-x)$$
$$x^2+80x = 8000-80x$$

$$x^2 + 160x - 8000 = 0$$

Factorizing $\Rightarrow$ $(x + 200)(x - 40) = 0$

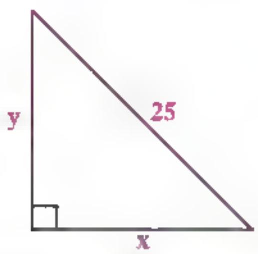
$x = 40$, [$x = -200$ is rejected]

then $100 - x = 100 - 40 = 60$

$\therefore$ The two women had 40 and 60 eggs in the beginning.

10. The hypotenuse of a right angled triangle is 25 cm and its perimeter 56 cm. Find the length of the smallest side.

Sol :



Given hypotenuse = 25 cm

Let the other sides be 'x' and 'y'

$$\therefore x^2 + y^2 = 25^2 \quad \text{... (1)}$$

Perimeter = 56 cm

$$x + y + 25 = 56$$
$$y = 56 - 25 - x$$
$$\Rightarrow y = 31 - x$$

Substituting in ... (1)

$$x^2 + (31 - x)^2 = 625$$
$$x^2 + 961 - 62x + x^2 - 625 = 0$$
$$2x^2 - 62x + 336 = 0$$

[$\because$ Dividing by 2]

$$x^2 - 31x + 168 = 0$$
$$x^2 - 24x - 7x + 168 = 0$$
$$x(x - 24) - 7(x - 24) = 0$$
$$(x - 24)(x - 7) = 0$$
$$x = 7, 24$$

$\therefore$ The smallest side of the triangle is 7 cm.

NATURE OF ROOTS OF A QUADRATIC EQUATION

Key Points

- Formula for finding roots of a quadratic equation $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The term $b^2 - 4ac$ is called "Discriminant" and it is denoted by Δ.
- If $\Delta > 0$ (i.e., $b^2 - 4ac > 0$), then the roots are real and unequal.
- If $\Delta = 0$ (i.e., $b^2 - 4ac = 0$), then the roots are real and equal.
- If $\Delta < 0$ (i.e., $b^2 - 4ac < 0$), then the roots are unreal.

Worked Examples

3.41. Determine the nature of roots for the following quadratic equations

(i) $x^2 - x - 20 = 0$

(ii) $9x^2 - 24x + 16 = 0$

(iii) $2x^2 - 2x + 9 = 0$

Sol :

(i) $x^2 - x - 20 = 0$

Here, $a = 1, b = -1, c = -20$

Now, $\Delta = b^2 - 4ac$

$\Delta = (-1)^2 - 4(1)(-20) = 81$

Here, $\Delta = 81 > 0$.

So, the equation will have real and unequal roots.

(ii) $9x^2 - 24x + 16 = 0$

Here, $a = 9, b = -24, c = 16$

Now, $\Delta = b^2 - 4ac$

$= (-24)^2 - 4(9)(16) = 0$

Here, $\Delta = 0$.

So, the equation will have real and equal roots.

(iii) $2x^2 - 2x + 9 = 0$

Here, $a = 2, b = -2, c = 9$

Now, $\Delta = b^2 - 4ac$

$= (-2)^2 - 4(2)(9) = -68$

Here, $\Delta = -68 < 0$.

So, the equation will have no real root.

3.42. (i) Find the values of 'k', for which the quadratic equation $kx^2 - (8k + 4) + 81 = 0$ has real and equal roots?

(ii) Find the values of 'k' such that quadratic equation $(k + 9)x^2 + (k + 1)x + 1 = 0$ has no real roots.

Sol :

(i) $kx^2 - (8k + 4) + 81 = 0$

Since the equation has real and equal roots $\Delta = 0$.

That is, $b^2 - 4ac = 0$

Here, $a = k$,

$b = -(8k + 4)$,

$c = 81$

That is, $[-(8k + 4)]^2 - 4(k)(81) = 0$

$$64k^2 + 64k + 16 - 324k = 0$$
$$64k^2 - 260k + 16 = 0$$
$$\div 4 \Rightarrow 16k^2 - 65k + 4 = 0$$
$$(16k - 1)(k - 4) = 0$$
$$\Rightarrow k = \frac{1}{16} \text{ or } k = 4$$

(ii) $(k + 9)x^2 + (k + 1)x + 1 = 0$

Since the equation has no real roots, $\Delta < 0$

That is, $b^2 - 4ac < 0$

Here, $a = k + 9, b = k + 1, c = 1$

Therefore, $(k + 1)^2 - 4(k + 9)(1) < 0$

$$k^2 + 2k + 1 - 4k - 36 < 0$$
$$k^2 - 2k - 35 < 0$$
$$(k + 5)(k - 7) < 0$$

Therefore, $-5 < k < 7$. {If $\alpha < \beta$ and if $(x-\alpha)(x-\beta) < 0$ then, $\alpha < x < \beta$}.

3.43. **Prove that the equation $x^2(p^2 + q^2) + 2x(pr + qs) + r^2 + s^2 = 0$ has no real roots. If $ps = qr$ then show that the roots are real and equal.**

Sol : The given quadratic equation is

$$x^2(p^2 + q^2) + 2x(pr + qs) + r^2 + s^2 = 0$$

Here, $a = p^2 + q^2$, $b = 2(pr + qs)$, $c = r^2 + s^2$

Now, $\Delta = b^2 - 4ac$

$$= [2(pr + qs)]^2 - 4(p^2 + q^2)(r^2 + s^2)$$
$$= 4[p^2r^2 + 2pqrs + q^2s^2 - p^2r^2 - p^2s^2 - q^2r^2 - q^2s^2]$$
$$= 4[-p^2s^2 + 2pqrs - q^2r^2]$$
$$= -4[(ps - qr)^2] < 0 \quad .....(1)$$

$\Delta = b^2 - 4ac < 0$, the roots are not real.

If $ps = qr$ then

$$\Delta = -4[ps - qr]^2$$
$$= -4[qr - qr]^2 = 0 \text{ (using (1))}.$$

Thus, $\Delta = 0$ if $ps = qr$ and so the roots will be real and equal.

Thinking Corner

1. Fill up the empty box in each of the given expression so that the resulting quadratic polynomial becomes a perfect square.

i) $x^2 + 14x +$ ______

Ans : 49

ii) $x^2 - 24x +$ ______

Ans : 144

iii) $p^2 + 2pq +$ ______

Ans : q^2

Exercise 3.13

1. Determine the nature of the roots for the following quadratic equations

(i) $15x^2 + 11x + 2 = 0$

(ii) $x^2 - x - 1 = 0$

(iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

(iv) $9y^2 - 6\sqrt{2}y + 2 = 0$

(v) $9a^2b^2x^2 - 24abc\,dx + 16c^2d^2 = 0$, $a \neq 0$, $b \neq 0$

Sol :

(i) $15x^2 + 11x + 2 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 15, b = 11, c = 2$

Now, $b^2 - 4ac = (11)^2 - 4(15)(2)$

$= 121 - 120 = 1 > 0$

$\therefore$ Roots are real and unequal.

(ii) $x^2 - x - 1 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 1, b = -1, c = -1$

$b^2 - 4ac = (-1)^2 - 4(1)(-1)$

$= 1 + 4 = 5 > 0$

$\therefore$ Roots are real and unequal.

(iii) $\sqrt{2}t^2 - 3t + 3\sqrt{2} = 0$

$a = \sqrt{2}$, $b = -3$, $c = 3\sqrt{2}$

$b^2 - 4ac = (-3)^2 - 4(\sqrt{2})(3\sqrt{2})$

$= 9 - 24 = -15 < 0$

$\therefore$ Roots are not real.

(iv) $9y^2 - 6\sqrt{2}y + 2 = 0$

$a = 9, b = -6\sqrt{2}, c = 2$

$$b^2 - 4ac = (-6\sqrt{2})^2 - 4(9)(2)$$
$$= 72 - 72 = 0$$
$\therefore$ Roots are real and equal.

(v) $9a^2b^2x^2 - 24abcd\,x + 16c^2d^2 = 0$

$A = 9a^2b^2$, $B = -24abcd$, $C = 16c^2d^2$

$$B^2 - 4AC = (-24abcd)^2 - 4(9a^2b^2)(16c^2d^2)$$
$$= 576\,a^2b^2c^2d^2 - 576\,a^2b^2c^2d^2$$
$$= 0$$
$\therefore$ Roots are real and equal.

2. Find the value(s) of 'k' for which the roots of the following equations are real and equal.

(i) $(5k-6)x^2 + 2kx + 1 = 0$

(ii) $kx^2 + (6k+2)x + 16 = 0$

Sol :

(i) $(5k-6)x^2 + 2kx + 1 = 0$

Comparing with $ax^2 + bx + c = 0$

$a = 5k - 6$,

$b = 2k$, $c = 1$

Given that roots are real and equal

$$b^2 - 4ac = 0$$
$$(2k)^2 - 4(5k-6)(1) = 0$$
$$4k^2 - 20k + 24 = 0$$

Dividing by 4

$$k^2 - 5k + 6 = 0$$
$$(k-3)(k-2) = 0$$
$$k = 2, 3$$

(ii) $kx^2 + (6k+2)x + 16 = 0$

$a = k$,

$b = 6k + 2$, $c = 16$

Roots are real and equal

$$b^2 - 4ac = 0$$
$$(6k+2)^2 - 4(k)(16) = 0$$
$$36k^2 + 24k + 4 - 64k = 0$$
$$36k^2 - 40k + 4 = 0$$

Dividing by 4

$$9k^2 - 10k + 1 = 0$$
$$9k^2 - 9k - k + 1 = 0$$
$$9k(k-1) - 1(k-1) = 0$$
$$(k-1)(9k-1) = 0$$
$$k - 1 = 0, \quad 9k - 1 = 0$$
$$k = 1, \quad k = \frac{1}{9}$$
$$k = 1, \frac{1}{9}$$

3. If the roots of $(a-b)x^2 + (b-c)x + (c-a) = 0$ are real and equal, then prove that b, a, c are in arithmetic progression.

Sol :

$$(a-b)x^2 + (b-c)x + (c-a) = 0$$

$A = a - b$,

$B = b - c$,

$C = c - a$

Given that roots are real and equal

$$B^2 - 4AC = 0$$
$$(b-c)^2 - 4(a-b)(c-a) = 0$$
$$b^2 - 2bc + c^2 - 4(ac - a^2 - bc + ab) = 0$$
$$b^2 - 2bc + c^2 - 4ac + 4a^2 + 4bc - 4ab = 0$$
$$4a^2 + b^2 + c^2 - 4ab - 2bc - 4ac = 0$$
$$(2a)^2 + (-b)^2 + (-c)^2 + 2(2a)(-b) + 2(-b)(-c) + 2(2a)(-c) = 0$$
$$(2a - b - c)^2 = 0$$
$$2a - b - c = 0$$
$$2a = b + c$$
$$a = \frac{b+c}{2}$$

$\therefore$ b, a and c are in arithmetic progression.

4. If a, b are real then show that the roots of the equation $(a-b)x^2 - 6(a+b)x - 9(a-b) = 0$ are real and unequal.

Sol :

$(a-b)x^2 - 6(a+b)x - 9(a-b) = 0$

Comparing with $Ax^2 + Bx + C = 0$

$A = a - b$, $B = -6(a+b)$, $C = -9(a-b)$

$$B^2 - 4AC = [-6(a+b)]^2 - 4(a-b)(-9)(a-b)$$
$$= 36(a+b)^2 + 36(a-b)^2$$
$$= 36[(a+b)^2 + (a-b)^2]$$
$$= 36[a^2 + 2ab + b^2 + a^2 - 2ab + b^2]$$
$$= 36(2a^2 + 2b^2) = 36 \times 2(a^2 + b^2)$$
$$= 72(a^2 + b^2)$$

Given that a, b are real

$\therefore$ $72(a^2 + b^2) \geq 0$

$\therefore$ Roots are real and unequal.

5. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + b^2 - ac = 0$ are real and equal prove that either $a = 0$ (or) $a^3 + b^3 + c^3 = 3abc$.

Sol :

$(c^2 - ab)x^2 - 2(a^2 - bc)x + b^2 - ac = 0$

Comparing with $Ax^2 + Bx + C = 0$

$A = c^2 - ab$, $B = -2(a^2 - bc)$, $C = b^2 - ac$

Given that the roots are real and equal

$\therefore$ $B^2 - 4AC = 0$

$$[-2(a^2-bc)]^2 - 4(c^2-ab)(b^2-ac) = 0$$
$$4(a^2-bc)^2 - 4(c^2-ab)(b^2-ac) = 0$$
$$4[a^4 - 2a^2bc + b^2c^2 - c^2b^2 + ac^3 + ab^3 - a^2bc] = 0$$
$$a^4 - 3a^2bc + ac^3 + ab^3 = 0$$
$$a(a^3+b^3+c^3-3abc) = 0$$
$$\therefore\ a = 0 \text{ (or) } a^3+b^3+c^3-3abc = 0$$
$$a^3+b^3+c^3 = 3abc$$

Hence proved.

THE RELATION BETWEEN ROOTS OF THE QUADRATIC EQUATION AND CO-EFFICIENTS

Key Points

- If α and β are the roots of the equation $ax^2+bx+c=0$, then
  Sum of the roots $\alpha+\beta = -b/a$
  Product of the roots $\alpha\beta = c/a$

Some More Important Results

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$
$$(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$$
$$\alpha^3+\beta^3 = (\alpha+\beta)[(\alpha+\beta)^2 - 3\alpha\beta]$$
$$\alpha^3-\beta^3 = (\alpha-\beta)[(\alpha+\beta)^2 - \alpha\beta]$$
$$\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2 - 2(\alpha\beta)^2$$
$$\alpha^4-\beta^4 = (\alpha^2+\beta^2)(\alpha+\beta)(\alpha-\beta)$$

Worked Examples

3.44. If the difference between the roots of the equation $x^2-13x+k=0$ is 17 find k.

Sol :

Comparing $x^2-13x+k=0$

here, $a=1, b=-13, c=k$

Let α, β be the roots of the equation

$$\alpha+\beta = \frac{-b}{a} = \frac{-(-13)}{1} = 13 \quad \text{... (1)}$$

Also $\alpha-\beta = 17$ ... (2)

(1) + (2) we get,

$$2\alpha = 30 \Rightarrow \alpha = 15$$

Therefore,

$$15+\beta = 13 \text{ (from (1))}$$
$$\Rightarrow \beta = -2$$

But, $\alpha\beta = \frac{c}{a} = \frac{k}{1}$

$$\Rightarrow 15\times(-2) = k \Rightarrow k = -30$$

3.45. If α and β are the roots of $x^2+7x+10=0$ find the value of

(i) $(\alpha-\beta)$ (ii) $\alpha^2+\beta^2$

(iii) $\alpha^3-\beta^3$ (iv) $\alpha^4+\beta^4$

(v) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (vi) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Sol :

$x^2+7x+10=0$, here, $a=1, b=7, c=10$

If α and β are roots of the equation then,

$$\alpha+\beta = \frac{-b}{a} = \frac{-7}{1} = -7;$$
$$\alpha\beta = \frac{c}{a} = \frac{10}{1} = 10$$

(i) $$\alpha-\beta = \sqrt{(\alpha+\beta)^2-4\alpha\beta} = \sqrt{(-7)^2-4\times 10} = \sqrt{9} = 3$$

(ii) $$\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = (-7)^2 - 2\times 10 = 29$$

(iii) $\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

$= (3)^3 + 3\,(10)\,(3) = 117$

(iv) $\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$

$= 29^2 - 2 \times (10)^2 = 641$

(since from **(ii)**, $\alpha^2+\beta^2 = 29$)

(v) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta}$

$= \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \dfrac{49-20}{10} = \dfrac{29}{10}$

(vi) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta}$

$= \dfrac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$= \dfrac{(-343) - 3\,(10\times(-7))}{10}$

$= \dfrac{-343+210}{10} = \dfrac{-133}{10}$

3.46. **If α, β are the roots of the equation $3x^2 + 7x - 2 = 0$, find the values of**

(i) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$ **(ii)** $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$

Sol :

$3x^2 + 7x - 2 = 0$

here, $a = 3, b = 7, c = -2$

Since, α, β are the roots of the equation

(i) $\alpha+\beta = \dfrac{-b}{a} = \dfrac{-7}{3}$;

$\alpha\beta = \dfrac{c}{a} = \dfrac{-2}{3}$

$\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta}$

$= \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \dfrac{\left(\dfrac{-7}{3}\right)^2 - 2\left(\dfrac{-2}{3}\right)}{\dfrac{-2}{3}} = \dfrac{-61}{6}$

(ii) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta}$

$= \dfrac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$= \dfrac{\left(-\dfrac{7}{3}\right)^3 - 3\left(-\dfrac{2}{3}\right)\left(-\dfrac{7}{3}\right)}{-\dfrac{7}{3}} = \dfrac{67}{9}$

3.47. **If α, β are the roots of the equation $2x^2 - x - 1 = 0$ then form the equation whose roots are**

(i) $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$ **(ii)** $\alpha^2\beta, \beta^2\alpha$ **(iii)** $2\alpha+\beta, 2\beta+\alpha$

Sol :

$2x^2 - x - 1 = 0$, here, $a = 2$, $b = -1, c = -1$

$\alpha+\beta = \dfrac{-b}{a} = \dfrac{-(-1)}{2} = \dfrac{1}{2}$

$\alpha\beta = \dfrac{c}{a} = -\dfrac{1}{2}$

(i) Given roots are $\dfrac{1}{\alpha}, \dfrac{1}{\beta}$

Sum of the roots $= \dfrac{1}{\alpha}+\dfrac{1}{\beta}$

$= \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{\frac{1}{2}}{-\frac{1}{2}} = -1$

Product of the roots $= \dfrac{1}{\alpha}\times\dfrac{1}{\beta}$

$= \dfrac{1}{\alpha\beta} = \dfrac{1}{-\frac{1}{2}} = -2$

The required equation is $x^2 -$ (Sum of the roots) $x +$ (Product of the roots) $= 0$

$x^2 - (-1)\,x - 2 = 0 \Rightarrow x^2 + x - 2 = 0$

(ii) Given roots are $\alpha^2\beta$, $\beta^2\alpha$

Sum of the roots $\alpha^2\beta + \beta^2\alpha = \alpha\beta(\alpha+\beta)$

$= -\dfrac{1}{2}\left(\dfrac{1}{2}\right) = -\dfrac{1}{4}$

Product of the roots $(\alpha^2\beta)\times(\beta^2\alpha) = \alpha^3\beta^3$

$= (\alpha\beta)^3 = \left(-\dfrac{1}{2}\right)^3 = -\dfrac{1}{8}$

The required equation is x^2 – (Sum of the roots) x + (Product of the roots) = 0

$$x^2-\left(-\frac{1}{4}\right)x-\frac{1}{8}=0 \Rightarrow 8x^2+2x-1=0$$

(iii) $2\alpha+\beta, 2\beta+\alpha$

Sum of the roots $2\alpha+\beta+2\beta+\alpha = 3(\alpha+\beta)$

$$= 3\left(\frac{1}{2}\right)=\frac{3}{2}$$

Product of the roots $= (2\alpha+\beta)(2\beta+\alpha)$

$$= 4\alpha\beta+2\alpha^2+2\beta^2+\alpha\beta$$
$$= 5\alpha\beta+2(\alpha^2+\beta^2)$$
$$= 5\alpha\beta+2\left[(\alpha+\beta)^2-2\alpha\beta\right]$$
$$= 5\left[-\frac{1}{2}\right]+2\left[\frac{1}{4}-2\times-\frac{1}{2}\right]$$
$$=0$$

The required equation is x^2 – (Sum of the roots) x + (Product of the roots) = 0

$$x^2-\frac{3}{2}x+0=0 \Rightarrow 2x^2-3x=0$$

Progress Check

1.

Quadratic equation	Roots of quadratic equation α and β	Co-efficients of x^2, x and constants	Sum of Roots $\alpha+\beta$	Product of roots $\alpha\beta$	$-\frac{b}{a}$	$\frac{c}{a}$	Conclusion
$4x^2-9x+2=0$							
$\left(x-\frac{4}{5}\right)^2=0$							
$2x^2-15x-27=0$							

Ans :

Quadratic equation	Roots of quadratic equation α and β	Co-efficients of x^2, x and constants	Sum of Roots $\alpha+\beta$	Product of roots $\alpha\beta$	$-\frac{b}{a}$	$\frac{c}{a}$	Conclusion
$4x^2-9x+2=0$	2, 1/4	4, –9, 2	9/4	1/2	9/4	1/2	Roots are real and distinct
$\left(x-\frac{4}{5}\right)^2=0$	$\frac{4}{5},\frac{4}{5}$	$25,-\frac{8}{5},\frac{16}{25}$	$\frac{8}{5}$	$\frac{16}{25}$	$\frac{8}{5}$	$\frac{16}{25}$	Roots are real and equal
$2x^2-15x-27=0$	– 3/2, 9	2, – 15, – 27	15/2	$-\frac{27}{2}$	15/2	– 27/2	Roots are real and distinct

Exercise 3.14

1. Write each of the following expression in terms of $\alpha+\beta$ and $\alpha\beta$.

(i) $\frac{\alpha}{3\beta}+\frac{\beta}{3\alpha}$ **(ii)** $\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}$

(iii) $(3\alpha-1)(3\beta-1)$ **(iv)** $\frac{\alpha+3}{\beta}+\frac{\beta+3}{\alpha}$

Sol :

(i) $$\frac{\alpha}{3\beta}+\frac{\beta}{3\alpha}=\frac{\alpha^2+\beta^2}{3\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{3\alpha\beta}$$

(ii) $$\frac{1}{\alpha^2\beta}+\frac{1}{\beta^2\alpha}=\frac{\beta+\alpha}{\alpha^2\beta^2}=\frac{\alpha+\beta}{(\alpha\beta)^2}$$

(iii) $$(3\alpha-1)(3\beta-1) = 9\alpha\beta-3\alpha-3\beta+1$$
$$= 9\alpha\beta-3(\alpha+\beta)+1$$

(iv) $\dfrac{\alpha+3}{\beta}+\dfrac{\beta+3}{\alpha} = \dfrac{\alpha(\alpha+3)+\beta(\beta+3)}{\alpha\beta}$

$= \dfrac{\alpha^2+3\alpha+\beta^2+3\beta}{\alpha\beta}$

$= \dfrac{(\alpha^2+\beta^2)+3(\alpha+\beta)}{\alpha\beta}$

$= \dfrac{(\alpha+\beta)^2-2\alpha\beta+3(\alpha+\beta)}{\alpha\beta}$

2. The roots of the equation $2x^2-7x+5=0$ are α and β. Without solving for the roots, find

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ **(ii)** $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

(iii) $\dfrac{\alpha+2}{\beta+2}+\dfrac{\beta+2}{\alpha+2}$

Sol :

Given equation $2x^2-7x+5=0$

Comparing with $ax^2+bx+c=0$

$a = 2,$

$b = -7,$

$c = 5$

Roots are α, β

Sum of the roots $= \alpha+\beta$

$= -\dfrac{b}{a} = \dfrac{7}{2}$

Product of the roots $= \alpha\beta$

$= \dfrac{c}{a} = \dfrac{5}{2}$

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta} = \dfrac{\beta+\alpha}{\alpha\beta}$

$= \dfrac{7/2}{5/2} = \dfrac{7}{5}$

(ii) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

$\dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

$= \dfrac{\left(\dfrac{7}{2}\right)^2-2\left(\dfrac{5}{2}\right)}{\dfrac{5}{2}}$

$= \dfrac{\dfrac{49}{4}-5}{\dfrac{5}{2}} = \left(\dfrac{49-20}{4}\right)\times\dfrac{2}{5}$

$= \dfrac{29}{4}\times\dfrac{2}{5} = \dfrac{29}{10}$

(iii) $\dfrac{\alpha+2}{\beta+2}+\dfrac{\beta+2}{\alpha+2} = \dfrac{(\alpha+2)^2+(\beta+2)^2}{(\alpha+2)(\beta+2)}$

$= \dfrac{\alpha^2+4\alpha+4+\beta^2+4\beta+4}{\alpha\beta+2\alpha+2\beta+4}$

$= \dfrac{(\alpha^2+\beta^2)+4(\alpha+\beta)+8}{\alpha\beta+2(\alpha+\beta)+4}$

$= \dfrac{(\alpha+\beta)^2-2\alpha\beta+4(\alpha+\beta)+8}{\alpha\beta+2(\alpha+\beta)+4}$

$= \dfrac{\left(\dfrac{7}{2}\right)^2-2\left(\dfrac{5}{2}\right)+4\left(\dfrac{7}{2}\right)+8}{\left(\dfrac{5}{2}\right)+2\left(\dfrac{7}{2}\right)+4}$

$= \dfrac{\dfrac{49}{4}+17}{\dfrac{5}{2}+11}$

$= \dfrac{117}{4}\times\dfrac{2}{27} = \dfrac{117}{54}$

3. The roots of the equation $x^2+6x-4=0$ are α, β. Find the quadratic equation whose roots are

(i) α^2 **and** β^2 **(ii)** $\dfrac{2}{\alpha}$ **and** $\dfrac{2}{\beta}$

(iii) $\alpha^2\beta$ **and** $\beta^2\alpha$

Sol :

Given equation $x^2+6x-4=0$

Comparing with $ax^2+bx+c=0$

$a = 1, b = 6, c = -4$

α, β are the roots

Sum of the roots $= \alpha+\beta$

$= -\dfrac{b}{a} = -\dfrac{6}{1} = -6$

Product of the roots $= \alpha\beta$

$= \dfrac{c}{a} = -\dfrac{4}{1} = -4$

(i) Given Roots are α^2 and β^2

Sum of the roots $= \alpha^2+\beta^2$

$= (\alpha+\beta)^2-2\alpha\beta$

$= (-6)^2-2(-4)$

$$= 44$$

Product of the roots $= \alpha^2 \beta^2$

$$= (\alpha\beta)^2$$
$$= (-4)^2 = 16$$

$\therefore$ Quadratic equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

$\therefore \quad x^2 - 44x + 16 = 0$

(ii) Given roots are $\frac{2}{\alpha}$ and $\frac{2}{\beta}$

Sum of the roots $= \frac{2}{\alpha} + \frac{2}{\beta}$

$$= \frac{2(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{2(-6)}{-4} = 3$$

Product of the roots $= \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$

$$= \frac{4}{\alpha\beta} = \frac{4}{-4} = -1$$

$\therefore$ Quadratic equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

$x^2 - 3x - 1 = 0$

(iii) Given roots are $\alpha^2\beta$ and $\beta^2\alpha$

Sum of the roots $= \alpha^2\beta + \beta^2\alpha$

$$= \alpha\beta(\alpha+\beta)$$
$$= (-4)(-6) = 24$$

Product of the roots $= (\alpha^2\beta)(\beta^2\alpha)$

$$= \alpha^3\beta^3 = (\alpha\beta)^3$$

$$= (-4)^3 = -64$$

$\therefore$ Quadratic equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

$x^2 - 24x - 64 = 0$

4. If α, β are the roots of $7x^2 + ax + 2 = 0$ and if $\beta - \alpha = \frac{-13}{7}$. Find the values of a.

Sol :

Given equation $7x^2 + ax + 2 = 0$

Comparing with $Ax^2 + Bx + C = 0$

$A = 7, B = a, C = 2$

α and β are the roots

Sum of the roots $\alpha+\beta = -\frac{B}{A} = -\frac{a}{7}$

Product of the roots $\alpha\beta = \frac{C}{A} = \frac{2}{7}$

and Given $\beta - \alpha = \frac{-13}{7}$

From the data, $(\alpha+\beta)^2 - (\beta-\alpha)^2 = 4\alpha\beta$

$$\left(\frac{-a}{7}\right)^2 - \left(\frac{-13}{7}\right)^2 = 4\left(\frac{2}{7}\right)$$

$$\frac{a^2}{49} - \frac{169}{49} = \frac{8}{7}$$

$$a^2 - 169 = \frac{8 \times 49}{7}$$

$$a^2 - 169 = 56$$
$$a^2 = 56 + 169$$
$$= 225$$
$$a = \pm 15$$

5. If one root of the equation $2y^2 - ay + 64 = 0$ is twice the other then find the values of a.

Sol :

Given equation $2y^2 - ay + 64 = 0$

$A = 2,$

$B = -a,$

$C = 64$

Given that one root is twice the other.

Let the roots be 'α' and '2α'

$\therefore$ Sum of the roots $\alpha + 2\alpha = -\frac{B}{A}$

$$3\alpha = \frac{a}{2}$$

$$\Rightarrow \alpha = \frac{a}{6}$$

Product of the roots $(\alpha)(2\alpha) = \frac{C}{A}$

$$2\alpha^2 = \frac{64}{2}$$
$$= 32$$
$$\alpha^2 = 16$$

$$\left(\frac{a}{6}\right)^2 = 16$$

$$\frac{a}{6} = \pm 4$$

$$\Rightarrow a = 24, -24$$

6. If one root of the equation $3x^2 + kx + 81 = 0$ (having real roots) is the square of the other then find k.

Sol :

Given equation $3x^2 + kx + 81 = 0$

$a = 3,$

$b = k,$

$c = 81$

Given that one root is square of the other.

$\therefore$ Let the root be 'α' and 'α^2'

Sum of the roots $\alpha + \alpha^2 = -b/a$

$= \frac{-k}{3}$

Product of the roots $(\alpha)(\alpha^2) = c/a$

$\alpha^3 = \frac{81}{3}$

$= 27$

$\therefore \alpha = 3$

Substituting in $\alpha + \alpha^2 = -k/3$

$\Rightarrow (3+9)3 = -k$

$k = -36$

QUADRATIC GRAPHS

Worked Examples

3.48. Discuss the nature of solutions of the following quadratic equations.

(i) $x^2 + x - 12 = 0$ **(ii) $x^2 - 8x + 16 = 0$**

(iii) $x^2 + 2x + 5 = 0$

Sol :

(i) $x^2 + x - 12 = 0$

Step 1: Prepare the table of values for the equation $y = x^2 + x - 12$

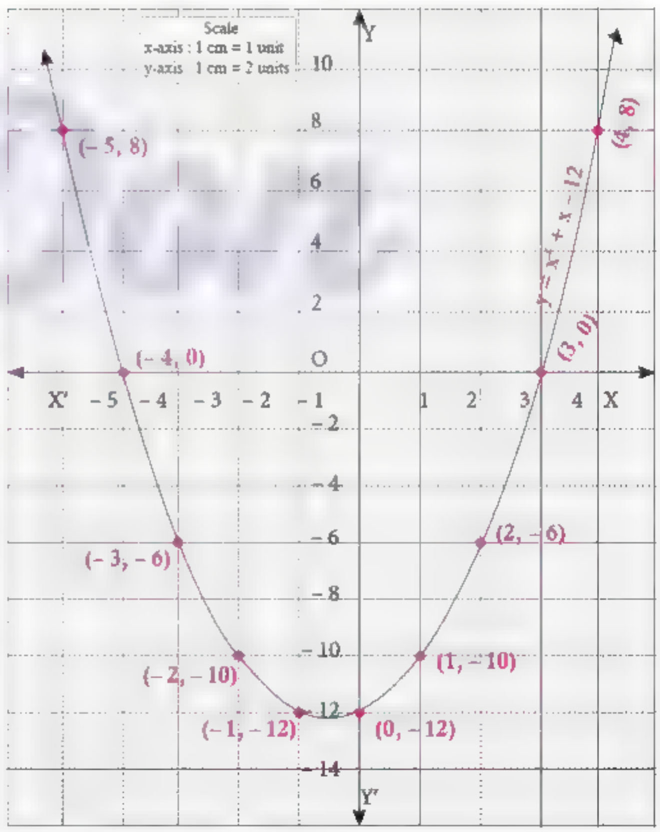
x	–5	–4	–3	–2	–1	0	1	2	3	4
y	8	0	–6	–10	–12	–12	–10	–6	0	8

Step 2: Plot the points for the above ordered pairs (x, y)

Step 3: Draw the parabola and mark the co-ordinates of the parabola which intersect with the X-axis.

Step 4: The roots of the equation are the x-coordinates of the intersecting points of the parabola with the X-axis (– 4, 0) and (3, 0) which are – 4 and 3.

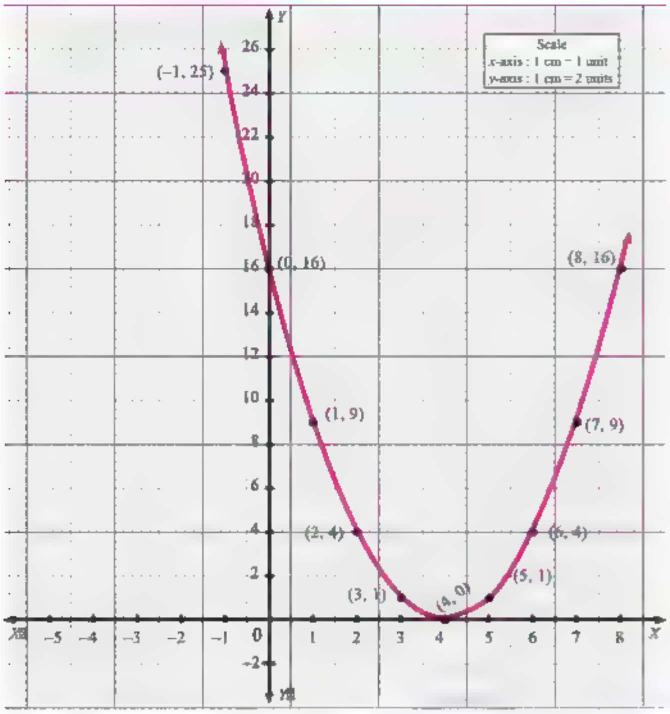
Since there are two points of intersection with the X-axis, the quadratic equation $x^2 + x - 12 = 0$ has real and unequal roots.



(ii) $x^2 - 8x + 16 = 0$

Step 1: Prepare the table of values for the equation $y = x^2 - 8x + 16$

x	–1	0	1	2	3	4	5	6	7	8
y	25	16	9	4	1	0	1	4	9	16



Step 2: Plot the points for the above ordered pairs (x, y)

Step 3: Draw the parabola and mark the co-ordinates of the parabola which intersect with the X-axis.

Step 4: The roots of the equation are the x-coordinates of the intersecting points of the parabola with the X-axis (4, 0) which is 4.

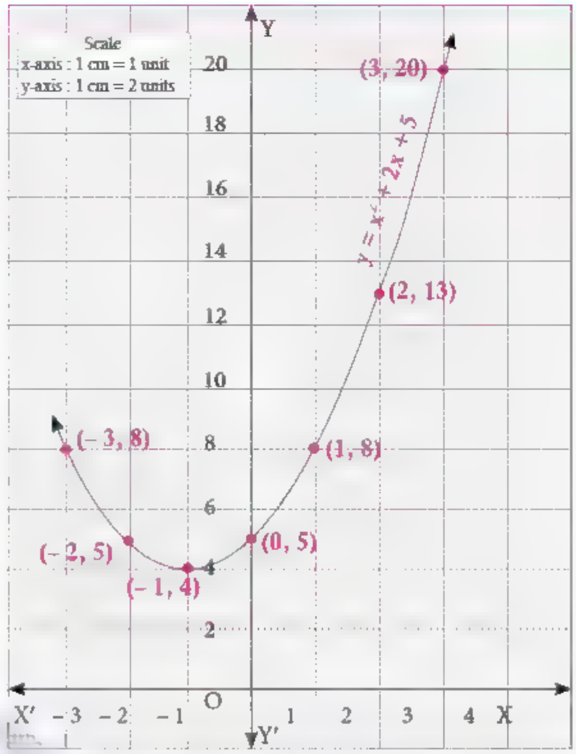
Since there is only one point of intersection with X-axis, the quadratic equation $x^2 - 8x + 16 = 0$ has real and equal roots.

(iii) $x^2 + 2x + 5 = 0$

Let $y = x^2 + 2x + 5$

Step 1: Prepare a table of values for

x	−3	−2	−1	0	1	2	3
y	8	5	4	5	8	13	20



Step 2: Plot the above ordered pairs (x, y) on the graph using suitable scale.

Step 3: Join the points by a free-hand smooth curve this smooth curve is the graph of $y = x^2 + 2x + 5$.

Step 4: The solutions of the given Quadratic equation are the x-coordinates of the intersecting points of the parabola with the X-axis.

Here the parabola doesn't intersect/touch the X-axis.

So, we conclude that there is no real root for the given quadratic equation.

3.49. Draw the graph of $y = 2x^2$ and hence solve $2x^2 - x - 6 = 0$.

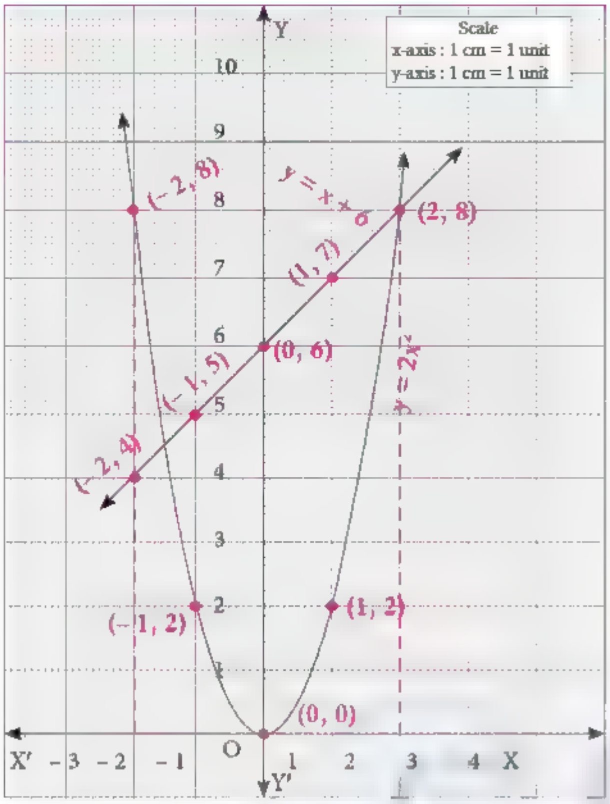
Sol :

Step 1: Draw the graph of $y = 2x^2$ by preparing the table of values as below

x	− 2	− 1	0	1	2
y	8	2	0	2	8

Step 2: To solve $2x^2 - x - 6 = 0$, subtract $2x^2 - x - 6 = 0$ from $y = 2x^2$

that is
$$\begin{aligned} y &= 2x^2 \\ 0 &= 2x^2 - x - 6 \quad (-) \\ \hline y &= x + 6 \end{aligned}$$



The equation $y = x + 6$ represents a straight line. Draw the graph of $y = x + 6$ by forming table of values as below

x	− 2	− 1	0	1	2
y	4	5	6	7	8

Step 3: Mark the points of intersection of the curve $y = 2x^2$ and $y = x + 6$. That is, $(-1.5, 4.5)$ and $(2, 8)$.

Step 4: The x-coordinates of the respective points forms the solution set $\{-1.5, 2\}$ for $2x^2 - x - 6 = 0$.

3.50. Draw the graph of $y = x^2 + 4x + 3$ and hence find the roots of $x^2 + x + 1 = 0$

Sol :

Step 1: Draw the graph of $y = x^2 + 4x + 3$ by preparing the table of values as below

x	− 4	− 3	− 2	− 1	0	1	2
y	3	0	− 1	0	3	8	15

Step 2: To solve $x^2 + x + 1 = 0$, subtract $x^2 + x + 1 = 0$ from $y = x^2 + 4x + 3$ that is,

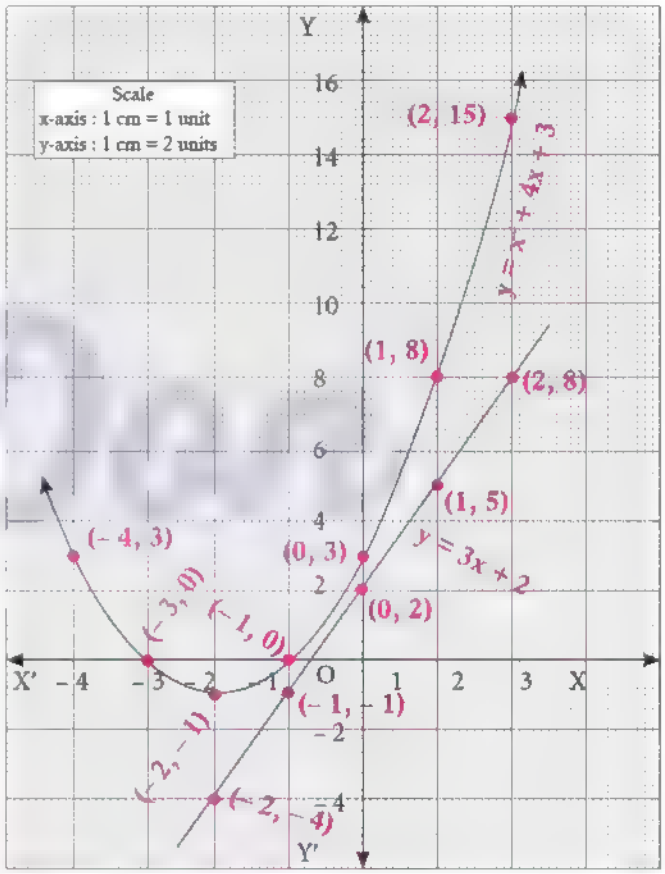
$$\begin{array}{rcl} y &=& x^2 + 4x + 3 \\ 0 &=& x^2 + x + 1 \quad (-) \\ \hline y &=& 3x + 2 \\ \hline \end{array}$$

The equation represent a straight line. Draw the graph of $y = 3x + 2$ forming the table of values as below

x	− 2	− 1	0	1	2
y	− 4	− 1	2	5	8

Step 3: Observe that the graph of $y = 3x + 2$ does not intersect/touch the graph of the parabola $y = x^2 + 4x + 3$.

Thus $x^2 + x + 1 = 0$ has no real roots.

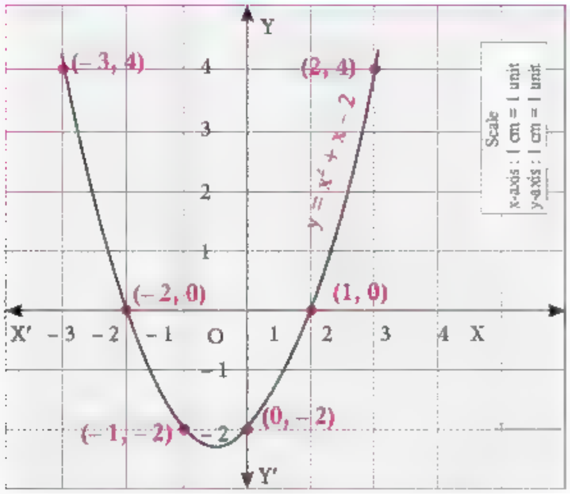


3.51. Draw the graph of $y = x^2 + x - 2$ and hence solve $x^2 + x - 2 = 0$.

Sol :

Step 1: Draw the graph $y = x^2 + x - 2$ by preparing the table of values as below

x	− 3	− 2	− 1	0	1	2
y	4	0	− 2	− 2	0	4



Step 2: To solve $x^2 + x - 2 = 0$ subtract $x^2 + x - 2 = 0$ from $y = x^2 + x - 2$.

that is
$$\begin{array}{rcll} y & = & x^2 + x - 2 & \\ 0 & = & x^2 + x - 2 & (-) \\ \hline y & = & 0 & \end{array}$$

The equation $y = 0$ represents the X-axis.

Step 3: Mark the point of intersection of the curve $x^2 + x - 2$ with the X-axis. That is $(-2, 0)$ and $(1, 0)$.

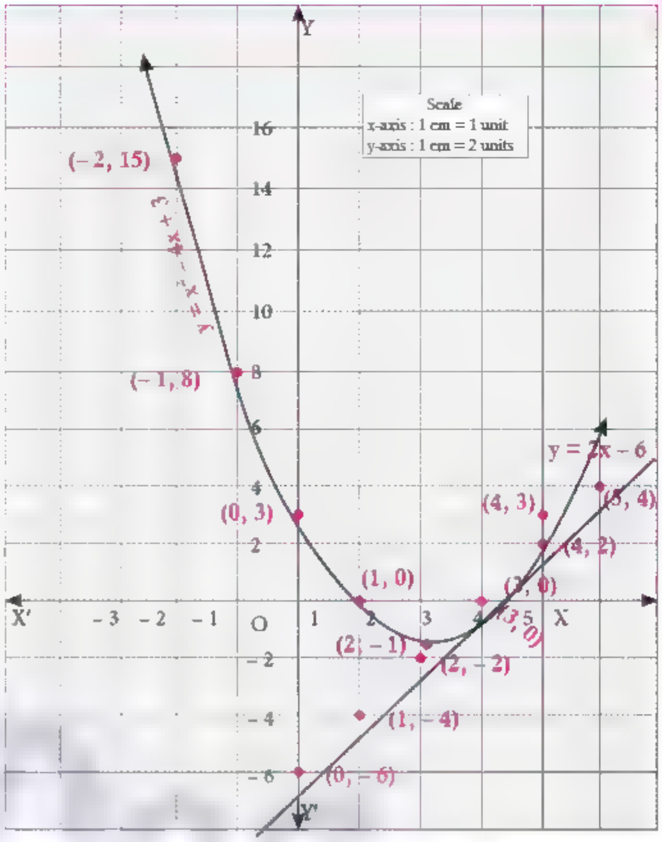
Step 4: The x co-ordinates of the respective points form the solution set $\{-2, 1\}$ for $x^2 + x - 2 = 0$

3.52. **Draw the graph of $y = x^2 - 4x + 3$ and use it to solve $x^2 - 6x + 9 = 0$.**

Sol :

Step 1: Draw the graph of $y = x^2 - 4x + 3$ by preparing the table of values as below

x	– 2	– 1	0	1	2	3	4
y	15	8	3	0	– 1	0	3



Step 2: To solve $x^2 - 6x + 9 = 0$, subtract $x^2 - 6x + 9 = 0$ from $y = x^2 - 4x + 3$

that is
$$\begin{array}{rcll} y & = & x^2 - 4x + 3 & \\ 0 & = & x^2 - 6x + 9 & (-) \\ \hline y & = & 2x - 6 & \end{array}$$

The equation $y = 2x - 6$ represent a straight line. Draw the graph of $y = 2x - 6$ forming the table of values as below.

x	0	1	2	3	4	5
y	– 6	– 4	– 2	0	2	4

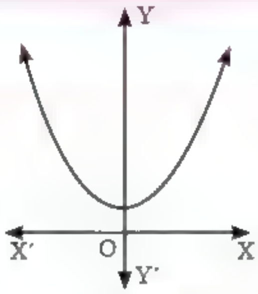
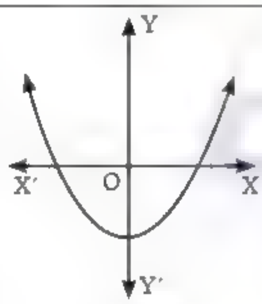
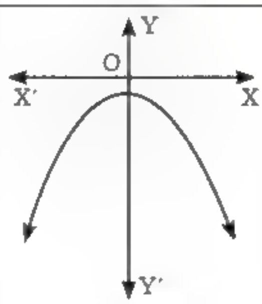
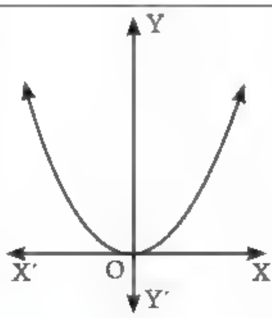
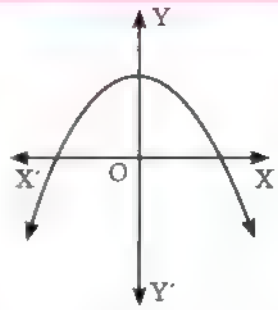
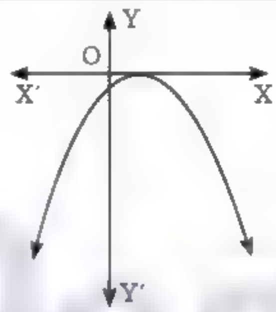
The line $y = 2x - 6$ intersects $y = x^2 - 4x + 3$ only at one point.

Step 3: Mark the point of intersection of the curve $y = x^2 - 4x + 3$ and $y = 2x - 6$, that is $(3, 0)$.

Therefore, the x-coordinate 3 is the only solution for the equation $x^2 - 6x + 9 = 0$.

Progress Check

1. Connect the graphs to its respective number of intersection with X-axis and to its corresponding nature of solutions which is given in the following table.

	Graphs	Number of points of Inter-section with x-axis	Nature of solutions
1		2	Real and Equal roots
2		1	No Real Roots
3		2	No Real Roots
4		0	Real and Equal Roots
5		0	Real and unequal roots
6		1	Real and unequal Roots

Ans :

	No. of points	Nature of solutions
1	0	No Real roots
2	2	Real and unequal roots
3	0	No Real roots
4	1	Real and equal roots
5	2	Real and unequal roots
6	1	Real and equal roots

Exercise 3.15

1. Graph the following quadratic equations and state their nature of solutions.

(i) $x^2 - 9x + 20 = 0$

(ii) $x^2 - 4x + 4 = 0$

(iii) $x^2 + x + 7 = 0$

(iv) $x^2 - 9 = 0$

(v) $x^2 - 6x + 9 = 0$

(vi) $(2x - 3)(x + 2) = 0$

Sol :

(i) $x^2 - 9x + 20 = 0$

Let $y = x^2 - 9x + 20$

Table of values

x	−1	0	1	2	3	4	5	6
x^2	1	0	1	4	9	16	25	36
−9x	9	0	−9	−18	−27	−36	−45	−54
20	20	20	20	20	20	20	20	20
y	30	20	12	6	2	0	0	2

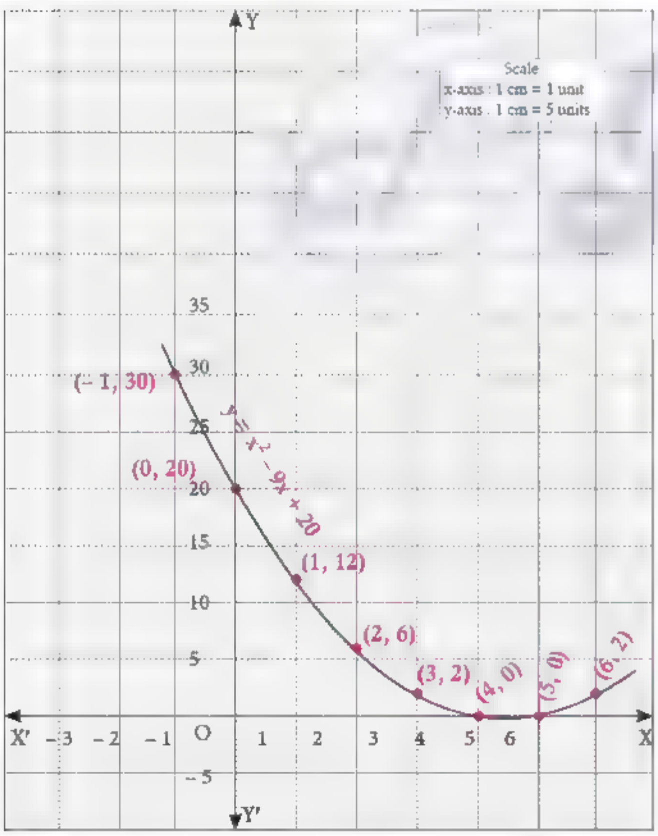
Now, plotting the points on the graph and joining them, we get the graph of

$$y = x^2 - 9x + 20$$

The parabola intersects X-axis at (4, 0) and (5, 0).

Hence the solution is x = 4, 5.

Since, the parabola intersects X-axis at two different points, the roots are real and unequal.



(ii) Solve $x^2 - 4x + 4 = 0$

Let $y = x^2 - 4x + 4$

Table of values

x	− 2	− 1	0	1	2	3
x^2	4	1	0	1	4	9
− 4x	8	4	0	− 4	− 8	− 12
4	4	4	4	4	4	4
y	16	9	4	1	0	1

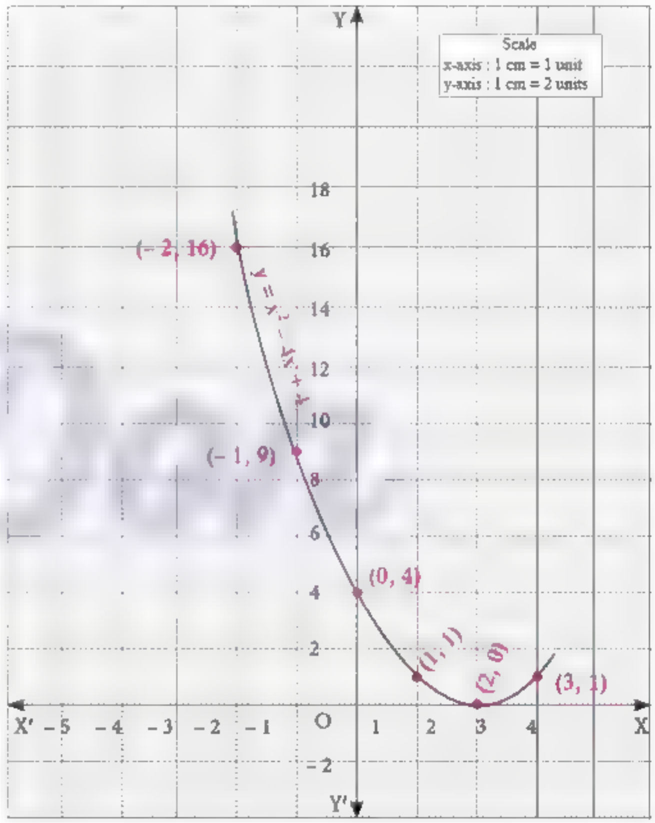
Now, plotting the points on the graph and joining them, we get the graph of

$$y = x^2 - 4x + 4$$

The parabola intersects the X-axis at (2, 0).

Hence the solution is x = 2

Since, the parabola intersects X-axis at only one point, the roots are real and equal.

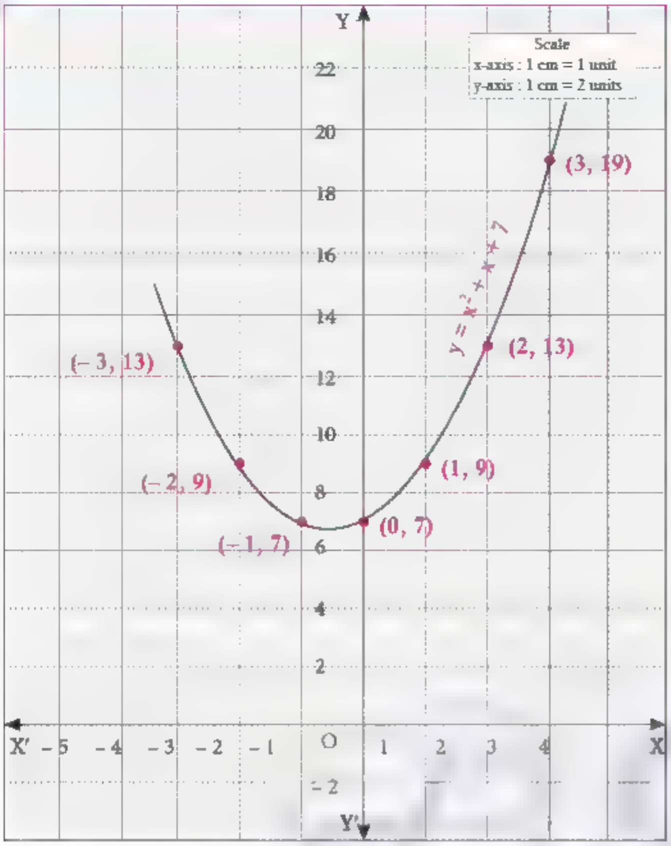


(iii) $x^2 + x + 7 = 0$

Let $y = x^2 + x + 7$

Table of values

x	− 3	− 2	− 1	0	1	2	3
x^2	9	4	1	0	1	4	9
7	7	7	7	7	7	7	7
y	13	9	7	7	9	13	19



Now, plotting the points on the graph and joining them, we get the graph of

$$y = x^2 + x + 7$$

Here, the parabola doesn't intersect the x - axis

$\therefore$ The given quadratic equation has no real roots.

(iv) $x^2 - 9 = 0$

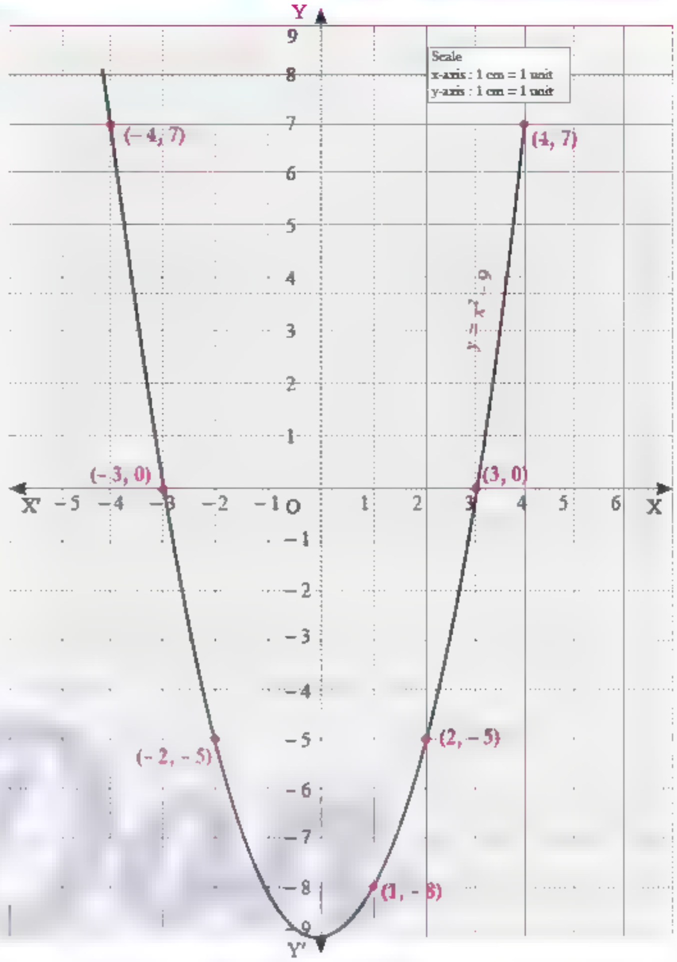
Let $y = x^2 - 9$

Table of Points:

x	–4	–3	–2	–1	0	1	2	3	4
x^2	16	9	4	1	0	1	3	9	16
–9	–9	–9	–9	–9	–9	–9	–9	–9	–9
y	7	0	–5	–8	–9	–8	–5	0	7

Points to be plotted in the graph are

(– 4, 7), (– 3, 0) (– 2, – 5), (0, – 9), (1, – 8), (2, – 5), (3, 0), (4, 7)



From the graph, we see that the curve $y = x^2 - 9$ intersects X - axis at (– 3, 0) and (3, 0).

$\therefore$ Solution: $x = -3, 3$

Roots are real and unequal.

(v)

$x^2 - 6x + 9 = 0$

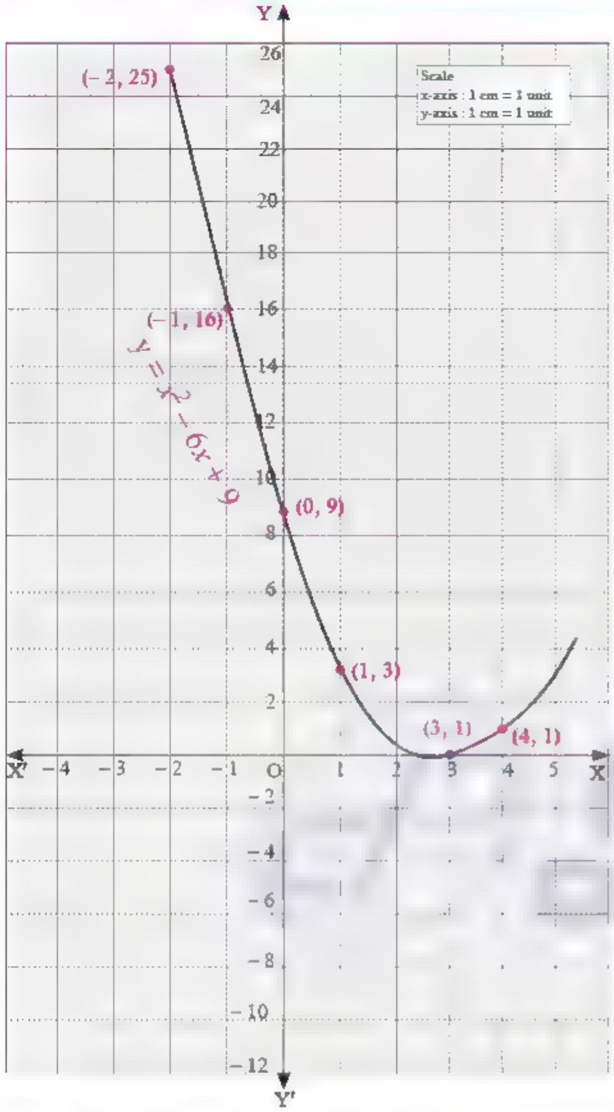
Let $y = x^2 - 6x + 9$

Table of Points:

x	–2	–1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
–6x	12	6	0	–6	–12	–18	–24
9	9	9	9	9	9	9	9
g	25	26	9	4	1	0	1

Points to be plotted in the graph are

(– 2, 25), (– 1, 26) (0, 9), (1, 4), (2, 1), (3, 0), (4, 1)



From the graph, we see that the curve

$y = x^2 - 6x + 9$ intersects

X - axis at (3, 0)

∴ Solution: $x = 3, 3$

Hence the roots are real and equal.

(vi) $(2x - 3)(x + 2) = 0$

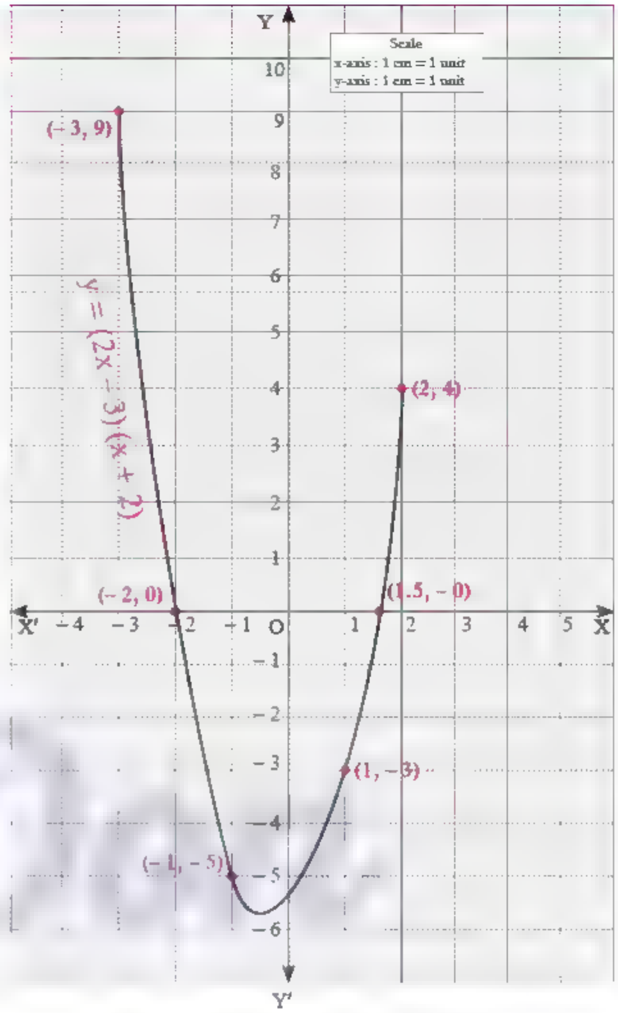
Let $y = (2x - 3)(x + 2)$

Table of Points:

x	– 3	– 2	– 1	0	1	1.5	2
2x – 3	– 9	– 7	– 5	– 3	– 1	0	1
x + 2	– 1	0	1	2	3	3.5	4
y	9	0	– 5	– 6	– 3	0	4

Points to be plotted in the graph are

(– 3, 9), (– 2, 0) (– 1, – 5), (0, – 6), (1, – 3), (1.5, 0) and (2, 4)



From the graph, we see that the curve

$y = (2x^2 - 3)$ (-2, 0) intersects

X - axis at (-2, 0) and (1.5, 0)

∴ Solution: $x = -2, 3/2$

Hence the roots are real and unequal.

2. Draw the graph of $y = x^2 - 4$ and hence solve $x^2 - x - 12 = 0$

Sol :

$$y = x^2 - 4$$

Table of points

x	– 4	– 3	– 2	– 1	0	1	2	3	4
x^2	16	9	4	1	0	1	4	9	16
– 4	– 4	–4	– 4	– 4	– 4	– 4	– 4	– 4	– 4
y	12	5	0	– 3	– 4	– 3	0	5	12

Now plotting the points on the graph and joining them we get the graph $y = x^2 - 4$

To solve $x^2 - x - 12 = 0$, subtract $x^2 - x - 12 = 0$ from $y = x^2 - 4$, that is

$$y = x^2 + 0x - 4$$
$$0 = x^2 - x - 12 \quad (-)$$
$$y = x + 8$$

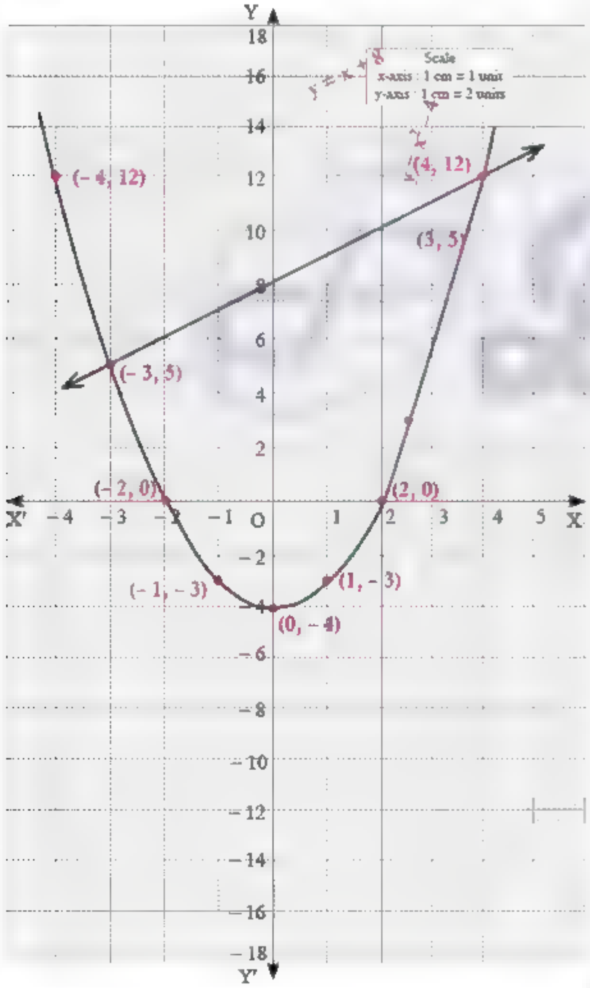
The equation represents a straight line.

Draw the graph of $y = x + 8$ by forming the table of values as below

x	-2	-1	0	1	2
y	6	7	8	9	10

It meets the parabola at (4, 12) and (-3, 5)

The solution x is = 4, -3



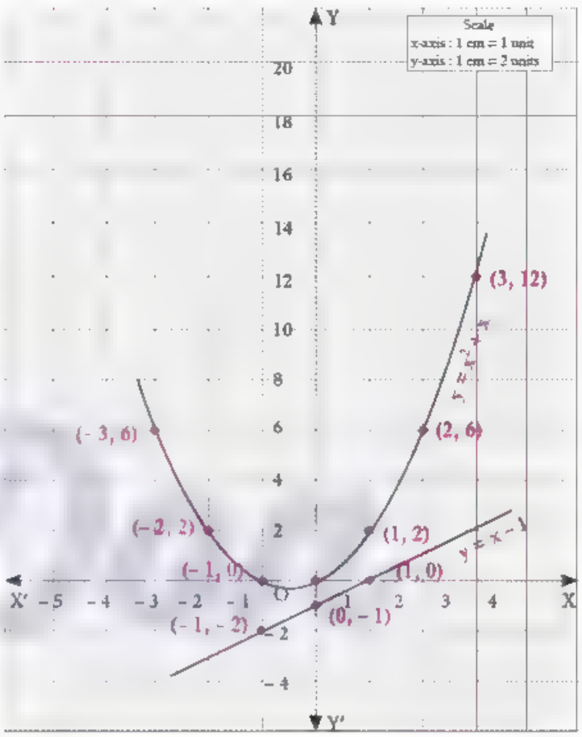
3. Draw the graph of $y = x^2 + x$ and hence solve $x^2 + 1 = 0$

Sol :

$y = x^2 + x$

Table of values

x	−3	−2	−1	0	1	2	3
x^2	9	4	1	0	1	4	9
y	6	2	0	0	2	6	12



Plotting and joining the points, we get the graph of

$y = x^2 + x$.

Now subtracting $x^2 + 1 = 0$ from $y = x^2 + x$

$$\therefore \; y = x^2 + x$$
$$0 = x^2 + 1 \quad (-)$$

$y = x - 1$ is a straight line

Table of values

x	0	1	−1
y	−1	0	−2

Let us draw a straight line by joining these points.

From the graph, we see that the straight line does not intersect the parabola.

$\therefore$ The Quadratic equation has no real roots.

4. Draw the graph of $y = x^2 + 3x + 2$ and use it to solve $x^2 + 2x + 1 = 0$

Sol :

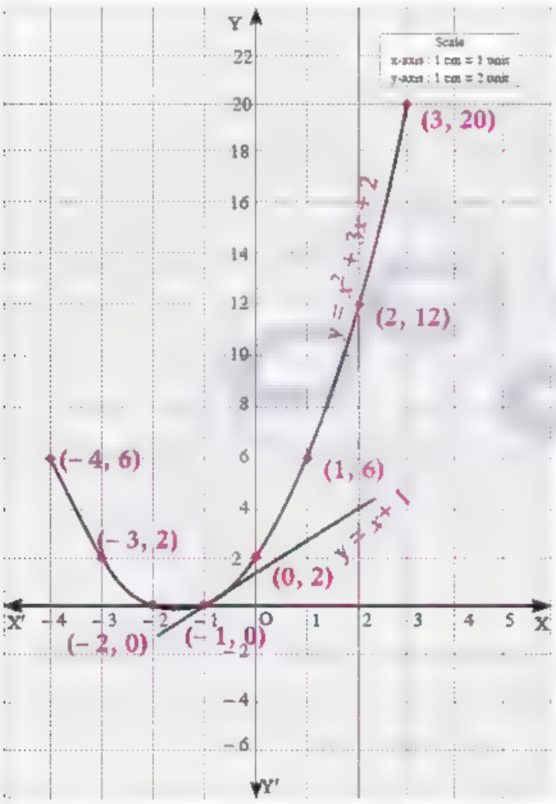
$y = x^2 + 3x + 2$

Table of Points:

x	– 4	– 3	– 2	– 1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
3x	–12	– 9	– 6	– 3	0	3	6	9
2	2	2	2	2	2	2	2	2
y	6	2	0	0	2	6	12	20

Points to be plotted in the graph are

(– 3, 2), (– 2, 0) (– 1, 0), (0, 2), (1, 6), (2, 12) and (3, 20)



Now, Subtracting $y = x^2 + 3x + 2$ and $x^2 + 2x + 1 = 0$, we get

$$\begin{array}{l} y = x^2 + 3x + 2 \\ 0 = x^2 + 2x + 1 \quad (-) \\ \hline y = x + 1 \end{array}$$ is a straight line.

Table of points

x	0	– 1	1
y	1	0	2

The curve $y = x^2 + 3x + 2$ and the line $y = x + 1$ intersects at (–1, 0)

Hence the solution of $x^2 + 2x + 1 = 0$ is $x = -1, -1$

Roots are real and equal.

5. Draw the graph of $y = x^2 + 3x - 4$ and hence use it to solve $x^2 + 3x - 4 = 0$

Sol :

$y = x^2 + 3x - 4$

Table of values

x	–5	–4	–3	–2	–1	0	1	2
x^2	25	16	9	4	1	0	1	4
3x	–15	–12	–9	–6	–3	0	3	6
–4	–4	–4	–4	–4	–4	–4	–4	–4
y	6	0	–4	–6	–6	–4	0	6

Plotting and joining the points on the graph, we get the parabola

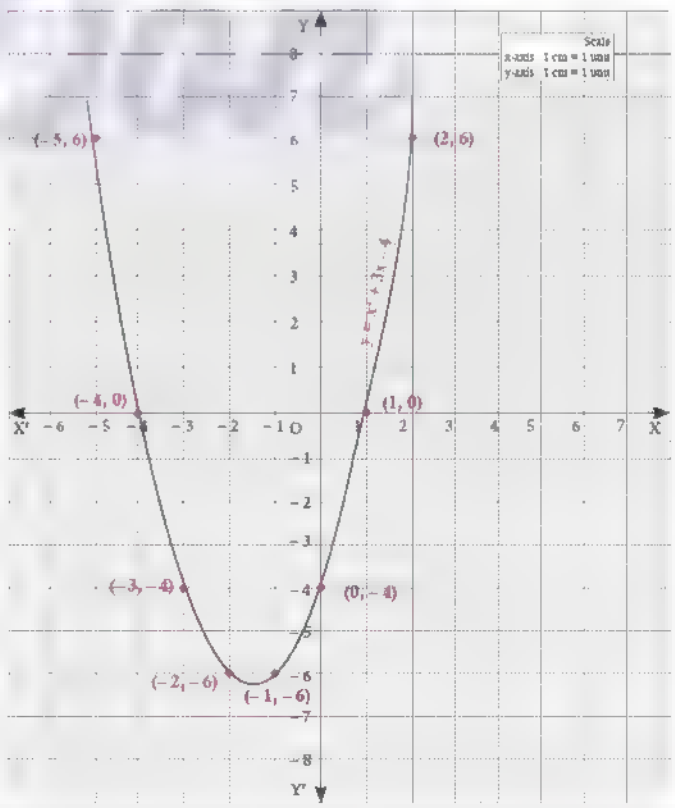
$$y = x^2 + 3x - 4$$

Now, subtracting $x^2 + 3x - 4 = 0$ from $y = x^2 + 3x - 4$

$$\begin{array}{l} y = x^2 + 3x - 4 \\ 0 = x^2 + 3x - 4 \quad (-) \\ \hline y = 0 \quad \Rightarrow \quad \text{X-axis} \end{array}$$

The parabola $y = x^2 + 3x - 4$ intersects X-axis at (–4, 0) and (1, 0)

$\therefore$ Solution $x = -4, 1$



6. Draw the graph of $y = x^2 - 5x - 6$ and hence solve $x^2 - 5x - 14 = 0$

Sol :

$y = x^2 - 5x - 6$

Table of values

x	–2	–1	0	1	2	3	4	5	6	7
x^2	4	1	0	1	4	9	16	25	36	49
–5x	10	5	0	–5	–10	–15	–20	–25	–30	–35
–6	–6	–6	–6	–6	–6	–6	–6	–6	–6	–6
y	8	0	–6	–10	–12	–12	–10	–6	0	8

Plotting and joining these points, we get the parabola

$y = x^2 - 5x - 6$

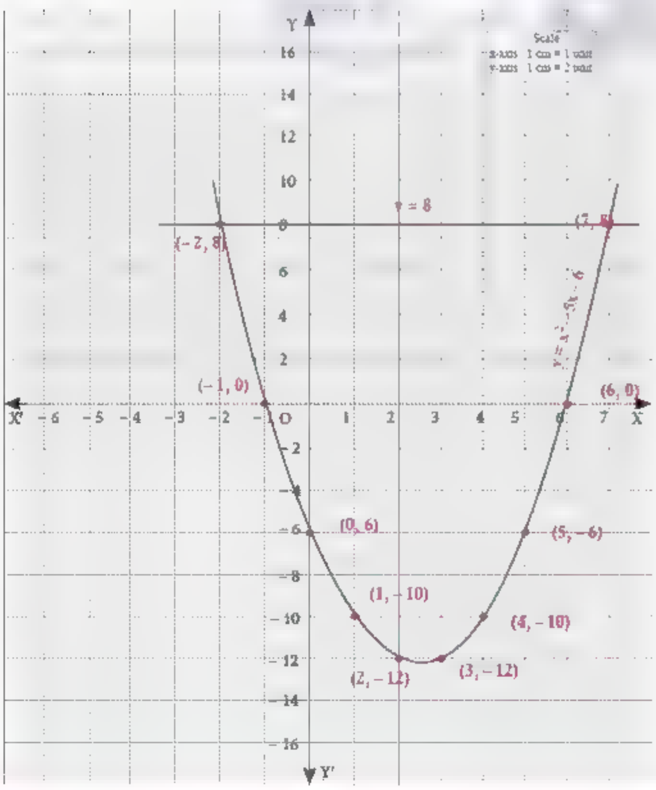
Now, Subtracting $x^2 - 5x - 14 = 0$ from $y = x^2 - 5x - 6$

$$\begin{aligned} y &= x^2 - 5x - 6 \\ 0 &= x^2 - 5x - 14 \ (-) \\ \hline y &= \quad 8 \end{aligned}$$

$y = 8$ is a straight line.

The straight line $y = 8$ and the parabola $y = x^2 - 5x - 6$ intersect at $(-2, 8)$ and $(7, 8)$.

$\therefore$ The solution is $x = -2, 7$.



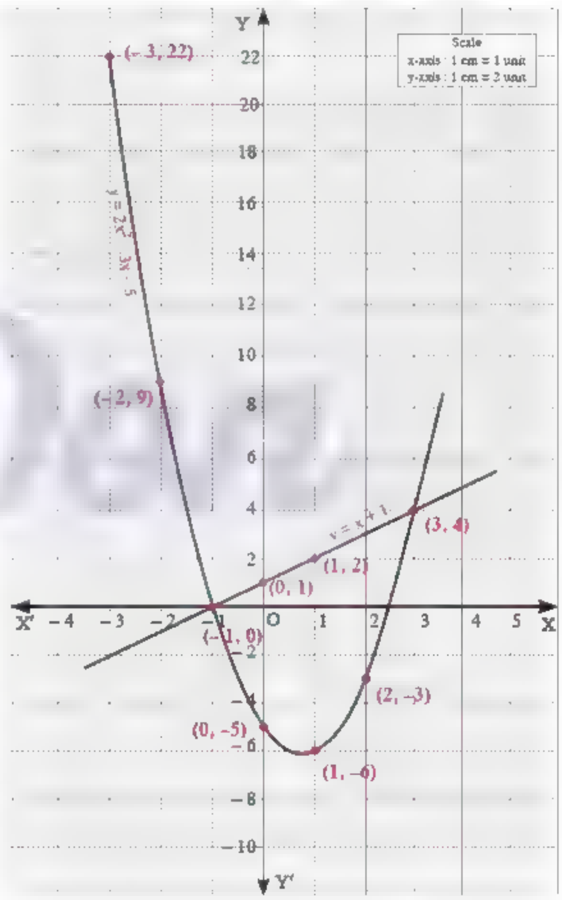
7. Draw the graph of $y = 2x^2 - 3x - 5$ and hence solve $2x^2 - 4x - 6 = 0$

Sol :

Table of Points:

x	–3	–2	–1	0	1	2	3
$2x^2$	18	8	2	0	2	8	18
$-3x$	9	6	3	0	–3	–6	–9
–5	–5	–5	–5	–5	–5	–5	–5
y	22	9	0	–5	–6	–3	–4

Points are (–3, 22), (–2, 9) (–1, 0), (0, –5), (1, –6), (2, –3) and (3, –4)



Now, Subtracting $y = 2x^2 - 3x - 5$ and $2x^2 - 4x - 6 = 0$, we get

$$\begin{aligned} y &= 2x^2 - 3x - 5 \\ (-) \quad 0 &= 2x^2 - 4x - 6 \\ \hline y &= x + 1 \end{aligned}$$

Table of points

x	0	–1	1
y	3	0	6

The curve $y = 2x^2 - 2x - 5$ and the line $y = x + 1$ intersect at (–1, 0) and (3, 4)

Hence the solution of $2x^2 - 4x - 6 = 0$ (or) $x^2 - 2x - 3 = 0$ is $x = -1, 3$.

Roots are real and unequal.

8. Draw the graph of $y = (x - 1)(x + 3)$ and hence solve $x^2 - x - 6 = 0$.

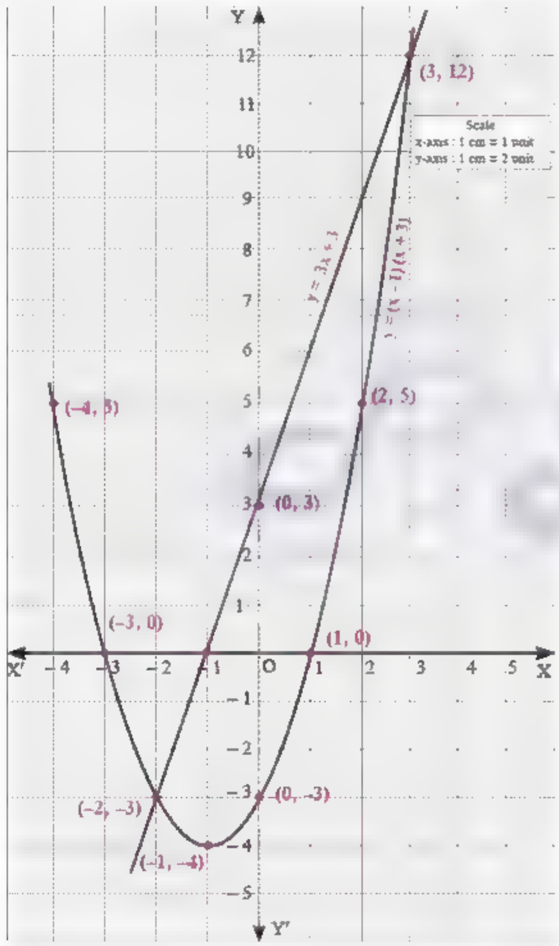
Sol :

Table of Points:

x	-4	-3	-2	-1	0	1	2	3
$x-1$	-5	-4	-3	-2	-1	0	1	2
$x+3$	-1	0	1	2	3	4	5	6
y	5	0	-3	-4	-3	0	5	12

Points to be plotted in the graph are

$(-4, 5)$, $(-3, 0)$ $(-2, -3)$, $(-1, -4)$, $(0, -3)$, $(1, 0)$ and $(2, 5)$



Now, Subtracting $y = x^2 + 2x - 3$ and $x^2 - x - 6 = 0$, we get

$$\begin{array}{rl} & y = x^2 + 2x - 3 \\ (-) & \underline{0 = x^2 - x - 6} \\ & \underline{y = 3x + 3} \quad \text{is a straight line} \end{array}$$

Table of points

x	0	-1	1
y	3	0	6

The curve $y = x^2 + 2x - 3$ and the line $y = 3x + 3$ intersect at $(-2, -3)$ and $(3, 12)$

Hence the solution of $x^2 - x - 6 = 0$ is $x = -2, 3$.

Roots are real and unequal.

MATRICES

Key Points

- A Rectangular array of numbers, variables is called a 'matrix'.
- Horizontal arrangement is 'Row' and vertical arrangement is Column.
- Order of a matrix = No. of Rows × No. of Columns = $m \times n$
- Number of elements in the matrix = mn.
- **Row matrix:** A matrix is having only one row is called a row matrix. Example: $\begin{bmatrix} 8 & 4 & -1 \end{bmatrix}$
- **Column matrix:** A matrix is having only one column is called a column matrix. Example: $\begin{pmatrix} 9 \\ 5 \\ 3 \end{pmatrix}$
- **Square matrix:** A matrix in which the number of rows and no. of columns are equal is called the square matrix. Example: $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 3 & -11 \end{bmatrix}$
- **Diagonal matrix:** A square matrix in which all the elements are zero, except the leading diagonal elements is called a Diagonal matrix.

  Example: $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$
- **Scalar matrix:** A diagonal matrix in which all the leading elements are equal is called the Scalar matrix. Example: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$
- **Identity (or) Unit matrix:**
  A Scalar matrix in which all the leading diagonal elements are '1', is called the Identity matrix and it is denoted by I. Example: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- **Zero matrix (or) Null matrix:**

  A matrix is called a zero matrix if all of its elements are zero. Example: $\begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Triangular matrix:

A square matrix in which all the entries above the leading diagonal are zero is called a lower triangular matrix. Example: $\begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 5 & -1 & 2 \end{bmatrix}$ If all the entries below the leading diagonal are zero, then it is called upper triangular matrix. Example: $\begin{bmatrix} 1 & 7 & -3 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

Transpose of a matrix:

The matrix obtained by interchanging the elements in rows and columns of the matrix A is called transpose of A and it is denoted by A^T. Example: $A = \begin{bmatrix} 1 & 5 \\ 8 & 9 \\ 4 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 8 & 4 \\ 5 & 9 & 3 \end{bmatrix}$ and $(A^T)^T = A$.

Equal matrices:

Two matrices A and B are equal if and only if they have the same order and corresponding elements are same.

Worked Examples

3.53. **Consider the following information regarding the number of men and women workers in three factories I, II and III.**

	Men	Women
I	23	18
II	47	36
III	15	16

Represent the above information in the form of a matrix. What does the entry in the second row and first column represent?

Sol :

The information is represented in the from of a 3×2 matrix as follows

$$A = \begin{bmatrix} 23 & 18 \\ 47 & 36 \\ 15 & 16 \end{bmatrix}$$

The entry in the second row and first column represent that there are 47 men workers in factory II.

3.54. **If a matrix has 16 elements, what are the possible orders it can have?**

Sol :

We know that a matrix is of order $m \times n$, has mn elements. Thus to find all possible orders of a matrix with 16 elements, we will find all ordered pairs of natural numbers whose product is 16.

Such ordered pairs are (1, 16), (16, 1), (4, 4), (8, 2), (2, 8)

Hence possible orders are 1×16, 16×1, 4×4, 2×8, 8×2

3.55. **Construct a 3×3 matrix whose elements are $a_{ij} = i^2 j^2$.**

Sol :

The general 3×3 matrix is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{ij} = i^2 j^2$$

$$\begin{aligned}
a_{11} &= 1^2 \times 1^2 = 1 \times 1 = 1; \\
a_{12} &= 1^2 \times 2^2 = 1 \times 4 = 4; \\
a_{13} &= 1^2 \times 3^2 = 1 \times 9 = 9; \\
a_{21} &= 2^2 \times 1^2 = 4 \times 1 = 4; \\
a_{22} &= 2^2 \times 2^2 = 4 \times 4 = 16; \\
a_{23} &= 2^2 \times 3^2 = 4 \times 9 = 36; \\
a_{31} &= 3^2 \times 1^2 = 9 \times 1 = 9;
\end{aligned}$$

$a_{32} = 3^2 \times 2^2 = 9 \times 4 = 36;$

$a_{33} = 3^2 \times 3^2 = 9 \times 9 = 81;$

Hence the required matrix is

$$A = \begin{bmatrix} 1 & 4 & 9 \\ 4 & 16 & 36 \\ 9 & 36 & 81 \end{bmatrix}$$

3.56. **Find the value of a, b, c, d from the equation**

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$$

Sol :

The given matrices are equal. Thus all corresponding elements are equal.

Therefore $a - b = 1$ ... (1)

$2a + c = 5$ ... (2)

$2a - b = 0$ ... (3)

$3c + d = 2$ ... (4)

(3) $\Rightarrow$ $2a - b = 0$

$2a = b$ ... (5)

Put $2a = b$ in equation (1), $a - 2a = 1 \Rightarrow a = -1$

Put $a = -1$ in equation (5), $2(-1) = b \Rightarrow b = -2$

Put $a = -1$ in equation (2), $2(-1) + c = 5 \Rightarrow c = 7$

Put $c = 7$ in equation (4), $3(7) + d = 2 \Rightarrow d = -19$

Therefore, $a = -1, b = -2, c = 7, d = -19$

Progress Check

1. Find the element in the second row and third column of the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$

Ans : $a_{23} = 5$

2. Find the order of the matrix $\begin{bmatrix} \sin\theta \\ \cos\theta \\ \tan\theta \end{bmatrix}$

Ans : Order $= 3 \times 1$

3. Determine the entries denoted by a_{11}, a_{22}, a_{33}, a_{44} from the matrix $\begin{bmatrix} 2 & 1 & 3 & 4 \\ 5 & 9 & -4 & \sqrt{7} \\ 3 & 5/2 & 8 & 9 \\ 7 & 0 & 1 & 4 \end{bmatrix}$

Ans : $a_{11} = 2,\quad a_{22} = 9,\quad a_{33} = 8,\quad a_{44} = 4$

4. The number of column (s) in a column matrix are ____

Ans : 1

5. The number of row (s) in a row matrix are ____

Ans : 1

6. The non-diagonal elements in any unit matrix are ____

Ans : 0

7. Does there exist a square matrix with 32 elements?

Ans : No

Exercise 3.16

1. In the matrix $A = \begin{bmatrix} 8 & 9 & 4 & 3 \\ -1 & \sqrt{7} & \frac{\sqrt{3}}{2} & 5 \\ 1 & 4 & 3 & 0 \\ 6 & 8 & -11 & 1 \end{bmatrix}$**, write**

(i) The number of elements

(ii) The order of the matrix

(iii) Write the elements corresponding to $a_{22}, a_{23}, a_{24}, a_{34}, a_{43}, a_{44}$

Sol :

$$A = \begin{bmatrix} 8 & 9 & 4 & 3 \\ -1 & \sqrt{7} & \frac{\sqrt{3}}{2} & 5 \\ 1 & 4 & 3 & 0 \\ 6 & 8 & -11 & 1 \end{bmatrix}$$

(i) The number of elements = 16

(ii) The order of the matrix = No of rows × No of columns = 4 × 4

(iii) $a_{22} = \sqrt{7}$, $a_{23} = \frac{\sqrt{3}}{2}$, $a_{24} = 5$, $a_{34} = 0$

$a_{43} = -11$, $a_{44} = 1$

2. If a matrix has 18 elements, what are the possible orders it can have? What if it has 6 elements?

Sol :

Given that a matrix has 18 elements.

$\therefore$ The possible orders are 1×18, 18×1, 9×2, 2×9, 6×3, 3×6.

If a matrix has 6 elements, then the possible orders are 1×6, 6×1, 2×3, 3×2.

3. Construct a 3 × 3 matrix whose elements are given by

(i) $a_{ij} = |i - 2j|$ **(ii)** $a_{ij} = \frac{(i+j)^3}{3}$

Sol :

(i) $a_{ij} = |i - 2j|$

Since, the matrix is of order 3 × 3, there will be 9 elements.

$$\begin{aligned}
a_{11} &= |1 - 2(1)| = |1 - 2| = |-1| = 1 \\
a_{12} &= |1 - 2(2)| = |1 - 4| = |-3| = 3 \\
a_{13} &= |1 - 2(3)| = |1 - 6| = |-5| = 5 \\
a_{21} &= |2 - 2(1)| = |2 - 2| = 0 \\
a_{22} &= |2 - 2(2)| = |2 - 4| = |-2| = 2 \\
a_{23} &= |2 - 2(3)| = |2 - 6| = |-4| = 4 \\
a_{31} &= |3 - 2(1)| = |3 - 2| = 1 \\
a_{32} &= |3 - 2(2)| = |3 - 4| = |-1| = 1 \\
a_{33} &= |3 - 2(3)| = |3 - 6| = |-3| = 3
\end{aligned}$$

$$\therefore \text{ The matrix is } \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

(ii) $aij = \frac{(i+j)^3}{3}$

$$\begin{aligned}
a_{11} &= \frac{(1+1)^3}{3} = \frac{(2)^3}{3} = 8/3 \\
a_{12} &= \frac{(1+2)^3}{3} = \frac{(3)^3}{3} = \frac{27}{3} = 9 \\
a_{13} &= \frac{(1+3)^3}{3} = \frac{(4)^3}{3} = \frac{64}{3} \\
a_{21} &= \frac{(2+1)^3}{3} = \frac{(3)^3}{3} = \frac{27}{3} = 9 \\
a_{22} &= \frac{(2+2)^3}{3} = \frac{(4)^3}{3} = \frac{64}{3} \\
a_{23} &= \frac{(2+3)^3}{3} = \frac{(5)^3}{3} = \frac{125}{3} \\
a_{31} &= \frac{(3+1)^3}{3} = \frac{(4)^3}{3} = \frac{64}{3} \\
a_{32} &= \frac{(3+2)^3}{3} = \frac{(5)^3}{3} = \frac{125}{3} \\
a_{33} &= \frac{(3+3)^3}{3} = \frac{(6)^3}{3} = \frac{216}{3} = 72
\end{aligned}$$

$$\therefore \text{ The matrix is } \begin{bmatrix} \frac{8}{3} & 9 & \frac{64}{3} \\ 9 & \frac{64}{3} & \frac{125}{3} \\ \frac{64}{3} & \frac{125}{3} & 72 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 5 & 4 & 3 \\ 1 & -7 & 9 \\ 3 & 8 & 2 \end{bmatrix}$ then find the transpose of A.

Sol :

$$A = \begin{bmatrix} 5 & 4 & 3 \\ 1 & -7 & 9 \\ 3 & 8 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 1 & 3 \\ 4 & -7 & 8 \\ 3 & 9 & 2 \end{bmatrix}$$

5. If $A = \begin{bmatrix} \sqrt{7} & -3 \\ -\sqrt{5} & 2 \\ \sqrt{3} & -5 \end{bmatrix}$ then find the transpose of –A.

Sol :

$$A = \begin{bmatrix} \sqrt{7} & -3 \\ -\sqrt{5} & 2 \\ \sqrt{3} & -5 \end{bmatrix}$$

$$-A = \begin{bmatrix} -\sqrt{7} & 3 \\ \sqrt{5} & -2 \\ -\sqrt{3} & 5 \end{bmatrix}$$

$$(-A)^T = \begin{bmatrix} -\sqrt{7} & \sqrt{5} & -\sqrt{3} \\ 3 & -2 & 5 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & \frac{5}{2} \\ 8 & 3 & 1 \end{bmatrix}$ then verify $(A^T)^T = A$

Sol :

$$A = \begin{bmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & \frac{5}{2} \\ 8 & 3 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & -\sqrt{17} & 8 \\ 2 & 0.7 & 3 \\ 2 & 5/2 & 1 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 5 & 2 & 2 \\ -\sqrt{17} & 0.7 & 5/2 \\ 8 & 3 & 1 \end{bmatrix} = A$$

$\therefore (A^T)^T = A$

Hence verified.

7. Find the values of x, y and z from the following equations

(i) $\begin{bmatrix} 12 & 3 \\ x & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} y & z \\ 3 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

(iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Sol :

(i) $\begin{bmatrix} 12 & 3 \\ x & 3/2 \end{bmatrix} = \begin{bmatrix} y & z \\ 3 & 5 \end{bmatrix}$

Equating the corresponding elements

$x = 3, y = 12, z = 3$

(ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$

$x + y = 6,\ xy = 8,\quad 5 + z = 5$

$x + 8/x = 6\quad y = 8/x,\quad z = 5 - 5 = 0$

$x^2 - 6x + 8 = 0$ when $x = 4, y = 2$

$(x - 4)(x - 2) = 0$ when $x = 2, y = 4,\ z = 0$

$x = 4, 2\ \therefore\ x = 4, y = 2, z = 0$ (or)

$x = 2, y = 4, z = 0$

(iii) $\begin{bmatrix} x+y+x \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

$x + y + z = 9$ ... (1)

$x + z = 5$ ... (2)

$y + z = 7$ ... (3)

$(1) + (2) + (3) \Rightarrow 2x + 2y + 3z = 21$ ... (4)

$(1) \times 2 \Rightarrow 2x + 2y + 2z = 18$ ... (5)

$(4) - (5) \Rightarrow z = 3$

Substituting in (3)

$y + 3 = 7$

$y = 7 - 3 = 4$

Substituting $z = 3$ in (2)

$x + 3 = 5$

$x = 5 - 3 = 2$

$\therefore\ x = 2, y = 4, z = 3.$

OPERATIONS ON MATRICES

Key Points

- Two matrices can be added or subtracted if they have the same order and to add or subtract the matrices just add or subtract the corresponding elements.
- When multiplying a matrix by a scalar (constant), we multiply all the elements in the matrix by a scalar.
- Matrix multiplication is possible only if the number of columns in the first matrix is equal to the number of rows in second matrix.
- If the matrix A is of order $m \times n$ and B is of order $p \times q$ then AB is possible only when $n = p$ and BA is possible only when $q = m$.

  Order of AB = $m \times q$

  Order of BA = $p \times n$

Worked Examples

3.57. **If** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}$, **find A + B.**

Sol :

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 7 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+7 & 3+0 \\ 4+1 & 5+3 & 6+1 \\ 7+2 & 8+4 & 9+0 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 3 \\ 5 & 8 & 7 \\ 9 & 12 & 9 \end{bmatrix}$$

3.58. **Two examinations were conducted for 3 groups of students namely group 1, group 2, group 3 and their data on average of marks for the subjects Tamil, English, Science and Mathematics are given below in the form of matrices A and B. Find the total marks of both the examinations for all the 3 groups.**

$$A = \begin{array}{c} \\ \textit{Group 1} \\ \textit{Group 2} \\ \textit{Group 3} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Tamil} & \textit{English} & \textit{Science} & \textit{Mathematics} \end{array} \\ \begin{bmatrix} 22 & 15 & 14 & 23 \\ 50 & 62 & 21 & 30 \\ 53 & 80 & 32 & 40 \end{bmatrix} \end{array}$$

$$B = \begin{array}{c} \\ \textit{Group 1} \\ \textit{Group 2} \\ \textit{Group 3} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Tamil} & \textit{English} & \textit{Science} & \textit{Mathematics} \end{array} \\ \begin{bmatrix} 20 & 38 & 15 & 40 \\ 18 & 12 & 17 & 80 \\ 81 & 47 & 52 & 18 \end{bmatrix} \end{array}$$

Sol :

The total marks in both the examinations for all the 3 groups is the sum of the given matrices.

$$A + B = \begin{bmatrix} 22+20 & 15+38 & 14+15 & 23+40 \\ 50+18 & 62+12 & 21+17 & 30+80 \\ 53+81 & 80+47 & 32+52 & 40+18 \end{bmatrix}$$

$$= \begin{bmatrix} 42 & 53 & 29 & 63 \\ 68 & 74 & 38 & 110 \\ 134 & 127 & 84 & 58 \end{bmatrix}$$

3.59. **If** $A = \begin{bmatrix} 1 & 3 & -2 \\ 5 & -4 & 6 \\ -3 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 8 \\ 3 & 4 \\ 9 & 6 \end{bmatrix}$, **find A + B.**

Sol :

It is not possible to add A and B because they have different orders.

3.60. **If** $A = \begin{bmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{bmatrix}$ **then Find 2A + B.**

Sol :

Since A and B have same order 3×3, 2A + B is defined.

We have

$$2A + B = 2\begin{bmatrix} 7 & 8 & 6 \\ 1 & 3 & 9 \\ -4 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 16 & 12 \\ 2 & 6 & 18 \\ -8 & 6 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 11 & -3 \\ -1 & 2 & 4 \\ 7 & 5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 27 & 9 \\ 1 & 8 & 22 \\ -1 & 11 & -2 \end{bmatrix}$$

3.61. **If** $A = \begin{bmatrix} 5 & 4 & -2 \\ \frac{1}{2} & \frac{3}{4} & \sqrt{2} \\ 1 & 9 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -7 & 4 & -3 \\ \frac{1}{4} & \frac{7}{2} & 3 \\ 5 & -6 & 9 \end{bmatrix}$, **find 4A − 3B.**

Sol :

Since A, B are of the same order 3×3, subtraction of 4A and 3B is defined.

$$4A - 3B = 4\begin{bmatrix} 5 & 4 & -2 \\ \frac{1}{2} & \frac{3}{4} & \sqrt{2} \\ 1 & 9 & 4 \end{bmatrix} - 3\begin{bmatrix} -7 & 4 & -3 \\ \frac{1}{4} & \frac{7}{2} & 3 \\ 5 & -6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 16 & -8 \\ 2 & 3 & 4\sqrt{2} \\ 4 & 36 & 16 \end{bmatrix} + \begin{bmatrix} 21 & -12 & 9 \\ -\frac{3}{4} & -\frac{21}{2} & -9 \\ -15 & 18 & -27 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 4 & 1 \\ \frac{5}{4} & -\frac{15}{2} & 4\sqrt{2}-9 \\ -11 & 54 & -11 \end{bmatrix}$$

3.62. Find the value of a, b, c, d, x, y from the following matrix equation.

$$\begin{pmatrix} d & 8 \\ 3b & a \end{pmatrix} + \begin{pmatrix} 3 & a \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 2a \\ b & 4c \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}$$

Sol :

First, we add the two matrices on both left, right hand sides to get

$$\begin{pmatrix} d+3 & 8+a \\ 3b-2 & a-4 \end{pmatrix} = \begin{pmatrix} 2 & 2a+1 \\ b-5 & 4c \end{pmatrix}$$

Equating the corresponding elements of the two matrices, we have

$d + 3 = 2 \Rightarrow d = -1$

$8 + a = 2a + 1 \Rightarrow a = 7$

$3b - 2 = b - 5 \Rightarrow b = \frac{-3}{2}$

Substituting $a = 7$ in $a - 4 = 4c \Rightarrow c = \frac{3}{4}$

Therefore, $a = 7, b = -\frac{3}{2}, c = \frac{3}{4}, d = -1$.

3.63. If $A = \begin{pmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{pmatrix}$,

$C = \begin{pmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{pmatrix}$

compute the following :

(i) 3A + 2B – C **(ii)** $\frac{1}{2}A - \frac{3}{2}B$

Sol :

(i) $3A + 2B - C$

$$= 3\begin{bmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{bmatrix} + 2\begin{bmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 0 \\ -1 & -7 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 24 & 9 \\ 9 & 15 & 0 \\ 24 & 21 & 18 \end{bmatrix} + \begin{bmatrix} 16 & -12 & -8 \\ 4 & 22 & -6 \\ 0 & 2 & 10 \end{bmatrix} + \begin{bmatrix} -5 & -3 & 0 \\ 1 & 7 & -2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 9 & 1 \\ 14 & 44 & -8 \\ 23 & 19 & 25 \end{bmatrix}$$

(ii) $\frac{1}{2}A - \frac{3}{2}B = \frac{1}{2}[A - 3B]$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 8 & -6 & -4 \\ 2 & 11 & -3 \\ 0 & 1 & 5 \end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 8 & 3 \\ 3 & 5 & 0 \\ 8 & 7 & 6 \end{bmatrix} + \begin{bmatrix} -24 & 18 & 12 \\ -6 & -33 & 9 \\ 0 & -3 & -15 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} -23 & 26 & 15 \\ -3 & -28 & 9 \\ 8 & 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{23}{2} & 13 & \frac{15}{2} \\ -\frac{3}{2} & -14 & \frac{9}{2} \\ 4 & 2 & -\frac{9}{2} \end{bmatrix}$$

Exercise 3.17

1. If $A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix}$ **then verify that**

(i) A + B = B + A

(ii) A + (– A) = (– A) + A = O.

Sol :

$$A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}, B = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix}$$

(i) $A + B = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix} + \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{pmatrix}$

$B + A = \begin{pmatrix} 5 & 7 \\ 3 & 3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix} = \begin{pmatrix} 6 & 16 \\ 6 & 7 \\ 9 & -3 \end{pmatrix}$

$\therefore \quad A + B = B + A$

(ii) $A = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}, -A = \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix}$

$$A + (-A) = \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix} + \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$$(-A) + A = \begin{pmatrix} -1 & -9 \\ -3 & -4 \\ -8 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 9 \\ 3 & 4 \\ 8 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

$\therefore \quad A + (-A) = (-A) + A = O$

Hence verified

2. If $A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$ then verify that $A + (B + C) = (A + B) + C$.

Sol :

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix},$$

$$C = \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$$

$$B + C = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix} + \begin{bmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= \begin{pmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{pmatrix}$$

$$A + (B + C) = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix} + \begin{pmatrix} 10 & 6 & 8 \\ 2 & 7 & 5 \\ -5 & 5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{pmatrix} \quad \ldots (1)$$

$$A + B = \begin{pmatrix} 4 & 3 & 1 \\ 2 & 3 & -8 \\ 1 & 0 & -4 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 4 \\ 1 & 9 & 2 \\ -7 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{pmatrix}$$

$$(A + B) + C = \begin{pmatrix} 6 & 6 & 5 \\ 3 & 12 & -6 \\ -6 & 1 & -5 \end{pmatrix} + \begin{pmatrix} 8 & 3 & 4 \\ 1 & -2 & 3 \\ 2 & 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 9 & 9 \\ 4 & 10 & -3 \\ -4 & 5 & -6 \end{pmatrix} \quad \ldots (2)$$

From (1) and (2)

$A + (B + C) = (A + B) + C$

Hence verified.

3. Find X and Y if $X + Y = \begin{bmatrix} 7 & 0 \\ 3 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

Sol :

$$X + Y = \begin{bmatrix} 7 & 0 \\ 3 & 5 \end{bmatrix} \quad \ldots (1)$$

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \quad \ldots (2)$$

$$(1)+(2) \Rightarrow \quad 2X = \begin{pmatrix} 10 & 0 \\ 3 & 9 \end{pmatrix}$$

$$X = \frac{1}{2}\begin{pmatrix} 10 & 0 \\ 3 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 \\ 3/2 & 9/2 \end{pmatrix}$$

Substituting in (1)

$$Y = \begin{pmatrix} 7 & 0 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 3/2 & 9/2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 5 & 0 \\ 3/2 & 9/2 \end{pmatrix},$$

$$Y = \begin{pmatrix} 2 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

4. If A = $\begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}$, B = $\begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$ find the value of (i) B – 5A (ii) 3A – 9B

Sol :

$$A = \begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}, B = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$$

(i) $$B - 5A = \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix} - 5\begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix} - \begin{pmatrix} 0 & 20 & 45 \\ 40 & 15 & 35 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -17 & -37 \\ -39 & -11 & -26 \end{pmatrix}$$

(ii) $$3A - 9B = 3\begin{pmatrix} 0 & 4 & 9 \\ 8 & 3 & 7 \end{pmatrix} - 9\begin{pmatrix} 7 & 3 & 8 \\ 1 & 4 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 12 & 27 \\ 24 & 9 & 21 \end{pmatrix} - \begin{pmatrix} 63 & 27 & 72 \\ 9 & 36 & 81 \end{pmatrix}$$

$$= \begin{pmatrix} -63 & -15 & -45 \\ 15 & -27 & -60 \end{pmatrix}$$

5. Find the values of x, y, z if

(i) $\begin{pmatrix} x-3 & 3x-z \\ x+y+7 & x+y+z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 6 \end{pmatrix}$

(ii) $\begin{bmatrix} x & y-z & z+3 \end{bmatrix} + \begin{bmatrix} y & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \end{bmatrix}$

Sol :

(i) $$\begin{bmatrix} x-3 & 3x-z \\ x+y+7 & x+y+z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 6 \end{bmatrix}$$

Equating the corresponding elements

$$x - 3 = 1$$
$$x = 1 + 3 = 4$$
$$3x - z = 0$$
$$3(4) - z = 0$$
$$12 = z$$
$$x + y + 7 = 1$$
$$4 + y + 7 = 1$$
$$y = -10$$
$$\therefore x = 4, y = -10, z = 12$$

(ii) $$\begin{bmatrix} x & y-z & z+3 \end{bmatrix} + \begin{bmatrix} y & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \end{bmatrix}$$

$$\begin{bmatrix} x+y & y-z+4 & z+6 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 16 \end{bmatrix}$$

$$\therefore x + y = 4, \; y - z + 4 = 8, \; z + 6 = 16$$
$$z = 16 - 6 = 10$$

Substituting $z = 10$

$$y - 10 + 4 = 8$$
$$y = 8 + 6 = 14$$
$$\therefore x + 14 = 4$$
$$x = 4 - 14 = -10$$

$\therefore$ Solution $x = -10, y = 14, z = 10$

6. Find x and y if $x\begin{pmatrix} 4 \\ -3 \end{pmatrix} + y\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

Sol :

$$x\begin{pmatrix} 4 \\ -3 \end{pmatrix} + y\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 4x \\ -3x \end{pmatrix} + \begin{pmatrix} -2y \\ 3y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$4x - 2y = 4 \Rightarrow 2x - y = 2$ ... (1)

$-3x + 3y = 6 \Rightarrow -x + y = 2$ ... (2)

$(1) + (2) \Rightarrow x = 4$

Substituting in (2)

$$-4 + y = 2$$
$$y = 2 + 4 = 6$$
$$\therefore x = 4, y = 6.$$

7. Find the non-zero values of x satisfying the matrix equation

$$x\begin{bmatrix}2x & 2\\ 3 & x\end{bmatrix}+2\begin{bmatrix}8 & 5x\\ 4 & 4x\end{bmatrix} = 2\begin{bmatrix}x^2+8 & 24\\ 10 & 6x\end{bmatrix}$$

Sol :

$$x\begin{bmatrix}2x & 2\\ 3 & x\end{bmatrix}+2\begin{bmatrix}8 & 5x\\ 4 & 4x\end{bmatrix} = 2\begin{bmatrix}x^2+8 & 24\\ 10 & 6x\end{bmatrix}$$

$$\begin{bmatrix}2x^2 & 2x\\ 3x & x^2\end{bmatrix}+\begin{bmatrix}16 & 10x\\ 8 & 8x\end{bmatrix} = \begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$$

$$\begin{bmatrix}2x^2+16 & 12x\\ 3x+8 & x^2+8x\end{bmatrix} = \begin{bmatrix}2x^2+16 & 48\\ 20 & 12x\end{bmatrix}$$

Equating the corresponding elements

$$3x^2 + 16 = 2x^2 + 16$$
$$3x^2 - 2x^2 = 16 - 16$$
$$x^2 = 0$$
$$x = 0$$

(Rejected as non zero value required).

$$12x = 48$$
$$x = \frac{48}{12} = 4$$
$$3x + 8 = 20$$
$$3x = 20 - 8 = 12$$
$$x = \frac{12}{3} = 4$$

$$x^2 + 8x = 12x$$
$$x^2 + 8x - 12x = 0$$
$$x^2 - 4x = 0$$
$$x(x-4) = 0$$

$x = 0$ (rejected) and $x = 4$

$\therefore$ The non-zero value of x is 4.

8. Solve for x, y: $\begin{pmatrix}x^2\\ y^2\end{pmatrix}+2\begin{pmatrix}-2x\\ -y\end{pmatrix} = \begin{pmatrix}5\\ 8\end{pmatrix}$

Sol :

$$\begin{pmatrix}x^2\\ y^2\end{pmatrix}+2\begin{pmatrix}-2x\\ -y\end{pmatrix} = \begin{pmatrix}5\\ 8\end{pmatrix}$$

$$\begin{pmatrix}x^2\\ y^2\end{pmatrix}+\begin{pmatrix}-4x\\ -2y\end{pmatrix} = \begin{pmatrix}5\\ 8\end{pmatrix}$$

$$\begin{pmatrix}x^2-4x\\ y^2-2y\end{pmatrix} = \begin{pmatrix}+5\\ 8\end{pmatrix}$$

Equating the corresponding elements,

$$x^2 - 4x = 5$$
$$x^2 - 4x - 5 = 0$$
$$(x-5)(x+1) = 0$$
$$x = 5, -1$$
$$y^2 - 2y = 8$$
$$y^2 - 2y - 8 = 0$$
$$(y-4)(y+2) = 0$$
$$y = 4, -2.$$

MULTIPLICATION OF MATRICES

Worked Example

3.64. If $A = \begin{bmatrix}1 & 2 & 0\\ 3 & 1 & 5\end{bmatrix}$, $B = \begin{bmatrix}8 & 3 & 1\\ 2 & 4 & 1\\ 5 & 3 & 1\end{bmatrix}$ **find AB.**

Sol :

We observe that A is a 2×3 matrix and B is a 3×3 matrix, hence AB is defined and it will be of the order 2×3.

$$\text{Given } A = \begin{bmatrix}1 & 2 & 0\\ 3 & 1 & 5\end{bmatrix}_{2\times 3}$$

$$B = \begin{bmatrix}8 & 3 & 1\\ 2 & 4 & 1\\ 5 & 3 & 1\end{bmatrix}_{3\times 3}$$

$$AB = \begin{bmatrix}1 & 2 & 0\\ 3 & 1 & 5\end{bmatrix}\times\begin{bmatrix}8 & 3 & 1\\ 2 & 4 & 1\\ 5 & 3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}(8+4+0) & (3+8+0) & (1+2+0)\\ (24+2+25) & (9+4+15) & (3+1+5)\end{bmatrix}$$

$$= \begin{bmatrix}12 & 11 & 3\\ 51 & 28 & 9\end{bmatrix}$$

3.65. If $A\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$, $B = \begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix}$ **find AB and BA. Check if AB = BA.**

Sol :

We observe that A is a 2×2 matrix and B is a 2×2 matrix, hence AB is defined and it will be of the order 2×2.

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 0+3 \\ 2+3 & 0+9 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 5 & 9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0 & 2+0 \\ 2+3 & 1+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & 10 \end{bmatrix}$$

Therefore, $AB \neq BA$

3.66. **If** $A = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$

Show that A and B satisfy commutative property with respect to matrix multiplication.

Sol :

We have to show that AB = BA

$$\text{LHS: } AB = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & 4\sqrt{2}-4\sqrt{2} \\ 2\sqrt{2}-2\sqrt{2} & 4+4 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

$$\text{RHS: } BA = \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+4 & -4\sqrt{2}+4\sqrt{2} \\ -2\sqrt{2}+2\sqrt{2} & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

Hence LHS = RHS i.e., AB = BA

3.67. **Solve** $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Sol :

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}_{2\times 2} \times \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

By matrix multiplication

$$= \begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Rewriting $2x + y = 4$ ... (1)

$x + 2y = 5$ ... (2)

(1) – 2 × (2) gives

$$\begin{array}{r} 2x + y = 4\ (-) \\ 2x + 4y = 10 \\ \hline -3y = -6 \end{array} \Rightarrow y = 2$$

Substituting y = 2 in (1), 2x + 2 = 4 ⇒ x = 1

Therefore x = 1, y = 2.

3.68. **If** $A = [1\ \ -1\ \ 2]$, $B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$ **and** $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

show that (AB) C = A (BC)

Sol :

LHS (AB) C

$$AB = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}_{1\times 3} \times \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}_{3\times 2}$$

$$= [1-2+2 \quad -1-1+6]$$

$$= [1 \quad 4]$$

$$(AB)\,C = \begin{bmatrix} 1 & 4 \end{bmatrix}_{1\times 2} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}_{2\times 2}$$

$$= [1+8 \quad 2-4] = [9 \quad -2] \quad \text{... (1)}$$

RHS A (BC)

$$BC = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}_{3\times 2} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1-2 & 2+1 \\ 2+2 & 4-1 \\ 1+6 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{bmatrix}$$

$$A\,(BC) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}_{1\times 3} \times \begin{bmatrix} -1 & 3 \\ 4 & 3 \\ 7 & -1 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} -1-4+14 & 3-3-2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -2 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2), (AB) C = A (BC)

3.69. If $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix}, C = \begin{bmatrix} -7 & 6 \\ 3 & 2 \end{bmatrix}$ verify that A (B + C) = AB + AC

Sol :

LHS A (B + C)

$$B + C = \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} -7 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 8 \\ -1 & 4 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} -6 & 8 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -6-1 & 8+4 \\ 6-3 & -8+12 \end{bmatrix} = \begin{bmatrix} -7 & 12 \\ 3 & 4 \end{bmatrix} \quad \text{... (1)}$$

RHS: AB + AC

$$AB = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 1-4 & 2+2 \\ -1-12 & -2+6 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -13 & 4 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} -7 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -7+3 & 6+2 \\ 7+9 & -6+6 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 16 & 0 \end{bmatrix}$$

$$\therefore AB + AC = \begin{bmatrix} -3 & 4 \\ -13 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 8 \\ 16 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 12 \\ 3 & 4 \end{bmatrix}$$

From (1) and (2), A (B + C) = AB + AC.

Hence proved.

3.70. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}$ show that $(AB)^T = B^T A^T$

Sol :

LHS: $(AB)^T$

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 0 & 2 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 2-2+0 & -1+8+2 \\ 4+1+0 & -2-4+2 \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 5 & -4 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 0 & 9 \\ 5 & -4 \end{bmatrix}^T = \begin{bmatrix} 0 & 5 \\ 9 & -4 \end{bmatrix} \quad \text{... (1)}$$

RHS: $(B^T A^T)$

$$B^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix},$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix}_{2\times 3} \times \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 2-2+0 & 4+1+0 \\ -1+8+2 & -2-4+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 9 & -4 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2), $(AB)^T = B^T A^T$.

Hence proved.

Exercise 3.18

1. Find the order of the product matrix AB if

	(i)	(ii)	(iii)	(iv)	(v)
Orders of A	3 × 3	4 × 3	4 × 2	4 × 5	1 × 1
Orders of B	3 × 3	3 × 2	2 × 2	5 × 1	1 × 3

Sol :

(i) Order of A = 3 × 3, order of B = 3 × 3
∴ Order of AB = 3 × 3

(ii) Order of A = 4 × 3, Order of B = 3 × 2
∴ Order of AB = 4 × 2

(iii) Order of A = 4 × 2, Order of B = 2 × 2
∴ Order of AB = 4 × 2

(iv) Order of A = 4 × 5, Order of B = 5 × 1
∴ Order of AB = 4 × 1

(v) Order of A = 1 × 1, Order of B = 1 × 3
∴ Order of AB = 1 × 3

2. If A is of order p × q and B is of order q × r what is the order of AB and BA?

Sol :

Order of A = p × q
Order of B = q × r
No.of columns in A = No of Rows in B
q = q

∴ The product 'AB' is possible.

Order of AB = Number of Rows in A × Number of Columns in B
= p × r

No. of columns in B ≠ No. of rows in A
i.e., r ≠ q

∴ BA is not possible.

3. A has 'a' rows and 'a + 3' columns. B has 'b' rows and '17 – b' columns and if both products AB and BA exist find a, b?

Sol :

Given A has 'a' rows and 'a + 3' columns
B has 'b' rows and '17–b' columns
Since AB and BA exist

For AB, No. of columns in A = No. of rows in B
∴ a + 3 = b

For BA, No. of columns in B = No of rows in A

$$
\begin{aligned}
17 - b &= a \\
\therefore\ 17 - (a + 3) &= a \\
17 - a - 3 &= a \\
2a &= 14 \\
a &= 7 \\
\therefore\ b &= a + 3 = 7 + 3 = 10 \\
\therefore\ a &= 7, b = 10.
\end{aligned}
$$

4. If $A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$ find AB, BA and check if AB = BA.

Sol :

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+10 & -6+25 \\ 4+6 & -12+15 \end{bmatrix} = \begin{bmatrix} 12 & 19 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-12 & 5-9 \\ 4+20 & 10+15 \end{bmatrix} = \begin{bmatrix} -10 & -4 \\ 24 & 25 \end{bmatrix}$$

$$\therefore AB \neq BA$$

5. Given that

$A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{bmatrix}$

verify that A (B + C) = AB + AC.

Sol :

$$A = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 2 \\ -4 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 4 \\ -1 & 6 & 5 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 1 & 3 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ -1 & 6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 2+18 & 4+15 \\ 10+1 & 10-6 & 20-5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 20 & 19 \\ 11 & 4 & 15 \end{bmatrix} \quad ...(1)$$

$$AB = \begin{bmatrix}1 & 3\\5 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 2\\3 & 5 & 2\end{bmatrix}$$

$$= \begin{bmatrix}1+9 & -1+15 & 2+6\\5-3 & -5-5 & 10-2\end{bmatrix}$$

$$= \begin{bmatrix}10 & 14 & 8\\2 & -10 & 8\end{bmatrix}$$

$$AC = \begin{bmatrix}1 & 3\\5 & -1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\-4 & 1 & 3\end{bmatrix}$$

$$= \begin{bmatrix}1-12 & 3+3 & 2+9\\5+4 & 15-1 & 10-3\end{bmatrix}$$

$$= \begin{bmatrix}-11 & 6 & 11\\9 & 14 & 7\end{bmatrix}$$

$$AB + AC = \begin{bmatrix}10 & 14 & 8\\2 & -10 & 8\end{bmatrix} + \begin{bmatrix}-11 & 6 & 11\\9 & 14 & 7\end{bmatrix}$$

$$= \begin{bmatrix}-1 & 20 & 19\\11 & 4 & 15\end{bmatrix} \quad \dots (2)$$

From (1), (2)

$\therefore$ A (B + C) = AB + AC , Hence verified.

6. Show that the matrices $A = \begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}$, $B = \begin{bmatrix}1 & -2\\-3 & 1\end{bmatrix}$ satisfy commutative property AB = BA.

Sol :

$$A = \begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}, \quad B = \begin{bmatrix}1 & -2\\-3 & 1\end{bmatrix}$$

$$AB = \begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\-3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1-6 & -2+2\\3-3 & -6+1\end{bmatrix}$$

$$= \begin{bmatrix}-5 & 0\\0 & -5\end{bmatrix}$$

$$BA = \begin{bmatrix}1 & -2\\-3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1-6 & 2-2\\-3+3 & -6+1\end{bmatrix}$$

$$= \begin{bmatrix}-5 & 0\\0 & -5\end{bmatrix}$$

$\therefore$ AB = BA

Hence proved.

7. Let $A = \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}$, $B = \begin{bmatrix}4 & 0\\1 & 5\end{bmatrix}$, $C = \begin{bmatrix}2 & 0\\1 & 2\end{bmatrix}$ Show that

(i) A(BC) = (AB)C

(ii) (A – B)C = AC – BC

(iii) $(A - B)^T = A^T - B^T$

Sol :

$$A = \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}, B = \begin{bmatrix}4 & 0\\1 & 5\end{bmatrix}, C = \begin{bmatrix}2 & 0\\1 & 2\end{bmatrix}$$

$$BC = \begin{bmatrix}4 & 0\\1 & 5\end{bmatrix}\begin{bmatrix}2 & 0\\1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}8+0 & 0+0\\2+5 & 0+10\end{bmatrix} = \begin{bmatrix}8 & 0\\7 & 10\end{bmatrix}$$

$$A(BC) = \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}8 & 0\\7 & 10\end{bmatrix}$$

$$= \begin{bmatrix}8+14 & 0+20\\8+21 & 0+30\end{bmatrix}$$

$$= \begin{bmatrix}22 & 20\\29 & 30\end{bmatrix} \quad \dots (1)$$

$$(AB) = \begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}4 & 0\\1 & 5\end{bmatrix}$$

$$= \begin{bmatrix}4+2 & 0+10\\4+3 & 0+15\end{bmatrix} = \begin{bmatrix}6 & 10\\7 & 15\end{bmatrix}$$

$$(AB)C = \begin{bmatrix}6 & 10\\7 & 15\end{bmatrix}\begin{bmatrix}2 & 0\\1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}12+10 & 0+20\\14+15 & 0+30\end{bmatrix}$$

$$= \begin{bmatrix}22 & 20\\29 & 30\end{bmatrix} \quad \dots (2)$$

From (1) and (2)

$$A(BC) = (AB)C$$

Hence proved.

(ii) $$(A-B) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & -2 \end{bmatrix}$$

$$(A-B)C = \begin{bmatrix} -3 & 2 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6+2 & 0+4 \\ 0-2 & 0-4 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & -4 \end{bmatrix} \quad \ldots(1)$$

$$AC = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+2 & 0+4 \\ 2+3 & 0+6 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 5 & 6 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8+0 & 0+0 \\ 2+5 & 0+10 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 7 & 10 \end{bmatrix}$$

$$AC - BC = \begin{bmatrix} 4 & 4 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & -4 \end{bmatrix} \quad \ldots(2)$$

From (1) and (2)

$(A-B)C = AC - BC$

Hence proved.

(iii) $$(A-B) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 0 & -2 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix} \quad \ldots(1)$$

$$A^T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix}$$

$$A^T - B^T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 2 & -2 \end{bmatrix} \quad \ldots(2)$$

From (1) and (2)

$$(A-B)^T = A^T - B^T$$

Hence proved.

8. If $A = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix}, B = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix}$ then show that $A^2 + B^2 = I$.

Sol :

$$A = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix},$$

$$B = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta+0 & 0+0 \\ 0+0 & 0+\cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{bmatrix}$$

$$B^2 = B.B = \begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix}\begin{bmatrix} \sin\theta & 0 \\ 0 & \sin\theta \end{bmatrix} = \begin{bmatrix} \sin^2\theta+0 & 0+0 \\ 0+0 & \sin^2\theta \end{bmatrix}$$

$$\text{Now } A^2 + B^2 = \begin{bmatrix} \cos^2\theta & 0 \\ 0 & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & 0 \\ 0 & \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta+\sin^2\theta & 0+0 \\ 0+0 & \cos^2\theta+\sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence Proved.

9. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ prove that $AA^T = I$.

Sol :

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$= I \qquad \therefore AA^T = I$$

Hence Proved.

10. Verify that $A^2 = I$ when $A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$

Sol :

$$A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24 & -20+20 \\ 30-30 & -24+25 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^2 = I$$

Hence proved.

11. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ show that $A^2 - (a + d) A = (bc - ad)I_2$

Sol :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = A. A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$(a + d) A = (a + d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{bmatrix}$$

$$= \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}$$

$$A^2 - (a + d) A$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix}$$

$$= \begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix} \qquad ...(1)$$

$$\text{Now } (bc - ad) I_2 = (bc-ad)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} bc-ad & 0 \\ 0 & bc-ad \end{bmatrix} \qquad ...(2)$$

From (1) and (2)

$$A^2 - (a + d) A = (bc - ad) I_2$$

Hence proved.

12. If $A = \begin{bmatrix} 5 & 2 & 9 \\ 1 & 2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 \\ 1 & 2 \\ 5 & -1 \end{bmatrix}$ verify that $(AB)^T = B^T A^T$

Sol :

$$A = \begin{bmatrix} 5 & 2 & 9 \\ 1 & 2 & 8 \end{bmatrix}, B = \begin{bmatrix} 1 & 7 \\ 1 & 2 \\ 5 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 2 & 9 \\ 1 & 2 & 8 \end{bmatrix}\begin{bmatrix} 1 & 7 \\ 1 & 2 \\ 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2+45 & 35+4-9 \\ 1+2+40 & 7+4-8 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 30 \\ 43 & 3 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 52 & 43 \\ 30 & 3 \end{bmatrix} \quad ...(1)$$

$$B^TA^T = \begin{bmatrix} 1 & 1 & 5 \\ 7 & 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 2 \\ 9 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5+2+45 & 1+2+40 \\ 35+4-9 & 7+4-8 \end{bmatrix}$$

$$= \begin{bmatrix} 52 & 43 \\ 30 & 3 \end{bmatrix} \quad ...(2)$$

From (1) and (2)

$$(AB)^T = B^TA^T$$

Hence proved.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ show that $A^2 - 5A + 7I_2 = 0$

Sol :

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = A.A$$

$$= \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$5A = 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$$

$$7I_2 = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I_2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence Proved.

Exercise 3.19

Choose the correct answer:

1. A system of 3 linear equations in 3 variables is inconsistent if their planes

(1) intersect only at a point
(2) intersect in a line
(3) coincides with each other
(4) do not intersect **[Ans : 4]**

2. The solution of the system $x + y - 3z = -6$, $-7y + 7z = 7$, $3z = 9$ is.

(1) $x = 1, y = 2, z = 3$
(2) $x = -1, y = 2, z = 3$
(3) $x = -1, y = -2, z = 3$
(4) $x = 1, y = 2, z = 3$ **[Ans : 1]**

Sol :

$$x + y - 3z = -6 \quad ...(1)$$
$$-7y + 7z = 7 \quad ...(2)$$
$$3z = 9 \Rightarrow z = 3 \quad ...(3)$$
$$(2) \Rightarrow -7y + 7z = 7 \quad (\div 7)$$
$$-y + z = 1$$
$$-y + 3 = 1$$
$$y = 3 - 1 = 2$$

Substituting in (1)

$$x + 2 - 3(3) = -6$$
$$x + 2 - 9 = -6$$
$$x = -6 + 7 = 1$$

$\therefore$ Solution is $x = 1, y = 2, z = 3$.

3. If $(x - 6)$ is the HCF of $x^2 - 2x - 24$ and $x^2 - kx - 6$ then the value of k is.

(1) 3 (2) 5
(3) 6 (4) 8 **[Ans : 2]**

Sol :

Given $(x - 6)$ is the HCF of $x^2 - kx - 6$ and $x^2 - 2x - 24$.

$\therefore$ $x^2 - kx - 6$ is divisible by $(x - 6)$

$$(6)^2 - k(6) - 6 = 0$$
$$\Rightarrow 36 - 6k - 6 = 0$$
$$6k = 30$$
$$\Rightarrow k = \frac{30}{6} = 5$$

4. $\frac{3y-3}{y} \div \frac{7y-7}{3y^2}$ is

(1) $\frac{9y}{7}$ (2) $\frac{9y^3}{(21y-21)}$

(3) $\frac{21y^2-42y+21}{3y^3}$ (4) $\frac{7(y^2-2y+1)}{y^2}$ **[Ans : 1]**

Sol :

$$\frac{3y-3}{y} \div \frac{7y-7}{3y^2} = \frac{3(y-1)}{y} \times \frac{3y^2}{7(y-1)} = \frac{9y}{7}$$

5. $y^2 + \dfrac{1}{y^2}$ is not equal to

(1) $\dfrac{y^4+1}{y^2}$ (2) $\left(y+\dfrac{1}{y}\right)^2$

(3) $\left(y-\dfrac{1}{y}\right)^2+2$ (4) $\left(y+\dfrac{1}{y}\right)^2-2$ **[Ans : 2]**

Sol :

$y^2 + \dfrac{1}{y^2}$ is not equal to $\left(y+\dfrac{1}{y}\right)^2$

i.e., $y^2 + \dfrac{1}{y^2} \neq y^2 + \dfrac{1}{y^2} + 2$

6. $\dfrac{x}{x^2-25} - \dfrac{8}{x^2+6x+5}$ gives

(1) $\dfrac{x^2-7x+40}{(x-5)(x+5}$ (2) $\dfrac{x^2+7x+40}{(x-5)(x+5)(x+1)}$

(3) $\dfrac{x^2-7x+40}{(x^2-25)(x+1)}$ (4) $\dfrac{x^2+10}{(x^2-25)(x+1)}$

[Ans : 3]

Sol :

$$\frac{x}{x^2-25} - \frac{8}{x^2+6x+5} = \frac{x}{(x+5)(x-5)} - \frac{8}{(x+5)(x+1)}$$

$$= \frac{x(x+1)-8(x-5)}{(x+5)(x-5)(x+1)}$$

$$= \frac{x^2+x-8x+40}{(x+5)(x-5)(x+1)}$$

$$= \frac{x^2-7x+40}{(x^2-25)(x+1)}$$

7. The square root of $\dfrac{256x^8y^4z^{10}}{25x^6y^6z^6}$ is equal to

(1) $\dfrac{16}{5}\left|\dfrac{x^2z^4}{y^2}\right|$ (2) $16\left|\dfrac{y^2}{x^2z^4}\right|$

(3) $\dfrac{16}{5}\left|\dfrac{y}{xz^2}\right|$ (4) $\dfrac{16}{5}\left|\dfrac{xz^2}{y}\right|$ **[Ans : 4]**

Sol :

$$\sqrt{\frac{256x^8y^4z^{10}}{25x^6y^6z^6}} = \frac{16}{5}\left|\frac{x^4y^2z^5}{x^3y^3z^3}\right| = \frac{16}{5}\left|\frac{xz^2}{y}\right|$$

8. Which of the following should be added to make $x^4 + 64$ a perfect square.

(1) $4x^2$ (2) $16x^2$

(3) $8x^2$ (4) $-8x^2$ **[Ans : 2]**

Sol :

$$\begin{aligned} x^4+64+16x^2 &= x^4+16x^2+64 \\ &= (x^2)^2+2(8)(x^2)+(8)^2 \\ &= (x^2+8)^2 \end{aligned}$$

9. The solution of $(2x-1)^2 = 9$ is equal to

(1) -1 (2) 2

(3) $-1, 2$ (4) None of these

[Ans : 3]

Sol :

$$\begin{aligned} (2x-1)^2 &= 9 \\ 2x-1 &= \pm 3 \end{aligned}$$

$$\begin{aligned} 2x-1 &= 3 & 2x-1 &= -3 \\ 2x &= 4 & 2x &= -2 \\ x &= 2 & x &= -1 \end{aligned}$$

Solution is $x = -1, 2$.

10. The values of a and b if $4x^4 - 24x^3 + 76x^2 + ax + b$ is a perfect square are

(1) 100, 120 (2) 10, 12

(3) -120, 100 (4) 12, 10 **[Ans : 3]**

Sol :

$\sqrt{4x^4-24x^3+76x^2+ax+b}$

	$2x^2 - 6x + 10$
$2x^2$	$4x^4 - 24x^3 + 76x^2 + ax + b$
	$4x^4$
	$(-)$
$4x^2 - 6x$	$-24x^3 + 76x^2$
	$-24x^3 + 36x^2$
	$(+)$ $(-)$
$4x^2 - 12x + 10$	$40x^2 + ax + b$
	$40x^2 - 120x + 100$
	$(-)$ $(+)$ $(-)$
	0

$\therefore$ a = -120, b = 100.

11. If the roots of the equation $q^2x^2 + p^2x + r^2 = 0$ are the squares of the roots of the equation $qx^2 + px + r = 0$ then, q, p, r are in ______

(1) A.P
(2) G.P
(3) Both A.P and G.P
(4) none of these. **[Ans : 2]**

Sol :

Let α, β be the roots of the equation $qx^2 + px + r = 0$

$$\text{Sum of the roots } \alpha+\beta = \frac{-p}{q}$$

$$\text{Product of the roots } \alpha\beta = \frac{r}{q}$$

Given that α^2, β^2 are the roots of $q^2x^2 + p^2x + r^2 = 0$

$$\therefore \text{Sum of the roots } \alpha^2+\beta^2 = \frac{-p^2}{q^2}$$

$$\text{But } \alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

$$\therefore \left(\frac{-p}{q}\right)^2 - \frac{2r}{q} = \frac{-p^2}{q^2}$$

$$\frac{p^2}{q^2} + \frac{p^2}{q^2} = \frac{2r}{q}$$

$$\frac{2p^2}{q^2} = \frac{2r}{q} \Rightarrow \frac{p^2}{q^2} = \frac{r}{q}$$

$p^2 = qr \Rightarrow$ p is a Geometric mean and q, p, r are in Geometric Progression.

12. Graph of a linear polynomial is a

(1) Straight line (2) Circle
(3) Parabola (4) Hyperbola **[Ans : 1]**

13. The number of points of intersection of the quadratic polynomial $x^2 + 4x + 4$ with the X -axis is

(1) 0 (2) 1
(3) 0 or 1 (4) 2 **[Ans : 2]**

Sol :

$$x^2 + 4x + 4 = 0$$
$$(x+2)^2 = 0$$
$$x = -2 \text{ (twice)}$$

Roots are real and equal.

$\therefore$ Point of intersection with X-axis is 1.

14. For the given matrix $A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 9 & 11 & 13 & 15 \end{bmatrix}$ the order of the matrix A^T is

(1) 2×3 (2) 3×2
(3) 3×4 (4) 4×3 **[Ans : 4]**

Sol :

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 9 & 11 & 13 & 15 \end{bmatrix}, \text{order of } A = 3 \times 4$$

$$A^T = \begin{bmatrix} 1 & 2 & 9 \\ 3 & 4 & 11 \\ 5 & 6 & 13 \\ 7 & 8 & 15 \end{bmatrix}, \text{Order of } A^T = 4 \times 3$$

15. If A is a 2×3 matrix and B is a 3×4 matrix, how many columns does AB have

(1) 3 (2) 4
(3) 2 (5) 5 **[Ans : 2]**

Sol :

Order of A $= 2 \times 3$
Order of B $= 3 \times 4$
Order of AB $= 2 \times 4$
No. of columns in AB $= 4$

16. If number of columns and rows are not equal in a matrix then it is said to be a

(1) Diagonal matrix
(2) Rectangular matrix
(3) Square matrix
(4) Identity matrix **[Ans : 2]**

17. Transpose of a column matrix is

(1) Unit matrix
(2) Diagonal matrix
(3) Column matrix
(4) Row matrix **[Ans : 4]**

18. Find the matrix X if $2X + \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 5 \end{bmatrix}$

(1) $\begin{pmatrix} -2 & -2 \\ 2 & -1 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ (4) $\begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}$ **[Ans : 2]**

Sol :

$$\text{Given } 2x + \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 9 & 5 \end{bmatrix}$$

$$2x = \begin{bmatrix} 5 & 7 \\ 9 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix}$$

$$x = \frac{1}{2}\begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

19. Which of the following can be calculated from the given matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

(i) A^2 (ii) B^2 (iii) AB (iv) BA

(1) (i) and (ii) only (2) (ii) and (iii) only

(3) (ii) and (iv) only (4) All of these **[Ans : 3]**

Sol :

Order of A = 3×2

Order of B = 3×3

(i) A^2 is not possible as it is not square matrix.

(ii) B^2 is possible as it is a square matrix.

(iii) AB is not possible. No of columns in A $\neq$ No. of Rows in B.

(iv) BA is possible. Since, No. of columns in B = No. of Rows in A.

20. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}$ Which of the following statements are correct?

(i) $AB + C = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$ (ii) $BC = \begin{bmatrix} 0 & 1 \\ 2 & -3 \\ -4 & 10 \end{bmatrix}$

(iii) $BA + C = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix}$ (iv) $(AB)C = \begin{bmatrix} -8 & 20 \\ -8 & 13 \end{bmatrix}$

(1) (i) and (ii) only

(2) (ii) and (iii) only

(3) (iii) and (iv) only

(3) all of these **[Ans : 1]**

Sol :

$$AB + C = \begin{pmatrix} 5 & 4 \\ 7 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -2 & 5 \end{pmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

$$BC = \begin{bmatrix} 0 & 1 \\ 2 & -3 \\ -4 & 10 \end{bmatrix}$$

UNIT EXERCISE - 3

1. Solve

$\frac{1}{3}(x + y - 5) = y - z = 2x - 11 = 9 - (x + 2z)$.

Sol :

$$\frac{1}{3}(x + y - 5) = y - z$$

$$x + y - 5 = 3y - 3z$$

$$\Rightarrow x - 2y + 3z = 5 \quad \text{... (1)}$$

$$y - z = 2x - 11$$

$$\Rightarrow 2x - y + z = 11 \quad \text{... (2)}$$

$$2x - 11 = 9 - (x + 2z)$$

$$2x + x + 2z = 9 + 11$$

$$3x + 2z = 20 \quad \text{... (3)}$$

Consider (1) and (2)

$(2) \times 2 \Rightarrow \quad 4x - 2y + 2z = 22 \quad \text{... (4)}$

$(4) - (1) \Rightarrow \quad x - 2y + 3z = 5$

$3x - z = 17 \quad \text{...(5)}$

Now, consider (3) and (5)

$3x + 2z = 20 \quad \text{...(3)}$

$3x - z = 17 \quad (5)$

$(3)-(5) \Rightarrow \quad 3z = 3$

$z = 3/3 = 1$

Substituting $z = 1$ in (3)

$3x + 2(1) = 20$

$3x = 20 - 2 = 18$

$$x = \frac{18}{3} = 6$$

Substituting $x = 6, z = 1$ is ...(1)

$$6 - 2y + 3(1) = 5$$
$$6 + 3 - 5 = 2y$$
$$4 = 2y$$
$$y = 2y$$
$$y = \frac{4}{2} = 2$$

$\therefore$ Solution : $x = 6, y = 2, z = 1.$

2. One hundred and fifty students are admitted to a school. They are distributed over three sections A, B and C. If 6 students are shifted from section A to section C, the sections will have equal number of students. If 4 times of students of C exceeds the number of students of section A by the number of students in section B, find the number of students in the three sections.

Sol :

Let the number of students in section A, B and C be 'x' , 'y' and 'z' respectively

$\therefore$ Given

$$x + y + z = 150 \quad \text{... (1)}$$
$$x - 6 = z + 6$$
$$x - z = 12 \quad \text{... (2)}$$
$$4z = x + y$$
$$x + y - 4z = 0 \quad \text{... (3)}$$

consider (1) and (3)

$$x + y + z = 150 \quad \text{... (1)}$$
$$x + y - 4z = 0 \quad \text{... (3)}$$
$$(1) - (3) \Rightarrow \quad 5z = 150$$
$$z = \frac{150}{5} = 30$$

substituting in (2)

$$x - 30 = 12$$
$$x = 30 + 12$$
$$x = 42$$

substituting $x = 42, z = 30$ in (1)

$$42 + y + 30 = 150$$
$$y = 150 - 72$$
$$y = 78$$

$\therefore$ The number of students in sections A, B and C are 42, 78, 30 respectively.

3. In a three-digit number, when the tens and the hundreds digit are interchanged the new number is 54 more than three times the original number. If 198 is added to the number, the digits are reversed. The tens digit exceeds the hundreds digit by twice as that of the tens digit exceeds the unit digit. Find the original number.

Sol :

Let the 100's digit be 'x'

10's digit be 'y'

Unit's digit be 'z'

Given $100y + 10x + z - 54 = 3(100x + 10y + z)$

Substituting $290x - 70y + 2z = -54 \ (\div 2)$

$$145x - 35y + z = -27 \quad \text{...(1)}$$
$$100x + 10y + z + 198 = 100z + 10y + x$$

Substituting $99x - 99z = -198 \ (\div 99)$

$$x - z = -2 \quad \text{...(2)}$$
$$y = x + 2(y - z)$$
$$\Rightarrow x + y - 2z = 0 \quad \text{...(3)}$$

Consider (1) and (3)

$$145x - 35y + z = -27 \quad \text{...(1)}$$
$$(3) \times 35 \Rightarrow \quad 35x + 35y - 70z = 0 \quad \text{...(4)}$$
$$(1) + (4) \Rightarrow \quad 180x - 69z = -27 \quad \text{...(5)}$$

consider (5) and (2)

$$180x - 69z = -27$$
$$(2) \times 69 \Rightarrow \quad 69x - 69z = -138 \quad \text{...(6)}$$
$$(5)-(6) \Rightarrow \quad 111x = 111$$
$$x = \frac{111}{111} = 1$$

Substituting $x = 1$ in ...(2)

$$1 - z = -2$$
$$z = 1 + 2 = 3$$

Substituting $x = 1, z = 3$ in (3)

$$1 + y - 6 = 0$$
$$y = 5$$

$\therefore$ solution : $x = 1, y = 5, z = 3.$

The number is 153.

4. Find the least common multiple of $xy(k^2 + 1) + k(x^2 + y^2)$ and $xy(k^2 - 1) + k(x^2 - y^2)$

Sol :

$xy(k^2 + 1) + k(x^2 + y^2)$

Factorizing $xyk^2 + xy + kx^2 + ky^2$

$$= xyk^2 + ky^2 + kx^2 + xy$$

$= ky(kx+y) + x(kx+y)$

$= (kx+y)(ky+x)$

and $xy(k^2-1) + k(x^2-y^2)$

$= xyk^2 - xy + kx^2 - ky^2$

$= xyk^2 + kx^2 - xy - ky^2$

$= kx(ky+x) - y(x+ky)$

$= (kx-y)(ky+x)$

$\therefore$ LCM $= (ky+x)(kx+y)(kx-y)$

$= (ky+x)(k^2x^2-y^2)$

5. Find the GCD of the following by division algorithm
$2x^4 + 13x^3 + 27x^2 + 23x + 7$, $x^3 + 3x^2 + 3x + 1$, $x^2 + 2x + 1$

Sol :

Let $f(x) = 2x^4 + 13x^3 + 27x^2 + 23x + 7$

$g(x) = x^3 + 3x^2 + 3x + 1$

$h(x) = x^2 + 2x + 1$ which is the least degree polynomial.

Now Dividing f (x) by h (x)

$$\begin{array}{r|ll}
 & 2x^2+9x+7 & \\
\hline
x^2+2x+1 & 2x^4+13x^3+27x^2+23x+7 & \\
 & 2x^4+4x^3+2x^2 & (-) \\
\hline
 & 9x^3+25x^2+23x & \\
 & 9x^3+18x^2+9x & (-) \\
\hline
 & 7x^2+14x+7 & \\
 & 7x^2+14x+7 & (-) \\
\hline
 & 0 &
\end{array}$$

Since the Remainder is zero, h (x) is the GCD of f (x) and h (x)

Now, dividing g (x) by h (x)

$$\begin{array}{r|ll}
 & x+1 & \\
\hline
x^2+2x+1 & x^3+3x^2+3x+1 & \\
 & x^3+2x^2+x & (-) \\
\hline
 & x^2+2x+1 & \\
 & x^2+2x+1 & (-) \\
\hline
 & 0 &
\end{array}$$

Remainder is zero, $\therefore$ h (x) is the GCD of g (x) and h (x)

$\therefore$ h (x) divides both f(x) and g (x) completely

$\therefore$ GCD $= x^2 + 2x + 1$

6. Reduce the given Rational expressions to its lowest form

(i) $\dfrac{x^{3a}-8}{x^{2a}+2x^a+4}$ **(ii)** $\dfrac{10x^3-25x^2+4x-10}{-4-10x^2}$

Sol :

(i) $\dfrac{x^{3a}-8}{x^{2a}+2x^a+4}$

$$= \frac{(x^a)^3-(2)^3}{x^{2a}+2x^a+4}$$

$$= \frac{(x^a-2)(x^{2a}+2x^a+4)}{x^{2a}+2x^a+4}$$

$[a^3-b^3 = (a-b)(a^2+ab+b^2)]$

$= x^a - 2$

(ii) $\dfrac{10x^3-25x^2+4x-10}{-4-10x^2}$

$$= \frac{10x^2(x-5/2)+4(x-5/2)}{-(4+10x^2)}$$

$$= \frac{(x-5/2)(10x^2+4)}{-(4+10x^2)}$$

$= -(x-5/2)$

$= 5/2 - x$

7. Simplify $\dfrac{\frac{1}{p}+\frac{1}{q+r}}{\frac{1}{p}-\frac{1}{q+r}} \times \left(1+\dfrac{q^2+r^2-p^2}{2qr}\right)$

Sol :

$$\frac{\frac{1}{p}+\frac{1}{q+r}}{\frac{1}{p}-\frac{1}{q+r}} \times \left(1+\frac{q^2+r^2-p^2}{2qr}\right)$$

$$= \frac{\frac{q+r+p}{p(q+r)}}{\frac{q+r-p}{p(q+r)}} \times \left[\frac{2qr+q^2+r^2-p^2}{2qr}\right]$$

$$= \frac{p+q+r}{q+r-p} \times \left[\frac{(q+r)^2-p^2}{2qr}\right]$$

$$= \frac{p+q+r}{q+r-p} \times \frac{[(q+r+p)(q+r-p)}{2qr}$$

$$= \frac{(p+q+r)^2}{2qr} = \frac{1}{2qr}$$

8. Arul, Ravi and Ram working together can clean a store in 6 hours. Working alone Ravi takes twice as long to clean the store as Arul does. Ram needs three times as long as Arul does. How long would it take each if they are working alone?

Sol :

Given that Arul and Ram working together to clean a store in 6 hours.

$\therefore$ Work done by Arul, Ravi and Ram working together in 1 hour is $\frac{1}{6}$

Let the time taken by Arul, Ravi and Ram be x, y and z respectively.

$\therefore$ Time taken by Arul alone $= \frac{1}{x}$

$\therefore$ Time taken by Ravi alone $= \frac{1}{y}$

$\therefore$ Time taken by Ram alone $= \frac{1}{z}$

Given y = 2x and

z = 3x

$\therefore$ Total work done in 1 hour is

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{6}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{2x}+\frac{1}{3x} = \frac{1}{6}$$

$$\frac{6+3+2}{6x} = \frac{1}{6}$$

$$\frac{11}{6x} = \frac{1}{6}$$

$$\Rightarrow x = 11$$

$$\therefore y = 22, z = 33.$$

$\therefore$ Time taken by three persons are 11 hrs, 22 hrs and 33 hrs respectively.

9. Find the square root of $289x^4 - 612x^3 + 970x^2 - 684x + 361$

$$\begin{array}{r|l}
 & 17x^2 - 18x + 19 \\ \hline
17x^2 & 289x^4 - 612x^3 + 970x^2 - 684x + 361 \\
 & 289x^4 \quad (-) \\ \hline
34x^2 - 18x & -612x^3 + 970x^2 \\
 & -612x^3 + 324x^2 \quad (-) \\ \hline
34x^2 - 36x + 19 & 646x^2 - 684x + 361 \\
 & 646x^2 - 684x + 361 \quad (-) \\ \hline
 & 0 \\ \hline
\end{array}$$

$$\therefore \sqrt{289x^4 - 612x^3 + 970x^2 - 684x + 361}$$

$$= |17x^2 - 18x + 19|$$

10. Solve $\sqrt{y+1} + \sqrt{2y-5} = 3$

Sol :

$$\sqrt{2y-5} = 3 - \sqrt{y+1}$$

Squaring on both sides.

$$\left(\sqrt{2y-5}\right)^2 = \left(3-\sqrt{y+1}\right)^2$$

$$2y - 5 = 9 + y + 1 - 2(3)\sqrt{y+1}$$

$$2y - 5 = y + 10 - 6\sqrt{y+1}$$

$$2y - y - 5 - 10 = -6\sqrt{y+1}$$

$$y - 15 = -6\sqrt{y+1}$$

Again, squaring on both rides.

$$(y-15)^2 = \left(-6\sqrt{y+1}\right)^2$$

$$y^2 - 30y + 225 = 36(y+1)$$

$$y^2 - 30y + 225 = 36y + 36$$

$$y^2 - 30y - 36y + 225 - 36 = 0$$

$$y^2 - 66y + 189 = 0$$

Factorizing $\Rightarrow (y-63)(y-3) = 0$

$$\therefore y = 3, 63$$

11. A boat takes 1.6 hours longer to go 36 kms up a river than down the river. If the speed of the water current is 4 km per hr, what is the speed of the boat in still water?

Sol :

Let the speed of the boat in still water be x km/hr.

Speed of the river = 4 km/hr

Speed of the boat upstream = Speed of the boat in still water – Speed of River

$\therefore$ Speed of the boat upstream = (x – 4) km/hr.

Speed of the boat down stream. = (Speed of boat in still water) + (speed of River) = (x + 4) km/hr.

Time of Upstream journey = Time for downstream journey + 1.6 hr.

$$\therefore \frac{\text{Distance covered upstream}}{\text{Speed upstream}} = \frac{\text{Distance covered down stream}}{\text{Speed upstream}} + 1.6 \text{ hr}$$

$$\frac{36}{x-4} = \frac{36}{x+4} + 1.6$$

$$\frac{36}{x-4}-\frac{36}{x+4} = 1.6$$

$$36\left[\frac{1}{x-4}-\frac{1}{x+4}\right] = 1.6$$

$$36\left[\frac{x+4-x+4}{(x-4)(x+4)}\right] = 1.6$$

$$\frac{36(8)}{x^2-16} = 1.6$$

$$\Rightarrow x^2-16 = \frac{288}{1.6}$$

$$x^2 - 16 = 180$$

$$x^2 = 180 + 16$$

$$x^2 = 196$$

$$x = 14$$

$\therefore$ Speed of the boat in still water is 14 km/hr.

12. Is it possible to design a rectangular park of perimeter 320 m and area 4800 m^2? If so find its length and breath.

Sol :

Let the length of the rectangular Park be 'x' m

breadth of the rectangular Park be 'y' m.

$$\text{Given Perimeter} = 320 \text{ m}$$

$$\text{i.e, } 2(x+y) = 320$$

$$x + y = 160$$

$$y = 160 - x$$

$$\text{Given, Area} = 4800$$

$$\Rightarrow xy = 4800$$

$$\Rightarrow x(160-x) = 4800$$

$$\Rightarrow 160x - x^2 = 4800$$

$$\Rightarrow x^2 - 160x + 4800 = 0$$

$$\text{Factorizing} \Rightarrow (x-120)(x-40) = 0$$

$$x - 120 = 0,$$

$$x - 40 = 0$$

$$x = 120,$$

$$x = 40$$

$\therefore$ length of the park = 120 m

breadth of the park = 40 m.

13. At t minutes past 2 pm. the time needed to 3 pm is 3 minute less than $\frac{t^2}{4}$. Find t.

Sol :

Given that, time needed by the minute hand to show 3 pm is $\frac{t^2}{4} - 3$.

$\therefore$ By the given data.

$$\frac{t^2}{4} - 3 = 60 - t$$

[$\because$ *From 2 pm to 3 pm = 1 hr = 60 min*]

$$t^2 - 12 = 4(60 - t)$$

$$t^2 - 12 = 240 - 4t$$

$$t^2 + 4t - 252 = 0$$

$$\text{Factorizing,} \quad (t+18)(t-14) = 0$$

$$t = -18 \text{ and}$$

$$t = 14$$

t = – 18 is rejected as 't' cannot be negative.

$\therefore$ t = 14 minutes.

14. The number of seats in a row is equal to the total number of rows in a hall. The total number of seats in the hall will increase by 375 if the number of rows is doubled and the number of seats in each row is reduced by 5. Find the number of rows in the hall at the beginning.

Sol :

Let the number of rows be 'x'

and number of seats in each row is also 'x'

$$\therefore \text{total no. of seats} = x.x = x^2$$

From the given data, we get

$$x^2 + 375 = 2x(x-5)$$

$$x^2 + 375 = 2x^2 - 10x$$

$$\text{Simplifying,} \quad 2x^2 - x^2 - 10x - 375 = 0$$

$$x^2 - 10x - 375 = 0$$

$$\text{Factorizing,} \quad (x-25)(x+15) = 0$$

$$x - 25 = 0, \quad x + 15 = 0$$

$x = 25,$ $x = -15$ is rejected as 'x cannot be negative.

$\therefore$ No of rows in the hall at the beginning is 25.

15. If α and β are the roots of the polynomial f (x) = x^2 – 2x + 3, find the polynomial whose roots are (i) $\alpha+2, \beta+2$ (ii) $\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$.

Sol :

Given α and β are the zeros of

$$f(x) = x^2 - 2x + 3$$

$$a = 1, b = -2, c = 3.$$

$$\text{Sum of the zeroes } \alpha+\beta = \frac{-b}{a} = \frac{2}{1} = 2$$

$$\text{Product of the zeroes } \alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$$

(i) $\alpha+2, \beta+2$ are the zeroes.

$$\text{sum of the zeroes} = \alpha+2+\beta+2 = \alpha+\beta+4 = 2+4 = 6$$

$$\text{Product of the zeroes} = (\alpha+2)(\beta+2) = \alpha\beta+2(\alpha+\beta)+4 = 3+2(2)+4 = 3+4+4 = 11$$

$\therefore$ the polynomial is

$x^2 -$ (sum of the zeroes) $x +$ (Product of the zeroes) $= 0$

$$x^2 - 6x + 11 = 0$$

(ii) $\dfrac{\alpha-1}{\alpha+1}, \dfrac{\beta-1}{\beta+1}$ are zeroes

$$\text{Sum of zeroes} = \frac{\alpha-1}{\alpha+1}+\frac{\beta-1}{\beta+1} = \frac{(\alpha-1)(\beta+1)+(\beta-1)(\alpha+1)}{(\alpha+1)(\beta+1)} = \frac{2\alpha\beta-2}{\alpha\beta+(\alpha+\beta)+1} = \frac{2(3)-2}{3+2+1} = \frac{6-2}{6} = \frac{4}{6} = \frac{2}{3}$$

$$\text{Product of zeroes} = \left(\frac{\alpha-1}{\alpha+1}\right)\left(\frac{\beta-1}{\beta+1}\right) = \frac{\alpha\beta-(\alpha+\beta)+1}{\alpha\beta+(\alpha+\beta)+1} = \frac{3-2+1}{3+2+1} = \frac{2}{6} = \frac{1}{3}$$

$\therefore$ the polynomial is

$x^2 -$ (sum of the zeroes) $x +$ (Product of the zeroes) $= 0$

$$x^2 - \frac{2}{3}x + \frac{1}{3} = 0$$

it is simplified as $3x^2 - 2x + 1$

16. If -4 is a root of the equation $x^2 + px - 4 = 0$ and if the equation $x^2 + px + q = 0$ has equal roots, find the values of p and q.

Sol :

Given that -4 is a root of the equation.

$$x^2 + px - 4 = 0$$

$$\therefore (-4)^2 + p(-4) - 4 = 0$$

$$16 - 4p - 4 = 0$$

$$\Rightarrow 4p = 12$$

$$p = \frac{12}{4} = 3 \text{ and}$$

$x^2 + px + q = 0$ has equal roots

$\therefore$ Discriminant $b^2 - 4ac = 0$

$[\because a = 1, b = p, c = q]$

$$p^2 - 4(1)(q) = 0$$

$$(3)^2 - 4q = 0$$

$$4q = 9$$

$$q = \frac{9}{4}$$

$$\therefore p = 3, \quad q = \frac{9}{4}$$

17. Two formers Senthil and Ravi cultivates three varieties of grains namely rice, wheat and ragi. If the sale (in ₹) of three varieties of grains by both the formers in the month of April is given by the matrix.

April sale in ₹

$$A = \begin{array}{c} \\ \left[\begin{array}{ccc} \textit{rice} & \textit{wheat} & \textit{ragi} \\ 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{array}\right] \end{array} \begin{array}{l} \textit{Senthil} \\ \textit{Ravi} \end{array}$$

and the may sale (in ₹) is exactly twice as that of the April month sale for each variety.

(i) What is the average sales of the months April and May.

(ii) If the sales continues to increase in the same way in the successive months, what will be sales in the month of August?

Sol :

Sale of 'April' month

$$A = \left[\begin{array}{ccc} \textit{rice} & \textit{wheat} & \textit{ragi} \\ 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{array}\right] \begin{array}{l} \textit{Senthil} \\ \textit{Ravi} \end{array}$$

Given that May month's sale is exactly twice the sale in April month.

$\therefore$ Sale of 'May' month

$$= \begin{bmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{bmatrix}$$

(i) The average sales of April and May

$$= \frac{1}{2}\begin{bmatrix} 500+1000 & 1000+2000 & 1500+3000 \\ 2500+5000 & 1500+3000 & 500+1000 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1500 & 3000 & 4500 \\ 7500 & 4500 & 1500 \end{bmatrix}$$

$$= \begin{bmatrix} 750 & 1500 & 2250 \\ 3750 & 2250 & 750 \end{bmatrix}$$

(ii) Sales in the month of April

$$= \begin{bmatrix} 500 & 1000 & 1500 \\ 2500 & 1500 & 500 \end{bmatrix}$$

Sales in the month of May

$$= \begin{bmatrix} 1000 & 2000 & 3000 \\ 5000 & 3000 & 1000 \end{bmatrix}$$

Similarly in the month of August

$$= \begin{bmatrix} 8000 & 16000 & 24000 \\ 40000 & 24000 & 8000 \end{bmatrix}$$

18. If $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} x & -\cos\theta \\ \cos\theta & x \end{bmatrix} = I_2$, find x.

Sol :

$$\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} x\sin\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & x\sin\theta \end{bmatrix} = I_2$$

$$\begin{bmatrix} \cos^2\theta + x\sin\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & x\sin\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x\sin\theta + \cos^2\theta & 0 \\ 0 & x\sin\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$x\sin\theta + \cos^2\theta = 1$$

$$x\sin\theta = 1 - \cos^2\theta$$

$$x = \frac{\sin^2\theta}{\sin\theta}$$

$$\therefore x = \sin\theta.$$

19. Given $A = \begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ and if BA = C^2. Find p and q.

Sol :

$$A = \begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & -q \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0-2q \\ p+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2q \\ p & 0 \end{bmatrix}$$

$$C^2 = C.C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-4 & -4-4 \\ 4+4 & -4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

Given BA = C^2

$$\begin{bmatrix} 0 & -2q \\ p & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$$

$$\therefore -2q = -8$$

$$q = \frac{8}{2} = 4$$

$$p = 8$$

20. $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 3 \\ 8 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ find the matrix D, such that CD – AB = 0.

Sol :

$$\left.\begin{aligned} \text{Order of } A &= 2\times 2 \\ \text{Order of } B &= 2\times 2 \end{aligned}\right\}$$

$\therefore$ Order of AB = Order of CD

$$\left.\begin{aligned} \text{Order of } C &= 2\times 2 \\ \text{and } CD - AB &= 0 \end{aligned}\right\}$$

$\therefore$ Order of CD = 2×2

$$\text{Let } D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Given CD – AB = 0

$$\begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 8 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3a+6c & 3b+6d \\ a+c & b+d \end{bmatrix} - \begin{bmatrix} 18+0 & 9+0 \\ 24+40 & 12+25 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3a+6c-18 & 3b+6d-9 \\ a+c-64 & b+d-37 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now, Equating the Corresponding elements.

$3a + 6c - 18 = 0$

$3a + 6c = 18$ (÷ 3)

$a + 2c = 6$ ...(1)

$a + c - 64 = 0$

$a + c = 64$ ...(2)

$3b + 6d - 9 = 0$

$3b + 6d = 9$ (÷ 3)

$b + 2d = 3$ ...(3)

$b + d - 37 = 0$

$b + d = 37$ ...(4)

Consider equation (1) and (2)

$a + 2c = 6$ (1)

$a + c = 64$ (2)

(1) – (2) ⇒ $c = -58$

Substituting in (2)

$a - 58 = 64$

$a = 64 + 58 = 122$

Now, Consider equation (3) and (4)

$b + 2d = 3$ (3)

$b + d = 37$ (4)

(3) – (4) ⇒ $d = -34$

Substituting in (4)

$b - 34 = 37$

$b = 37 + 34 = 71$

∴ The matrix $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} 122 & 71 \\ -58 & -34 \end{bmatrix}$

CREATIVE QUESTIONS

I. Multiple Choice Questions:

Simultaneous Linear Equations in three variables

1. The value of x in (x + 2) + 2 (x – 1) = 4x – 3

(1) 2 (2) 3

(3) – 2 (4) – 3 **[Ans: (2)]**

Sol :

$(x + 2) + 2(x - 1) = 4x - 3$

$x + 2 + 2x - 2 = 4x - 3$

$3x = 4x - 3$

$4x - 3x = 3$

$x = 3$

2. Solve for x: $\left(x-\frac{1}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 = x + 2$

(1) 4 (2) 8

(3) – 4 (4) – 8 **[Ans: (1)]**

Sol : $\left(x-\frac{1}{2}\right)^2 - \left(x-\frac{3}{2}\right)^2 = x + 2$

$x^2 + \frac{1}{4} - x - x^2 - \frac{9}{4} + 3x = x + 2$

Simplifying we get, $2x - 2 = x + 2$

$x = 4$

3. If x men can do a piece of work in y days, in how many days will z men do the same work?

(1) $\frac{xz}{y}$ (2) $\frac{xy}{z}$

(3) $\frac{yz}{x}$ (4) xyz **[Ans : (2)]**

Sol :

x men can do the work in y days

1 man can do the work in xy days.

∴ z men can do the work in $\frac{xy}{z}$ days.

4. Find the value of x and y if $\frac{5}{y}-\frac{2}{x}=\frac{7}{6}$ and $\frac{36}{x}-\frac{24}{y}=1$

(1) x = 4, y = 3 (2) x = – 4, y = 3
(3) x = – 4, y = – 3 (4) x = 4, y = – 3

[Ans : (1)]

Sol :

$$(1)\times 18 \Rightarrow \quad \frac{-36}{x}+\frac{90}{y} = 21$$

$$(2) \Rightarrow \quad \frac{36}{x}-\frac{24}{y} = 1$$

Adding

$$\frac{66}{y} = 22$$

$$y = 3$$

Substituting in (2) we get x = 4; y = 3.

5. What should be the value of p if 3x + 2y = 8 and 6x + 4y = 9 have infinitely many solutions?

(1) 3 (2) 16
(3) 5 (4) 6 **[Ans : (2)]**

Sol :

If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$ are the two equations, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ should be true for infinitely many solutions.

$$\therefore \frac{3}{6} = \frac{2}{4} = \frac{8}{p}$$

$$p = 16$$

6. What should be the value of m in the pair of equations 4x + my + 9 = 0 and 3x + 4y + 18 = 0 to have unique solution?

(1) $m \neq 16$ (2) $m \neq 15$
(3) $m \neq \frac{16}{3}$ (4) $m \neq \frac{15}{4}$ **[Ans : (3)]**

Sol :

For unique solution, $a_1b_2 \neq b_1a_2$

$$4 \times 4 \neq 3m$$

$$3m \neq 16$$

$$m \neq \frac{16}{3}$$

7. If the sum of two numbers is 640 and their difference is 280, then the numbers are

(1) 140, 500 (2) 180, 460
(3) 130, 510 (4) 150, 490 **[Ans : (2)]**

Sol :

Let the numbers be x and y

Then, Given $x + y = 640$ ... (1)

$x - y = 280$ ... (2)

Adding $2x = 920$

$\Rightarrow x = 460$

Substituting in (1) $460 + y = 640$

$y = 640 - 460 = 180$

8. The total salary of 15 men and 8 women in ₹ 3050. The difference of salaries of 5 women and 3 men is ₹ 50. Find the sum of the salaries of 3 men and 5 women.

(1) ₹ 900 (2) ₹ 850
(3) ₹ 950 (4) ₹ 1000 **[Ans : (3)]**

Sol :

Let the man's salary be 'x'

Let the woman's salary be 'y'

Given $15x + 8y = 3050$ ... (1)

$3x - 5y = -50$ ... (2)

$(2) \times 5 \Rightarrow 15x - 25y = -250$

$15x + 8y = 3050$

Subtracting $-33y = -3300$

$y = 100$

Substituting in (2), $3x - 500 = -50$

$3x = 450$

$x = 150$

Total salary $= 3(150) + 5(100)$

$= 450 + 500 =$ ₹ 950

9. Find the solution to the system x + y + z = 2, 6x – 4y + 5z = 31 and 5x + 2y + 2z = 13

(1) (3, – 2, 1) (2) (2, – 3, 1)
(3) (1, 2, 3) (4) (–1, – 2, – 3)

[Ans : (1)]

Sol :

Substituting the given answers in all the 3 equations, we get the answer.

(3, – 2, 1) satisfying all the three equations.

10. The solution of the system of equations 4x + 2y – 4z = – 18, 8x – 2y – 5z = – 18 and – 16x – 2y – z = – 2

(1) (1, 0, 4) (2) (4, 0, 1)
(3) (0, – 1, – 4) (4) (0, – 1, 4) **[Ans : (4)]**

Sol :

(0, – 1, 4) Satisfying all the three equations.

GCD and LCM

11. The GCD of two numbers is 36 and their LCM is 648. The product of two numbers is

(1) 23328 (2) 648
(3) 3888 (4) 23348 **[Ans : (1)]**

Sol :

Product of two numbers = GCD × LCM
= 36 × 648
= 23328

12. The GCD of $10(x^2 + x - 20)$, $15(x^2 - 3x - 4)$ and $20(x^2 + 2x + 1)$ is

(1) $5(x - 4)$ (2) 5
(3) $5(x + 1)$ (4) $5(x + 1)(x - 1)$

[Ans : (2)]

Sol :

GCD of 10, 15 and 20 is 5

Now $x^2 + x - 20 = (x + 5)(x - 4)$
$x^2 - 3x - 4 = (x - 4)(x + 1)$
$x^2 + 2x + 1 = (x + 1)^2$
$\therefore$ GCD = 5

13. The LCM of $a^2 + 3a + 2$, $a^2 + 5a + 6$ and $a^2 + 4a + 4$ is

(1) $(a + 2)^2 (a + 3)$
(2) $(a + 2)^2 (a + 1)$
(3) $(a + 2)^2 (a + 3)(a + 1)$
(4) $(a + 3)(a + 2)(a + 1)$ **[Ans : (3)]**

Sol :

$a^2 + 3a + 2 = (a + 2)(a + 1)$
$a^2 + 5a + 6 = (a + 3)(a + 2)$
$a^2 + 4a + 4 = (a + 2)^2$
LCM $= (a + 2)^2 (a + 3)(a + 1)$

14. How many times 5 bells ring together in 1 hour if they start together and ring at intervals of 2, 3, 4, 5 and 6 sec respectively?

(1) 71 times (2) 60 times
(3) 59 times (4) 61 times **[Ans : (2)]**

Sol :

LCM of 2, 3, 4, 5 and 6 is 60 sec = 1 min.

Therefore, they ring together once in a minute and hence, 60 times in an hour.

15. GCD of $x^2 - \frac{1}{x^2}$, $x^2 - 2 + \frac{1}{x^2}$ and $x^3 - \frac{1}{x^3} - 3x - \frac{3}{x}$ is

(1) $x^2 - \frac{1}{x^2}$ (2) $\left(x - \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)$

(3) $x - \frac{1}{x}$ (4) $\left(x - \frac{1}{x}\right)^2$ **[Ans : (3)]**

Sol :

$$x^2 - \frac{1}{x^2} = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$$

$$x^2 - 2 + \frac{1}{x^2} = x^2 - 2x\frac{1}{x^2} + \left(\frac{1}{x}\right)^2$$

$$= \left(x - \frac{1}{x}\right)^2$$

$$x^3 - \frac{1}{x^3} - 3x - \frac{3}{x} = x^3 - \left(\frac{1}{x}\right)^3 - 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2$$

$$= \left(x - \frac{1}{x}\right)^3$$

$$\therefore \text{GCD} = \left(x - \frac{1}{x}\right)$$

16. If the GCD and LCM of two expressions are $x + 2$ and $(x + 2)^2 (x - 2)$ respectively, then the two expressions are

(1) $(x + 2), (x - 2)$ (2) $(x + 2)^2, (x^2 - 4)$
(3) $(x + 2), (x^2 - 4)$ (4) $(x + 2)^2, (x - 2)$

[Ans : (2)]

Sol :

GCD = $x + 2$,
LCM = $(x + 2)^2 (x - 2)$

$$\frac{LCM}{GCD} = \frac{(x+2)^2 (x-2)}{(x+2)}$$

$= (x + 2)(x - 2)$

The required expressions are $(x + 2)(x + 2)$ and $(x + 2)(x - 2)$

i.e., $(x + 2)^2$ and $(x^2 - 4)$

17. The GCD of $x^2 + 3x + 2$ and $x^3 + 9x^2 + 23x + 15$ is

(1) $x + 1$ (2) $x + 2$
(3) $(x + 1)(x + 2)$ (4) $(x + 1)(x - 1)$

[Ans : (1)]

Sol :

$x^2 + 3x + 2 = (x + 2)(x + 1)$

$(x + 1)$ is a factor of $x^3 + 9x^2 + 23x + 15$ as $x + 1$

divides it completely whereas (x + 2) does not divide it.

$\therefore$ (x + 1) is the common divisor.

18. The biggest length of a tape that can be used for measuring cloth of lengths 3 m, 5 m, 10 m and 90 m is

(1) 30 (2) 50

(3) 90 (4) 100 **[Ans : (3)]**

Sol :

LCM of 3, 5, 10 and 90 is 90

19. The LCM of two numbers is a + b and the GCD is k (a – b). If one of the numbers is k, the other number is

(1) $\frac{ka}{b}$ (2) kab

(3) $a^2 - b^2$ (4) $\frac{ka+b}{ka-b}$ **[Ans : (3)]**

Sol :

$$kx = (a+b)\,k\,(a-b)$$

$$\therefore\ x = \frac{(a+b)\,k\,(a-b)}{k}$$

$$= (a+b)(a-b)$$

$$= a^2 - b^2$$

20. The GCD and LCM of a and b are 27 and 2079 respectively. If a is divided by 9, the quotient is 21. Then b is

(1) 243 (2) 189

(3) 113 (4) 297 **[Ans : (4)]**

Sol :

$$ab = 27 \times 2079$$

$$a = 9 \times 21 = 189$$

$$\therefore\ b = \frac{27 \times 2079}{189} = 297$$

21. The LCM of $8x^4y^2z^3$, $10xy^3z^5$ and $12x^2y^2z^4$ is

(1) $120x^2y^2z^2$ (2) $120x^4y^3z^5$

(3) $2xy^2z^3$ (4) $120x^3y^3z^5$ **[Ans : (2)]**

Sol :

LCM of 8, 10 and 12 is 120

LCM of $x^4y^2z^3$, xy^3z^5 and $x^2y^2z^4$ is $x^4y^3z^5$

22. The LCM of $x^2 - 3ax + 2a^2$, $x^2 - 4ax + 4a^2$ and $x^2 - ax - 2a^2$ is

(1) $(x-2a)^2(x^2-a^2)$

(2) $(x-a)^2(x-2a)$

(3) $(x-a)(x-2a)(x-3a)$

(4) $(x-2a)^3$ **[Ans : (1)]**

Sol :

$$x^2 - 3ax + 2a^2 = (x-a)(x-2a)$$

$$x^2 - 4xa + 4a^2 = (x-2a)^2$$

$$x^2 - ax - 2a^2 = (x-2a)(x+a)$$

$$\text{LCM} = (x-2a)^2(x+a)(x-a)$$

$$= (x-2a)^2(x^2-a^2)$$

Rational expressions

23. Simplified form of $\frac{x^3-3x^2}{9x^2-x^4}$ is

(1) $\frac{1}{x+3}$ (2) $-\frac{1}{x+3}$

(3) $\frac{1}{3(x+1)}$ (4) $\frac{1}{3(x-1)}$ **[Ans : (2)]**

Sol :

$$\frac{x^3-3x^2}{9x^2-x^4} = \frac{x^2(x-3)}{x^2(3^2-x^2)}$$

$$= \frac{(x-3)}{(3+x)(3-x)} = \frac{-1}{x+3}$$

24. Simplest form of $\frac{a^2-b^2}{a^2-3ab+2b^2}$ is

(1) $\frac{(a+b)^2}{a-2b}$ (2) $\frac{a+b}{a-2b}$

(3) $\frac{a-b}{a-2b}$ (4) $\frac{a+b}{a+2b}$ **[Ans : (2)]**

Sol :

$$\frac{a^2-b^2}{a^2-3ab+2ab^2} = \frac{(a+b)(a-b)}{(a-2b)(a-b)} = \frac{a+b}{a-2b}$$

25. Simplest form of

$$\frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \frac{1}{(x+3)(x+1)}$$ **is**

(1) $\frac{1}{(x+1)(x+3)}$ (2) $\frac{2}{(x+1)(x+3)}$

(3) $\frac{3}{(x+2)(x+3)}$ (4) $\frac{3}{(x+1)(x+3)}$ **[Ans : (4)]**

Sol :

$$\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)(x+3)}+\frac{1}{(x+3)(x+1)}$$
$$=\frac{(x+3)+(x+1)+(x+2)}{(x+1)(x+2)(x+3)}$$
$$=\frac{3x+6}{(x+1)(x+2)(x+3)}$$
$$=\frac{3(x+2)}{(x+1)(x+2)(x+3)}=\frac{3}{(x+1)(x+3)}$$

26. If $x = 2\left(t+\frac{1}{t}\right)$ and $y = 3\left(t-\frac{1}{t}\right)$ and then $\frac{x^2}{4}-\frac{y^2}{9}$ is

(1) 3 (2) -4
(3) 4 (4) -3 **[Ans : (3)]**

Sol :

$$\frac{x^2}{4}-\frac{y^2}{9}=\frac{2^2\left[t+\frac{1}{t}\right]^2}{4}-\frac{3^2\left[t-\frac{1}{t}\right]^2}{9}$$
$$=\left[t+\frac{1}{t}\right]^2-\left[t-\frac{1}{t}\right]^2$$
$$=2+2=4$$

27. Simplified form of $\frac{p+p^2+p^3+p^4+p^5+p^6+p^7}{p^{-3}+p^{-4}+p^{-5}+p^{-6}+p^{-7}+p^{-8}+p^{-9}}$

(1) p^{10} (2) p^{-10}
(3) p^9 (4) p^{-9} **[Ans : (1)]**

Sol :

$$\frac{p+p^2+p^3+p^4+p^5+p^6+p^7}{p^{-3}+p^{-4}+p^{-5}+p^{-6}+p^{-7}+p^{-8}+p^{-9}}$$
$$=\frac{p(1+p+p^2+p^3+p^4+p^5+p^6)}{p^{-9}(p^6+p^5+p^4+p^3+p^2+p+1)}$$
$$=p^1.p^9=p^{1+9}=p^{10}$$

28. Simplest form of $\frac{x^7+2x^6+x^5}{x^3(x+1)^8}$ is

(1) $\frac{x^2}{(x^6+1)}$ (2) $\frac{x^2}{(x+1)^6}$

(3) $\frac{x^3}{x+1}$ (4) $\frac{x^4}{x+2}$ **[Ans : (2)]**

Sol :

$$\frac{x^7+2x^6+x^5}{x^3(x+1)^8}=\frac{x^5(x^2+2x+1)}{x^3(x+1)^8}$$
$$=\frac{x^2(x+1)^2}{(x+1)^8}=\frac{x^2}{(x+1)^6}$$

29. $\frac{x^2-5x-14}{x^2-3x+2}\times\frac{x^2-4}{x^2-14x+49}=$

(1) $\frac{x+2}{x+7}$ (2) $\frac{(x+2)^2}{x-7}$

(3) $\frac{(x+2)^2}{(x-1)(x-7)}$ (4) $\frac{x-2}{(x-1)(x-7)}$

[Ans : (3)]

Sol :

$$\frac{x^2-5x-14}{x^2-3x+2}\times\frac{x^2-4}{x^2-14x+49}$$
$$=\frac{(x-7)(x+2)}{(x-2)(x-1)}\cdot\frac{(x+2)(x-2)}{(x-7)^2}$$
$$=\frac{(x+2)^2}{(x-1)(x-7)}$$

30. $\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}=?$

(1) 1 (2) 3
(3) -3 (4) -1 **[Ans : (4)]**

Sol :

$$\frac{m^2-9}{m^2+5m+6}\div\frac{3-m}{m+2}$$
$$=\frac{(m+3)(m-3)}{(m+3)(m+2)}\times\frac{(m+2)}{-(m-3)}$$
$$=\frac{m-3}{-(m-3)}=-1$$

31. Simplify : $\frac{(y^2+5y+4)}{\left(\frac{y^2-1}{y+5}\right)}$

(1) $\frac{y-1}{y-4}$ (2) $\frac{y+5}{y-1}$

(3) $\frac{(y+4)(y+3)}{y-1}$ (4) $\frac{y^2+9y+20}{y-1}$ **[Ans : (4)]**

Sol :

$$\frac{(y^2+5y+4)}{\left(\frac{y^2-1}{y+5}\right)} = \frac{(y+4)(y+1)}{(y+1)(y-1)} \times (y+5)$$

$$= \frac{(y+4)(y+5)}{y-1} = \frac{y^2+9y+20}{y-1}$$

32. $\frac{x^2}{x+3} + \frac{11x+24}{x+3} =$

(1) $x + 8$ (2) $x - 8$
(3) $8 - x$ (4) $x + 3$ **[Ans : (1)]**

Sol :

$$\frac{x^2}{x+3} + \frac{11x+24}{x+3} = \frac{x^2+11x+24}{x+3}$$

$$= \frac{(x+3)(x+8)}{x+3} = (x+8)$$

33. What is the result in simplest form when $\frac{4x-5}{x^2-64}$ is subtracted from $\frac{5x+3}{x^2-64}$

(1) $x - 8$ (2) $(x-8)^{-1}$
(3) $(x-8)^{-2}$ (4) $(x-8)^{-3}$ **[Ans : (2)]**

Sol :

$$\frac{5x+3}{x^2-64} - \frac{4x-5}{x^2-64} = \frac{5x+3-4x+5}{x^2-64}$$

$$= \frac{x+8}{(x+8)(x-8)}$$

$$= \frac{1}{x-8} = (x-8)^{-1}$$

34. Excluded values of $\frac{2x+1}{x^2-x-6}$ are

(1) $1, -2$ (2) $-2, 3$
(3) $2, -3$ (4) $2, 3$ **Ans : (2)]**

Sol :

$$\frac{2x+1}{x^2-x-6} = \frac{2x+1}{(x-3)(x+2)}$$

When $x = -2$ and $x = 3$, denominator becomes zero.

$\therefore$ Excluded values are $-2, 3$.

35. Excluded values of $\frac{4x-2}{2x^2+x-1}$ is / are

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$

(3) $-\frac{1}{2}$ (4) -1 **[Ans : (4)]**

Sol :

$$\frac{4x-2}{2x^2+x-1} = \frac{2(2x-1)}{(2x-1)(x+1)} = \frac{2}{x+1}$$

When $x = -1$, the denominator becomes zero.

$\therefore$ -1 is the excluded value.

36. If $a + b + c = 0$, then the value of

$\frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca}$ **is**

(1) 0 (2) 1
(3) 2 (4) 3 **[Ans : (4)]**

Sol :

$$\frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca}$$

Given $a + b + c = 0 \Rightarrow a + b = -c, b + c = -a,$
$c + a = -b$

$$\therefore \frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} =$$

$$\frac{c^2}{ab} + \frac{a^2}{bc} + \frac{b^2}{ca} = \frac{a^3+b^3+c^3}{abc}$$

If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$
(Algebraic Identity)

$$\therefore = \frac{3abc}{abc} = 3$$

Square root of Rational Expression and Polynomials

37. Number of methods to find square root of an algebraic expression are

(1) 3 (2) 4
(3) 5 (4) 2 **[Ans : (4)]**

38. **The square root of (x + 1) (x + 2) (x + 3) (x + 4) + 1 is**

(1) $x^2 + 2x + 3$ (2) $x^2 + 5x + 5$
(3) $x^2 + 3x + 2$ (4) $x^2 + 2x + 1$ [Ans : (2)]

Sol :

$(x + 1)(x + 2)(x + 3)(x + 4) + 1$

Rearranging $= (x + 1)(x + 4)(x + 2)(x + 3) + 1$
$= (x^2 + 5x + 4)(x^2 + 5x + 6) + 1$

Substituting $x^2 + 5x = a$

Then

$(a + 4)(a + 6) + 1 = a^2 + 10a + 24 + 1$
$= a^2 + 10a + 25 = (a + 5)^2$

$\therefore$ Square root of

$(a + 5)^2 = |a + 5|$

i.e., $= |x^2 + 5x + 5|$

39. **If the polynomial $16x^4 - 24x^3 + 41x^2 - mx + 16$ be a perfect square, then the value of 'm' is**

(1) 12 (2) – 12
(3) 24 (4) – 24 [Ans : (3)]

40. $\sqrt{\dfrac{x^{-7}\,y^{14}}{x^{14}\,y^{-28}}} \div \sqrt{\dfrac{x^{-15}\,y^{25}}{x^{10}\,y^{-15}}}$

(1) $\sqrt{x^3 y^5}$ (2) x y
(3) x^2y (4) xy^2 [Ans : (3)]

Sol :

$$\sqrt{\frac{x^{-7}y^{14}}{x^{14}y^{-28}}} \div \sqrt{\frac{x^{-15}y^{25}}{x^{10}y^{-15}}} = \sqrt{\frac{x^{-7}y^{14}}{x^{14}y^{-28}} \times \frac{x^{10}y^{-15}}{x^{-15}y^{25}}}$$
$$= \sqrt{\frac{y^{-1}}{x^{-1}} \times \frac{x^3}{y^{-3}}}$$
$$= \sqrt{x^4y^2} = x^2y$$

41. $\sqrt{(4a^2)(6b^2)(3a^2b^2)} =$

(1) a^2b^2 (2) $6\sqrt{2}\,a^2\,b^2$
(3) $72a^4b^4$ (4) a^4b^4 [Ans : (2)]

Quadratic equation

42. **Which of the following is a quadratic equation?**

(1) $x^{1/2} + 2x + 3 = 0$
(2) $(x - 1)(x + 4) = x^2 + 1$
(3) $x^2 - 3x + 5 = 0$
(4) $(2x + 1)(3x - 4) = 6x^2 + 3$ [Ans : (3)]

43. **The quadratic equation whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$ is**

(1) $x^2 - 4x + 2 = 0$ (2) $x^2 - 2x + 2 = 0$
(3) $x^2 + 2x - 4 = 0$ (4) $x^2 - 2x + 4 = 0$

[Ans : (1)]

Sol :

Sum of the roots $= 2 + \sqrt{2} + 2 - \sqrt{2} = 4$

Product of the roots $= \left(2 + \sqrt{2}\right)\left(2 - \sqrt{2}\right)$
$= 4 - 2 = 2$

Quadratic Equation : $x^2 - (S\text{ o }R)\,x + P\text{ o }R = 0$
$\Rightarrow x^2 - 4x + 2 = 0$

44. **The Quadratic equation whose roots are $\dfrac{p}{q}, \dfrac{-q}{p}$ is**

(1) $qx^2 - (q^2 + p^2)\,x - pq = 0$
(2) $pqx^2 - (p^2 - q^2)\,x - pq = 0$
(3) $px^2 - (p^2 + 1)\,x + p = 0$
(4) $p^2x^2 - (p^2 - q^2)\,x - pq = 0$ [Ans : (2)]

Sol :

Sum of the roots $= \dfrac{p}{q} - \dfrac{q}{p} = \dfrac{p^2 - q^2}{pq}$

Product of the roots $= \left(\dfrac{p}{q}\right)\left(\dfrac{-q}{p}\right) = -1$

Quadratic Equation: $x^2 - (\text{Sum})\,x + \text{Product} = 0$

$$x^2 - \left(\frac{p^2 - q^2}{pq}\right)x - 1 = 0$$

$pq\,x^2 - (p^2 - q^2)\,x - pq = 0$

45. **If $ax^2 + bx + c$ is a perfect square, then $b^2 =$**

(1) 2ac (2) ac
(3) 4ac (4) $\sqrt{2ac}$ [Ans : (3)]

Sol :

$b^2 = 4ac$

46. **One root of $px^2 + qx + r = 0$ is r, then the second root is**

(1) p (2) q
(3) $\dfrac{1}{q}$ (4) $\dfrac{1}{p}$ [Ans : (4)]

Sol :

Product of roots $\alpha\beta = \dfrac{r}{p}$

$$r(\beta) = \frac{r}{p}$$

$$\beta = \frac{1}{p}$$

47. The condition for $px^2 + qx + r = 0$ to be a pure quadratic equation is

(1) $p = 0$ (2) $q = 0$

(3) $r = 0$ (4) $p = q = 0$ [Ans : (2)]

48. Common root of $x^2 + x - 6 = 0$ and $x^2 + 3x - 10 = 0$ is

(1) -2 (2) 2

(3) -3 (4) -5 [Ans : (2)]

Sol :

Let the common factor be $x - k$

then, $k^2 + k - 6 = k^2 + 3k - 10$

$4 = 2k$

$k = 2$ (common root)

49. Ratio of the sum of the roots of $x^2 - 9x + 18 = 0$ to the product of the roots is

(1) $1 : 2$ (2) $2 : 1$

(3) $-1 : 2$ (4) $-2 : 1$ [Ans : (1)]

Sol :

$$x^2 - 9x + 18 = 0$$

$$\text{Sum of the roots} = -\left(\frac{-9}{1}\right) = 9$$

$$\text{Product of roots} = \frac{18}{1} = 18$$

Ratio $9 : 18 = 1 : 2$

50. If the discriminant of $3x^2 - 14x + k = 0$ is 100, then k =

(1) 8 (2) 32

(3) 16 (4) 24 [Ans : (1)]

Sol :

$$\begin{aligned} \text{Discriminant} &= 100 \\ b^2 - 4ac &= 100 \\ (-14)^2 - 4(3)(k) &= 100 \\ 196 - 12k &= 100 \\ k &= 8 \end{aligned}$$

51. The roots of the equation $4x^2 - 2x + 8 = 0$ are

(1) Real and equal

(2) Rational and not equal

(3) Irrational

(4) Not real [Ans : (4)]

Sol :

$$\begin{aligned} b^2 - 4ac &= (-2)^2 - 4(4)(8) \\ &= 4 - 128 = -124 < 0 \end{aligned}$$

52. The roots of the equation $(x - a)(x - b) = b^2$ are

(1) Real and equal (2) Real and unequal

(3) Imaginary (4) equal [Ans : (2)]

Sol :

$$\begin{aligned} (x-a)(x-b) &= b^2 \\ x^2 - (a+b)x + ab - b^2 &= 0 \\ \text{Discriminant } B^2 - 4AC &= (a+b)^2 - 4(1)(ab - b^2) \\ &= a^2 + 2ab + b^2 - 4ab + 4b^2 \\ &= a^2 - 2ab + b^2 + 4b^2 \\ &= (a-b)^2 + 4b^2 > 0 \end{aligned}$$

$\therefore$ Roots are real and unequal.

53. The Discriminant of $\sqrt{x^2 + x + 1} = 2$ is

(1) -3 (2) 13

(3) 11 (4) 12 [Ans : (2)]

Sol :

$$\sqrt{x^2 + x + 1} = 2$$

Squaring $\Rightarrow$ $x^2 + x + 1 = 4$

$x^2 + x - 3 = 0$

Discriminant $b^2 - 4ac = (1)^2 - 4(1)(-3)$

$= 1 + 12 = 13.$

54. If a and b are the roots of the equation $x^2 - 6x + 6 = 0$, then the value of $a^2 + b^2$ is

(1) 36 (2) 24

(3) 12 (4) 6 [Ans : (2)]

Sol :

Sum of the roots $= a + b = 6$

Product of roots $= ab = 6$

$\therefore a^2 + b^2 = (a+b)^2 - 2ab = 36 - 12 = 24$

55. The roots of the equation $x^2 + kx + 12 = 0$ will differ by unity only when

(1) $k = \pm\sqrt{12}$ (2) $k = \pm\sqrt{48}$

(3) $k = \pm\sqrt{47}$ (4) $k = \pm\sqrt{49}$ [Ans : (4)]

Sol :

Roots are $\alpha, \alpha + 1$

Sum of the roots $= \alpha + \alpha + 1 = -k$

$$2\alpha + 1 = -k$$

$$\alpha = \frac{-k-1}{2}$$

Product of roots $= \alpha(\alpha + 1) = 12$

$$\alpha^2+\alpha = 12$$

$$\left(\frac{-k-1}{2}\right)^2+\left(\frac{-k-1}{2}\right) = 12$$

$$k^2+2k+1-2k-2 = 48$$

$$k^2-49 = 0$$

simplifying and factorizing

$$(k+7)(k-7) = 0$$

56. Ajay and Vijay Solved an equation. In solving it, Ajay made a mistake in the constant term only and got the roots as 8 and 2, while Vijay made a mistake in the coefficient of x only and obtained the roots as – 9 and – 1. The correct roots of the equation are

(1) 8, – 1 (2) – 9, 2
(3) – 8, – 2 (4) 9, 1 **[Ans : (4)]**

Sol :

Let the equation be $x^2+ax+b=0$

Ajay made a mistake in 'b' and roots are 8, 2,

Hence the equation is $x^2-10x+16=0$

Vijay made a mistake in the coefficient of 'x' and roots are – 9, – 1.

Hence the equation is $x^2+10x+16=0$

Vijay made a mistake in the coefficient of 'x' and roots are – 9, –1.

Hence the equation $x^2+10x+9=0$

$\therefore$ The real equation is $x^2-10x+9=(x-9)(x-1)=0$

$\therefore$ The roots are 9 and 1.

57. If the sum of the squares of two consecutive even numbers is 100, then the numbers are

(1) 6, 8 or – 8, –6
(2) – 6, 8 or – 8, –6
(3) – 6, – 8 or 6, – 8
(4) –6, – 8 or 7, 8 **[Ans : (1)]**

Sol :

Let the numbers be x and x + 2

Then, $x^2+(x+2)^2 = 100$

$$2x^2+4x-96 = 0$$

$$x^2+2x-48 = 0$$

$$(x+8)(x-6) = 0$$

$$x = 6 \text{ or } x = -8$$

$\therefore$ The numbers are 6, 8 and – 8, –6.

58. What is the smallest integral value of k such that $2x(kx-4)-x^2+6=0$ has no real roots?

(1) 3 (2) 1
(3) 2 (4) 4 **[Ans : (3)]**

Sol :

Equation is $2x(kx-4)-x^2+6 = 0$

$$(2k-1)x^2-8x+6 = 0$$

$$\text{Discriminant } \Delta = b^2-4ac$$

$$= 64-4(2k-1)(6)$$

$$= 88-48k$$

$$= 8(11-6k)$$

A quadratic equation has no real roots if Δ is negative. Δ is negative if $11-6k<0$ i.e., when $k>\frac{11}{6}$, the smallest integral value of k for which the equation has no real roots is 2.

59. If α and β are the roots of $ax^2+2bx+c=0$ then $\sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}}=$

(1) $\frac{2b}{ac}$ (2) $\frac{2b}{\sqrt{ac}}$
(3) $\frac{-2b}{\sqrt{ac}}$ (4) $\frac{-b}{\sqrt{ac}}$ **[Ans : (3)]**

Sol :

$$ax^2+2bx+c = 0$$

α and β are the roots

$$\text{Sum of roots} = \alpha+\beta = \frac{-2b}{a}$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

$$\therefore \sqrt{\frac{\alpha}{\beta}}+\sqrt{\frac{\beta}{\alpha}} = \frac{\alpha+\beta}{\sqrt{\alpha\beta}} = \frac{\frac{-2b}{a}}{\sqrt{\frac{c}{a}}} = \frac{-2b}{\sqrt{ac}}$$

60. If α, β are the roots of the equation $x^2+kx+12=0$ such that $\alpha-\beta=1$, then the value of k is.

(1) 0 (2) ±5
(3) ±1 (4) ±7 **[Ans : (4)]**

Sol :

α, β are the roots of $x^2+kx+12 = 0$

$$\therefore \alpha+\beta = -k \text{ and}$$

$$\alpha\beta = 12 \text{ and}$$

$$\text{given } \alpha-\beta = 1$$

$$\text{Now } (\alpha-\beta)^2 = (\alpha+\beta)^2-4\alpha\beta$$

$$(1)^2 = k^2-48$$

$$k^2 = 49; \quad k = \pm 7$$

61. If α and β are the roots of the equation $ax^2 + bx + c = 0$, identify the quadratic equation whose roots are $\alpha+\beta$ and $\alpha\beta$

(1) $a^2x^2 + a(b-c)x + bc = 0$
(2) $a^2x^2 + a(b-c)x - bc = 0$
(3) $ax^2 + (b+c)x + bc = 0$
(4) $ax^2 - (b+c)x - bc = 0$ [Ans : (2)]

Sol :

$$\alpha+\beta = \frac{-b}{a} \quad \text{----- 1}^{\text{st}} \text{ root}$$

$$\alpha\beta = \frac{c}{a} \quad \text{----- 2}^{\text{nd}} \text{ root}$$

$\therefore$ Required equation is $x^2 - (\text{sum})\,x + \text{product} = 0$

i.e., $x^2 - (\alpha+\beta+\alpha\beta)x + (\alpha+\beta)(\alpha\beta) = 0$

$$x^2 - \left(\frac{-b}{a} + \frac{c}{a}\right)x + \left(\frac{-b}{a}\right)\left(\frac{c}{a}\right) = 0$$

$$a^2x^2 + a(b-c)x - bc = 0$$

Matrices

62. The order of the matrix A is 3×5 and that of B is 2×3. The order of the matrix BA is

(1) 2×3 (2) 3×2
(3) 2×5 (4) 5×2 [Ans : (3)]

63. If $A \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$, then the values of k, a, b are respectively

(1) $-6, -12, -18$ (2) $-6, 4, 9$
(3) $-6, -4, -9$ (4) $-6, 12, 18$ [Ans : (3)]

Sol :

$$k\begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$-4k = 24 \Rightarrow k = -6$
$2k = 3a \Rightarrow a = -4$
$3k = 2b \Rightarrow b = -9$

64. If $m\begin{bmatrix} -3 & 4 \end{bmatrix} + n\begin{bmatrix} 4 & -3 \end{bmatrix} =$ then find $3m + 7n$

(1) 3 (2) 5
(3) 10 (4) 1 [Ans : (4)]

Sol :

$-3m + 4n = 10$
$4m - 3n = -11$

Solving them, we get $m = -2$ and $n = 1$

$\therefore 3m + 7n = -6 + 7 = 1$

65. If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and $A^2 = I$, then $x =$

(1) 0 (2) 1
(3) -1 (4) 2 [Ans : (1)]

Sol :

$$A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix}$$

Given $A^2 = I$

$$\therefore \begin{bmatrix} x^2+1 & x \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$x = 0$

66. If $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$ then $A + A^T =$

(1) $\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & -4 \\ 10 & 6 \end{bmatrix}$

(3) $\begin{bmatrix} 2 & 4 \\ -10 & 6 \end{bmatrix}$ (4) None of these [Ans : (1)]

Sol :

$$A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

67. If $U = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, then $UV + XY =$

(1) 20 (2) $[-20]$
(3) -20 (4) $[20]$ [Ans : (4)]

Sol :

$$UV = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= [6 - 6 + 4] = [4]$$

$$XY = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

$$= [0 + 4 + 12] = [16]$$

$$UV + XY = [4] + [16] = [20]$$

68. If A $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = kA$, then k =

(1) 4 (2) 5

(3) 6 (4) 7 [Ans : (3)]

Sol :

$$A^2 = A\,A$$

$$= \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+9 & -9-9 \\ -9-9 & 9+9 \end{bmatrix} = \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix}$$

$$kA = \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix}$$

$$\text{Given } A^2 = kA$$

$$\begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \begin{bmatrix} 3k & -3k \\ -3k & 3k \end{bmatrix}$$

$$3k = 18 \Rightarrow k = 6$$

69. If A + B = $\begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$ and A – B = $\begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix}$, then A =

(1) $\begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & 2 \\ 4 & 6 \end{bmatrix}$

(3) $\begin{bmatrix} 4 & 6 \\ 4 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$ [Ans : (1)]

Sol :

$$A + B = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix} \quad \text{... (1)}$$

$$A - B = \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix} \quad \text{... (2)}$$

$$(1) + (2) \Rightarrow 2A = \begin{bmatrix} 12 & 4 \\ 8 & 10 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 6 & 2 \\ 4 & 5 \end{bmatrix}$$

70. If A = $\begin{bmatrix} 5 & x \\ y & 6 \end{bmatrix}$, B = $\begin{bmatrix} -4 & y \\ -4 & -5 \end{bmatrix}$ and A + B = I, then the values of x and y respectively are

(1) – 4, 4 (2) – 4, – 4

(3) 4, 4 (4) 4, – 4 [Ans : (1)]

Sol :

$$A + B = I$$

$$\begin{bmatrix} 5 & x \\ y & 6 \end{bmatrix} + \begin{bmatrix} -4 & y \\ -4 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & x+y \\ y-4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x + y = 0, \quad y - 4 = 0$$

$$x + 4 = 0 \qquad y = 4$$

$$x = -4$$

71. Number of matrices obtained with 36 elements is

(1) 10 (2) 9

(3) 8 (4) 7 [Ans : (2)]

Sol :

With 36 elements, The Possible orders are 6 × 6, 9 × 4, 4 × 9, 12 × 3, 3 × 12, 2 × 18, 18 × 2, 36 × 1 and 1 × 36.

72. If A = $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $f(x) = x^2 - 5x + 4I$, then f(A) =

(1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (4) $\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$ [Ans : (4)]

Sol :

$$f(x) = x^2 - 5x + 4I$$

$$\therefore f(A) = A^2 - 5A + 4I$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^2 - 5\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

73. If order of A, B, C are 3 × 4, 5 × 4 and 5 × 8, then the order of (AB^TC) is

(1) 8 × 3 (2) 3 × 8

(3) 3 × 4 (4) 4 × 5 [Ans : (2)]

Sol :

$$\text{Order of A} = 3 \times 4, \text{Order of } B^T = 4 \times 5$$

$$\text{Order of } AB^T = 3 \times 5$$

$$\text{Order of C} = 5 \times 8$$

$$\therefore \text{Order of } AB^TC \text{ is } 3 \times 8$$

74. Given $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, then $A^3 - A^2 =$

(1) 2A (2) 2I
(3) A (4) I **[Ans : (1)]**

Sol :

$$A^2 = A\,A$$
$$= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$A^3 = A^2A$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix}$$
$$A^3 - A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= 2\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = 2A$$

75. If AB = A, BA = B then $A^2 + B^2 =$

(1) A + B (2) A − B
(3) AB (4) 0 **[Ans : (1)]**

Sol :

$$\begin{aligned} A^2 + B^2 &= A \times A + B \times B \\ &= (AB)\,(AB) + (BA)\,(BA) \\ &= A\,(BA)\,B + B\,(AB)\,A \\ &= (AB)\,B + (BA)\,A \quad [\because AB = A, BA = B] \\ &= AB + BA \\ &= A + B \end{aligned}$$

II. Very Short Answer Questions

1. Solve: 5x + 2y = 3, 3x + 2y = 5

Sol :

$$5x + 2y = 3 \qquad \dots (1)$$
$$3x + 2y = 5 \qquad \dots (2)$$

$(1) - (2) \Rightarrow \quad 2x = -2$

$$x = \frac{-2}{2} = -1$$

Substituting in (1)

$$5(-1) + 2y = 3$$
$$2y = 3 + 5$$
$$2y = \frac{8}{2} = 4$$

$\therefore$ Solution: $x = -1, y = 4.$

2. Find the GCD of $x^3 + 8y^3$ and $x + 2y$.

Sol :

$$x^3 + 8y^3 = x^3 + (2y)^3$$
$$= (x + 2y)\,)\,(x^2 - 2xy + 4y^2)$$

by using $a^3 + b^3 = (a + b)\,(a^2 - ab + b^2)$

$$x + 2y = x + 2y.$$
$$\therefore \text{GCD} = (x + 2y)$$

3. $m^2 - 5m - 14$ is an expression. Find out another similar expression such that their HCF is $(m - 7)$ and LCM is $m^3 - 10m^2 + 11m + 70$.

Sol :

We know $P(x) \times Q(x) = \text{GCD} \times \text{LCM}$

$$\therefore Q(x) = \frac{GCD \times LCM}{P(x)}$$
$$Q(x) = \frac{(m-7)(m^3 - 10m^2 + 11m + 70)}{m^2 - 5m - 14}$$
$$= \frac{(m-7)(m-5)(m^2 - 5m - 14)}{m^2 - 5m - 14}$$
$$= m^2 - 12m + 35$$

4. Find the GCD of $10(x^2 + x - 20)$, $15(x^2 - 3x - 4)$ and $20(x^2 + 2x + 1)$

Sol :

GCD of 10, 15 and 20 is 5

$$x^2 + x - 20 = (x + 5)\,(x - 4)$$
$$x^2 - 3x - 4 = (x - 4)\,(x + 1)$$
$$x^2 + 2x + 1 = (x + 1)\,(x + 1)$$
$$\therefore \text{GCD} = 5$$

5. Find the LCM of $a^2 + 3a + 2$, $a^2 + 5a + 6$ and $a^2 + 4a + 4$

Sol :

$$a^2 + 3a + 2 = (a + 2)\,(a + 1)$$
$$a^2 + 5a + 6 = (a + 2)\,(a + 3)$$
$$a^2 + 4a + 4 = (a + 2)^2$$
$$\text{LCM} = (a + 2)^2\,(a + 1)\,(a + 3)$$

6. Simplify: $\dfrac{3x^2 - 3x}{3x^3 - 6x^2 + 3x}$

Sol :

$$\frac{3x^2 - 3x}{3x^3 - 6x^2 + 3x} = \frac{3x\,(x-1)}{3x\,(x^2 - 2x + 1)}$$
$$= \frac{3x\,(x-1)}{3x\,(x-1)^2} = \frac{1}{x-1}$$

7. Simplify: $\dfrac{x^2+3x}{x^2-4x-21}$

Sol :

$$\frac{x^2+3x}{x^2-4x-21} = \frac{x(x+3)}{(x-7)(x+3)} = \frac{x}{x-7}$$

8. Simplify: $\dfrac{2x^2+26x+84}{2x^2+12x-14}$

Sol :

$$\frac{2x^2+26x+84}{2x^2+12x-14} = \frac{2(x^2+13x+42)}{2(x^2+6x-7)} = \frac{(x+6)(x+7)}{(x+7)(x-1)} = \frac{x+6}{x-1}$$

9. Simplify: $\dfrac{1}{4x^2}+\dfrac{5}{6xy^2}$

Sol :

$$\frac{1}{4x^2}+\frac{5}{6xy^2} = \frac{1(3y^2)+5(2x)}{12x^2y^2} = \frac{3y^2+10x}{12x^2y^2}$$

10. Simplify: $\dfrac{3x}{x^2+3x-10}-\dfrac{6}{x^2+3x-10}$

Sol :

$$\frac{3x}{x^2+3x-10}-\frac{6}{x^2+3x-10} = \frac{3x-6}{x^2+3x-10} = \frac{3(x-2)}{(x+5)(x-2)} = \frac{3}{x+5}$$

11. Simplify: $\dfrac{x+4}{2x}-\dfrac{x-1}{x^2}$

Sol :

$$\frac{x+4}{2x}-\frac{x-1}{x^2} = \frac{x(x+4)-2(x-1)}{2x^2} = \frac{x^2+4x-2x+2}{2x^2} = \frac{x^2+2x+2}{2x^2}$$

12. Find the Quadratic equation whose sum and product of the roots are (i) – 12, 32 (ii) 2a, a^2-b^2.

Sol :

(i) Sum = – 12,

Product = 32.

Quadratic equation:

x^2 – (Sum of Roots) x + Product of Roots = 0

$x^2 + 12x + 32 = 0$

(ii) Sum of the roots '2a'

Product of the roots '$a^2 - b^2$'

Quadratic equation:

x^2 – (Sum of Roots) x + Product of Roots = 0

$x^2 - 2a\,x + a^2 - b^2 = 0.$

13. Find the sum and product of the quadratic equation

(i) $x^2 + 2x - 360 = 0$

(ii) $\dfrac{a^2d^2}{b}x^2+2acdx+c^2b = 0.$

Sol :

(i) $x^2 + 2x - 360 = 0,$

$a = 1, b = 2, c = -360$

Sum of the Roots $= \dfrac{-b}{a} = \dfrac{-2}{1} = -2.$

Product of the Roots $= \dfrac{c}{a} = \dfrac{-360}{1} = -360.$

(ii) $\dfrac{a^2d^2}{b}x^2+2ac\,dx+c^2b = 0$

Sum of the roots $= \dfrac{-B}{A}$

$$= \frac{-2acd}{\left(\frac{a^2d^2}{b}\right)} = \frac{-2bc}{ad}$$

Product of the roots $= \dfrac{C}{A}$

$$= \frac{c^2b}{\left(\frac{a^2d^2}{b}\right)} = \frac{c^2b^2}{a^2d^2}$$

14. Solve: $25p^2 - 49 = 0$ by factorization method.

Sol :

$$25p^2 - 49 = 0$$
$$(5p)^2 - (7)^2 = 0 \quad [\because a^2 - b^2 = (a+b)(a-b)]$$
$$(5p+7)(5p-7) = 0$$
$$5p + 7 = 0, \quad 5p - 7 = 0$$
$$p = -7/5, \quad p = 7/5$$
$$P = -7/5, 7/5$$

15. Solve: $8x^2 - 22x - 21 = 0$ by factorization method.

Sol :

$$8x^2 - 22x - 21 = 0$$
$$8x^2 - 28x + 6x - 21 = 0$$
$$4x(2x-7) + 3(2x-7) = 0$$
$$(2x-7)(4x+3) = 0$$
$$2x - 7 = 0, \quad 4x + 3 = 0$$
$$x = 7/2, \quad x = -3/4$$
$$\text{Solution : } x = -3/4, 7/2.$$

16. Solve: $x^2 - (1+\sqrt{2})x + \sqrt{2} = 0$ by factorization method.

Sol :

$$x^2 - (1+\sqrt{2})x + \sqrt{2} = 0$$
$$x^2 - x - \sqrt{2}x + \sqrt{2} = 0$$
$$x(x-1) - \sqrt{2}(x-1) = 0$$
$$(x-1)(x-\sqrt{2}) = 0$$
$$x - 1 = 0 \text{ or } x - \sqrt{2} = 0$$
$$x = 1 \text{ or } x = \sqrt{2}$$
$$\text{Solution : } x = 1, \sqrt{2}$$

17. Solve: $21x^2 - 2x + \frac{1}{21} = 0$ by factorization method.

Sol :

$$21x^2 - 2x + \frac{1}{21} = 0$$
$$441x^2 - 42x + 1 = 0$$
$$441x^2 - 21x - 21x + 1 = 0$$
$$21x(21x-1) - 1(21x-1) = 0$$
$$(21x-1)(21x-1) = 0$$
$$x = \frac{1}{21} \text{ (twice)}$$
$$\text{Solution: } x = \frac{1}{21}, \frac{1}{21}$$

18. Solve the following Quadratic equation by the method of completing the square.

(i) $x^2 - 4x + 1 = 0$,

(ii) $x^2 + 3x - 5 = 0$,

(iii) $4x^2 + 4\sqrt{3}x + 3 = 0$,

(iv) $2x^2 + x - 4 = 0$

Sol :

(i)
$$x^2 - 4x + 1 = 0$$
$$x^2 - 4x = -1$$
$$x^2 - 4x + 4 = -1 + 4$$
$$(x-2)^2 = 3$$
$$x - 2 = \pm\sqrt{3}$$
$$x - 2 = +\sqrt{3} \text{ and } x - 2 = -\sqrt{3}$$
$$\text{Solution is } x = 2 \pm \sqrt{3}$$

(ii)
$$x^2 + 3x - 5 = 0$$
$$x^2 + 3x = 5$$
$$x^2 + 3x + \frac{9}{4} = 5 + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = \frac{29}{4}$$
$$x + \frac{3}{2} = \pm\frac{\sqrt{29}}{2}$$
$$\therefore x + \frac{3}{2} = \frac{\sqrt{29}}{2}, \quad x + \frac{3}{2} = \frac{-\sqrt{29}}{2}$$
$$\therefore \text{Solution is } x = \frac{3}{2} + \frac{\sqrt{29}}{2}, \quad x = -\frac{3}{2} - \frac{\sqrt{29}}{2}$$

(iii)
$$4x^2 + 4\sqrt{3}x + 3 = 0$$
$$4x^2 + 4\sqrt{3}x = -3$$
$$\text{Dividing by 4} \quad x^2 + \sqrt{3}x = \frac{-3}{4}$$
$$x^2 + \sqrt{3}x + \frac{3}{4} = \frac{-3}{4} + \frac{3}{4}$$
$$\left(x + \frac{\sqrt{3}}{2}\right)^2 = 0$$
$$x = -\frac{\sqrt{3}}{2} \text{ (twice)}$$

(iv)
$$2x^2 + x - 4 = 0$$
$$2x^2 + x = 4$$
$$\text{Dividing by 2} \quad x^2 + \frac{x}{2} = 2$$
$$x^2 + \frac{x}{2} + \frac{1}{16} = 2 + \frac{1}{16}$$

$$\left(x+\frac{1}{4}\right)^2 = \frac{33}{16}$$

$$x+\frac{1}{4} = \pm\frac{\sqrt{33}}{4}$$

$$\therefore \quad x+\frac{1}{4} = \frac{\sqrt{33}}{4} \text{ and}$$

$$x+\frac{1}{4} = \frac{-\sqrt{33}}{4}$$

$$x = \frac{\sqrt{33}-1}{4},\ x = \frac{\sqrt{33}+1}{4}$$

Hence the solution.

19. Solve the following Quadratic equation by using formula method.

(i) $9x^2 - 15x + 6 = 0$,

(ii) $4x^2 - 3 = 2x$,

(iii) $2t^2 - 5t + 3 = 0$,

(iv) $3x^2 = -7x - 2$

Sol :

(i) $9x^2 - 15x + 6 = 0$

Comparing this with $ax^2 + bx + c = 0$

$a = 9, b = -15, c = 6$

$$\therefore x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{15\pm\sqrt{225-216}}{18}$$

$$= \frac{15\pm\sqrt{9}}{18} = \frac{15\pm 3}{18}$$

$$= \frac{15+3}{18}, \frac{15-3}{18}$$

$$x = \frac{18}{18}, \frac{12}{18}$$

Solution: $x = 1, \frac{2}{3}$

(ii) $4x^2 - 3 = 2x$

$4x^2 - 2x - 3 = 0$

comparing with $ax^2 + bx + c = 0$, we get

$a = 4, b = -2, c = -3$

Now, $$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{2\pm\sqrt{4+48}}{8}$$

$$= \frac{2\pm\sqrt{52}}{8}$$

$$= \frac{2\pm\sqrt{4\times 13}}{8}$$

$$= \frac{2\pm 2\sqrt{13}}{8}$$

$$= \frac{1\pm\sqrt{13}}{4}$$

$$\therefore x = \frac{1+\sqrt{13}}{4}, \frac{1-\sqrt{13}}{4}$$

(iii) $2t^2 - 5t + 3 = 0$

comparing with $ax^2 + bx + c = 0$, we get

$a = 2, b = -5, c = 3$

Now, $$t = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{5\pm\sqrt{25-24}}{4} = \frac{5\pm 1}{4}$$

$$= \frac{5+1}{4}, \frac{5-1}{4} = \frac{6}{4}, \frac{4}{4}$$

$$\therefore t = \frac{3}{2}, 1$$

(iv) $3x^2 = -7x - 2$

$\Rightarrow 3x^2 + 7x + 2 = 0$

Comparing with $ax^2 + bx + c = 0$, we get

$a = 3, b = 7, c = 2$

Now, $$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-7\pm\sqrt{49-24}}{6}$$

$$= \frac{-7\pm\sqrt{25}}{6}$$

$$= \frac{-7\pm 5}{6}$$

$$= \frac{-7+5}{6}, \frac{-7-5}{6}$$

$$= -\frac{2}{6}, -\frac{12}{6}$$

$$= \frac{-1}{3}, -2$$

20. Determine the nature of the roots of
(i) $3x^2 - 5x + 2 = 0$,
(ii) $x^2 - 2\sqrt{2}x - 6 = 0$,
(iii) $2x^2 - 4x + 3 = 0$,
(iv) $x^2 - 4x + 4 = 0$

Sol :

(i) $3x^2 - 5x + 2 = 0$

Comparing this with $ax^2 + bx + c = 0$,

We get $a = 3, b = -5, c = 2$.

$$\begin{aligned} \text{Discriminant } \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(3)(2) \\ &= 25 - 24 \\ &= 1 > 0 \end{aligned}$$

$\therefore$ The roots are real and distinct.

(ii) $x^2 - 2\sqrt{2}x - 6 = 0$

Comparing this with $ax^2 + bx + c = 0$, we get

$a = 1, b = -2\sqrt{2}$,
$c = -6$

$$\begin{aligned} \text{Discriminant } \Delta &= b^2 - 4ac \\ &= \left(-2\sqrt{2}\right)^2 - 4(1)(-6) \\ &= 8 + 24 = 32 > 0. \end{aligned}$$

Hence the roots are real and distinct.

(iii) $2x^2 - 4x + 3 = 0$

Comparing this with $ax^2 + bx + c = 0$,

We get $a = 2, b = -4, c = 3$.

$$\begin{aligned} \text{Discriminant } \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(2)(3) \\ &= 16 - 24 \\ &= -8 < 0. \end{aligned}$$

Hence the roots are unreal.

(iv) $x^2 - 4x + 4 = 0$

Comparing with $ax^2 + bx + c = 0$,

We get $a = 1, b = -4, c = 4$.

$$\begin{aligned} \text{Discriminant } \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(1)(4) \\ &= 16 - 16 = 0 \end{aligned}$$

Hence, the roots are real and equal.

21. If one root of the equation $5x^2 + 13x + k = 0$ is reciprocal of the other, then what is the value of k?

Sol :

Let the roots be α and $\frac{1}{\alpha}$

Equation $5x^2 + 13x + k = 0$

Comparing with $ax^2 + bx + c = 0$,

we get $a = 5, b = 13, c = k$

$$\alpha + \beta = \frac{-b}{a}$$

$$\Rightarrow \quad \alpha + \frac{1}{\alpha} = \frac{-13}{5}$$

$$\alpha\beta = \frac{c}{a}$$

$$\Rightarrow \quad \alpha\left(\frac{1}{\alpha}\right) = \frac{k}{5}$$

$$1 = \frac{k}{5} \Rightarrow k = 5.$$

22. Construct a 2×2 matrix where $a_{ij} = -2i + 3j$.

Sol :

$$\begin{aligned} a_{ij} &= -2i + 3j \\ a_{11} &= -2(1) + 3(1) = 1 \\ a_{12} &= -2(1) + 3(2) = 4 \\ a_{21} &= -2(2) + 3(1) = -1 \\ a_{22} &= -2(2) + 3(2) = 2 \end{aligned}$$

Hence the matrix of order 2×2 is $\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix}$

23. Find the transpose of $A = \begin{bmatrix} 0 & 0 \\ 3 & 8 \\ 8 & 7 \end{bmatrix}$

Sol :

$$A = \begin{bmatrix} 0 & 0 \\ 3 & 8 \\ 8 & 7 \end{bmatrix}$$

$$\therefore A^T = \begin{bmatrix} 0 & 3 & 8 \\ 0 & 8 & 7 \end{bmatrix}$$

24. If $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$. Find $(A^T)^T$.

Sol :

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 5/2 & 4 \\ 1/2 & 2 & 7/2 \\ 0 & 3/2 & 3 \\ 1/2 & 1 & 5/2 \end{bmatrix}$$

$$(A^T)^T = A$$

$$= \begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ 5/2 & 2 & 3/2 & 1 \\ 4 & 7/2 & 3 & 5/2 \end{bmatrix}$$

25. If $\begin{bmatrix} x-y & 2y \\ 2y+z & x+y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 5 \end{bmatrix}$, then find the value of x + y + z.

Sol :

Since the matrices are equal,

Let us equate the corresponding elements.

Now, $x - y = 1$, $2y = 4$

$2y + z = 9$, $x + y = 5$

$2y = 4$ $\quad x + y = 5$ $\quad 2y + z = 9$

$y = 2$ $\quad x + 2 = 5$ $\quad 2(2) + z = 9$

$x = 3$ $\quad z = 9 - 4 = 5$

$\therefore x + y + z = 3 + 2 + 5 = 10.$

26. If A = $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$, B = $\begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$ and C = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find 5A – 3B + 2C.

Sol :

$5A - 3B + 2C$

$$= 5\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix} - 3\begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -10 \\ 15 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 12 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -20 \\ 7 & -9 \end{bmatrix}$$

27. If A = $\begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$, verify that A + (B + C) = (A + B) + C.

Sol :

$$B + C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ -3 & 3 \end{bmatrix}$$

$$\text{LHS} : A + (B + C) = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 \\ 1 & 5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\text{RHS} : (A + B) + C = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 \\ 1 & 5 \end{bmatrix}$$

$\therefore$ LHS = RHS

$A + (B + C) = (A + B) + C$

Hence verified.

28. Simplify:

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

Sol :

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ -\cos\theta\sin\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

29. Find a matrix 'X' such that 2A + B + X = 0, where A = $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$

Sol :

Given $2A + B + X = 0$

$$2\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix} + X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 7 & 13 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$$

30. Given A = $\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$. Find AB and BA.

Sol :

$$AB = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2-8 & 6+20 \\ 3-4 & 9+10 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & 26 \\ -1 & 19 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2+9 & 4+6 \\ -4+15 & -8+10 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & 10 \\ 11 & 2 \end{bmatrix}$$

31. If A = $\begin{bmatrix} 3 & 2 \\ 12 & 8 \end{bmatrix}$ and B = $\begin{bmatrix} 8 & 4 \\ -12 & -6 \end{bmatrix}$. Show that AB = 0.

Sol :

$$AB = \begin{bmatrix} 3 & 2 \\ 12 & 8 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ -12 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} 24-24 & 12-12 \\ 96-96 & 48-48 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence proved.

32. If A = $\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$, find $-A^2 + 6A$.

Sol :

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$$
$$A^2 = A.A$$
$$= \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 4+6 & -4-8 \\ -6-12 & 6+16 \end{bmatrix}$$
$$= \begin{bmatrix} 10 & -12 \\ -18 & 22 \end{bmatrix}$$
$$\therefore \quad -A^2 + 6A = \begin{bmatrix} -10 & 12 \\ 18 & -22 \end{bmatrix} + 6\begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} -10 & 12 \\ 18 & -22 \end{bmatrix} + \begin{bmatrix} 12 & -12 \\ -18 & 24 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

33. Solve for 'x' if $\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$.

Sol :

$$\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} x-2 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Equating the corresponding elements

$$x - 2 = 0$$
$$x = 2$$

III. Short Answer Questions

1. If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes 1/2 if we only add 1 to the denominator. What is the fraction?

Sol :

Let the fraction be $\frac{x}{y}$

$$\text{Given, } \frac{x+1}{y-1} = 1$$
$$x - y + 2 = 0$$
$$x - y = -2 \quad \dots (1)$$
$$\text{and } \frac{x}{y+1} = \frac{1}{2}$$
$$\Rightarrow \quad 2x - y - 1 = 0 \quad \dots (2)$$
$$\Rightarrow \quad 2x - y = 1$$
$$x - y = -2$$
$$(1) - (2) \Rightarrow \quad -x = -3$$
$$x = 3$$

Substituting in (1)

$$3 - y = -2 \quad \Rightarrow y = 5$$

Hence, the required fraction is 3/5.

2. Five years ago, Nuri was thrice as old as Sonu. Ten years later, Nuri will be twice as old as Sonu. How old are Nuri and Sonu?

Sol :

Let the present age of Nuri be 'x' years and that of Sonu be 'y' years.

Given, $x - 5 = 3(y - 5)$

$\Rightarrow \quad x - 3y = -10 \quad \ldots (1)$

and $x + 10 = 2(y + 10)$

$\Rightarrow \quad x - 2y = 10 \quad \ldots (2)$

Solving (1) and (2)

$x - 3y = -10$

$x - 2y = 10$

$(1) - (2) \Rightarrow \quad -y = -20$

$y = 20$

Substituting in (2)

$x - 2(20) = 10$

$\Rightarrow x = 10 + 40 = 50$

$\therefore$ Present age of Nuri is 50 years

Present age of Sonu is 20 years.

3. Find the LCM of the polynomial $f(x) = 18x^4 - 36x^3 + 18x^2$ and $g(x) = 45x^6 - 45x^3$.

Sol :

$$\begin{aligned} f(x) &= 18x^4 - 36x^3 + 18x^2 \\ &= 18x^2(x^2 - 2x + 1) \\ &= 18x^2(x-1)^2 \\ g(x) &= 45x^6 - 45x^3 = 45x^3(x^3 - 1) \\ &= 45x^3(x-1)(x^2 + x + 1) \\ \therefore \text{LCM} &= 90x^3(x-1)^2(x^2 + x + 1) \end{aligned}$$

4. Find the GCD of the following polynomials using division algorithm

Sol :

$$\begin{aligned} f(x) &= x^3 - 9x^2 + 23x - 15, \\ g(x) &= 4x^2 - 16x + 12 \\ &= 4(x^2 - 4x + 3) \end{aligned}$$

Now dividing $x^3 - 9x^2 + 23x - 15$ by $x^2 - 4x + 3$

$$\begin{array}{r|l} & x - 5 \\ \hline x^2 - 4x + 3 & x^3 - 9x^2 + 23x - 15 \\ & x^3 - 4x^2 + 3x \quad (-) \\ \hline & -5x^2 + 20x - 15 \\ & -5x^2 + 20x - 15 \quad (-) \\ \hline & 0 \end{array}$$

$\therefore$ GCD $= x^2 - 4x + 3$

5. Find the excluded values, if any of the expression $\dfrac{6x^3 + 57x^2 + 72x}{10x^3 + 85x^2 + 40x}$

Sol :

$$\begin{aligned} \frac{6x^3 + 57x^2 + 72x}{10x^3 + 85x^2 + 40x} &= \frac{3x(2x^2 + 19x + 24)}{5x(2x^2 + 17x + 8)} \\ &= \frac{3(x+8)(2x+3)}{5(x+8)(2x+1)} \\ &= \frac{3(2x+3)}{5(2x+1)} \end{aligned}$$

when $x = -1/2$, $2x + 1 = 0$, then the fraction becomes undefined.

$\therefore$ The excluded value is $-1/2$

6. Simplify: $\dfrac{x^3 + 27}{x^2 + 12x + 27} \times \dfrac{x^2 + 3x}{x^2 - 4x - 21}$

Sol :

$$\begin{aligned} \frac{x^3 + 27}{x^2 + 12x + 27} \times \frac{x^2 + 3x}{x^2 - 4x - 21} &= \frac{x^3 + 3^3}{(x+3)(x+9)} \times \frac{x(x+3)}{(x-7)(x+3)} \\ &= \frac{(x+3)(x^2 - 3x + 9)}{(x+3)(x+9)} \times \frac{x(x+3)}{(x-7)(x+3)} \\ &= \frac{x(x^2 - 3x + 9)}{(x+9)(x-7)} = \frac{x(x^2 - 3x + 9)}{x^2 + 2x - 63} \end{aligned}$$

7. Simplify: $\dfrac{x^2 - 25}{5x + x^2} \div \dfrac{x^2 - 10x + 25}{x^2 + 8x + 15}$

Sol :

$$\begin{aligned} \frac{x^2 - 25}{5x + x^2} \div \frac{x^2 - 10x + 25}{x^2 + 8x + 15} &= \frac{x^2 - 5^2}{x(5+x)} \times \frac{x^2 + 8x + 15}{x^2 - 10x + 25} \\ &= \frac{(x+5)(x-5)}{x(5+x)} \times \frac{(x+3)(x+5)}{(x-5)^2} \\ &= \frac{(x+3)(x+5)}{x(x-5)} \end{aligned}$$

8. Find the Square Root of $x^4 - 4x^3 + 10x^2 - 12x + 9$ by division method.

Sol :

$$
\begin{array}{r|l}
 & x^2 - 2x + 3 \\
\hline
x^2 & x^4 - 4x^3 + 10x^2 - 12x + 9 \\
 & x^4 \\
 & (-) \\
\hline
2x^2 - 2x & -4x^3 + 10x^2 \\
 & -4x^3 + 4x^2 \\
 & (+) \quad (-) \\
\hline
2x^2 - 4x + 3 & \quad 6x^2 - 12x + 9 \\
 & \quad 6x^2 - 12x + 9 \\
 & \quad (-) \quad (+) \quad (-) \\
\hline
 & \qquad 0 \\
\hline
\end{array}
$$

$\therefore \sqrt{x^4 - 4x^3 + 10x^2 - 12x + 9} = |x^2 - 2x + 3|$

9. Find the square root of $x^4 - 2x^3 + 3x^2 - 2x + 1$ by division method.

$$
\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
x^2 & x^4 - 2x^3 + 3x^2 - 2x + 1 \\
 & x^4 \\
 & (-) \\
\hline
2x^2 - x & -2x^3 + 3x^2 \\
 & -2x^3 + x^2 \\
 & (+) \quad (-) \\
\hline
2x^2 - 2x + 1 & \quad 2x^2 - 2x + 1 \\
 & \quad 2x^2 - 2x + 1 \\
 & \quad (-) \quad (+) \quad (-) \\
\hline
 & \qquad 0 \\
\end{array}
$$

$\therefore \sqrt{x^4 - 2x^3 + 3x^2 - 2x + 1} = |x^2 - x + 1|$

10. The sum of the reciprocals of Raman's ages (in years) 3 years ago and 5 years hence is $\frac{1}{3}$. Find his present age.

Sol :

Let the present age of Raman be 'x' years.

$\therefore$ 3 years ago, his age was $(x - 3)$ and 5 years hence, age will be $(x + 5)$

By the Given data,

$$\frac{1}{x-3} + \frac{1}{x+5} = \frac{1}{3}$$

$$\frac{x+5+x-3}{(x-3)(x+5)} = \frac{1}{3}$$

$$3(2x + 2) = (x - 3)(x + 5)$$

$$6x + 6 = x^2 + 2x - 15$$

$$x^2 - 4x - 21 = 0$$

$$(x - 7)(x + 3) = 0$$

$x - 7 = 0$ and

$x + 3 = 0,$

$x = -3$ is not possible.

$\therefore$ Present age of Raman is 7 years.

11. The sum of two natural numbers is 8. Determine the numbers if the sum of their reciprocals is $\frac{8}{15}$.

Sol :

Let the first number be 'x'

$\therefore$ Other number is $8 - x$ [$\because$ Sum is 8]

Given, $\frac{1}{x} + \frac{1}{8-x} = \frac{8}{15}$

$$\frac{8-x+x}{x(8-x)} = \frac{8}{15}$$

$$\frac{8}{x(8-x)} = \frac{8}{15}$$

$$x^2 - 8x + 15 = 0$$

$$(x - 5)(x - 3) = 0$$

$$x = 3, 5$$

$\therefore$ The two natural numbers are 3, 5

12. For what value of k, the roots of the Quadratic equation $(k + 1)x^2 - 2(k - 1)x + 1 = 0$ are real and equal?

Sol :

$$(k + 1)x^2 - 2(k - 1)x + 1 = 0$$

Comparing this with $ax^2 + bx + c = 0$

We get $a = k + 1, b = -2(k - 1), c = 1$

For real and equal roots $\Delta = 0$

i.e., $b^2 - 4ac = 0$

$$[-2(k - 1)]^2 - 4(k + 1)(1) = 0$$

$$4(k - 1)^2 - 4(k + 1) = 0$$

$$(k - 1)^2 - (k + 1) = 0$$

$$k^2 - 2k + 1 - k - 1 = 0$$

$$k^2 - 3k = 0$$

$$k(k - 3) = 0$$

$k = 0; \quad k - 3 = 0$

$k = 0$ (or) $k = 3$

13. Show that the equation $(x-a)(x-b) = h^2$ has real roots.

Sol :

The given equation is

$$(x-a)(x-b) = h^2$$
$$x^2-(a+b)x+ab-h^2 = 0$$
$$\text{Discriminant } \Delta = B^2-4AC$$
$$= [-(a+b)]^2-4(1)(ab-h^2)$$
$$= (a+b)^2-4ab+4h^2$$
$$= (a-b)^2+4h^2 > 0$$

Hence the roots are real.

14. If the sum of the roots of $ax^2+bx+c = 0$ is equal to the sum of the squares of the roots. Find the condition.

Sol :

Let 'α' and 'β' are the roots of $ax^2+bx+c = 0$

then $\alpha+\beta = -b/a$ and $\alpha\beta = c/a$

Given that $\alpha+\beta = \alpha^2+\beta^2$

$$\alpha+\beta = (\alpha+\beta)^2-2\alpha\beta$$
$$\frac{-b}{a} = \left(\frac{-b}{a}\right)^2-2\left(\frac{c}{a}\right)$$
$$\frac{-b}{a} = \frac{-b^2}{a^2}-\frac{2c}{a}$$
$$\frac{-b}{a} = \frac{b^2-2ac}{a^2}$$
$$-ab = b^2-2ac$$

$2ac = b^2+ab$ is the required condition.

15. Given that α,β are the roots of the equation $2x^2+3x+7 = 0$, then find

(i) $\alpha^2+\beta^2$, (ii) $\frac{1}{\alpha}+\frac{1}{\beta}$, (iii) $\alpha^3+\beta^3$

Sol :

$$2x^2+3x+7 = 0$$

Comparing with $ax^2+bx+c = 0$,

we get $a = 2, b = 3, c = 7$

$$\alpha+\beta = \frac{-b}{a} = \frac{-3}{2},$$
$$\alpha\beta = \frac{c}{a} = \frac{7}{2}$$

(i) $$\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta$$
$$= \left(\frac{-3}{2}\right)^2-2\left(\frac{7}{2}\right) = \frac{-19}{4}$$

(ii) $$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$$
$$= \frac{-3}{2}\times\frac{2}{7} = \frac{-3}{7}$$

(iii) $$\alpha^3+\beta^3 = (\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]$$
$$= \left(\frac{-3}{2}\right)\left[\left(\frac{-3}{2}\right)^2-3\left(\frac{7}{2}\right)\right]$$
$$= -\frac{3}{2}\left[\frac{9}{4}-\frac{21}{2}\right]$$
$$= \frac{-3}{2}\left[\frac{9-42}{4}\right]$$
$$= \frac{-3}{2}\left(\frac{-33}{4}\right) = \frac{99}{8}$$

16. α,β are the roots of the equation $x^2-3ax+a^2 = 0$ such that $\alpha^2+\beta^2 = 1.75$. Find the value of 'a'.

Sol :

Equation is $x^2-3ax+a^2 = 0$

Comparing with $ax^2+bx+c = 0$,

we get $a = 1$, $b = -3a$, $c = a^2$

$$\alpha+\beta = \frac{-b}{a}$$
$$\Rightarrow \quad = \frac{-(-3a)}{1} = 3a$$
$$\alpha\beta = \frac{c}{a}$$
$$\Rightarrow \quad \frac{a^2}{1} = a^2$$

Given $\alpha^2+\beta^2 = 1.75$

$$(\alpha+\beta)^2-2\alpha\beta = 1.75$$
$$(3a)^2-2a^2 = 1.75$$
$$9a^2-2a^2 = 1.75$$
$$7a^2 = 1.75$$
$$a^2 = \frac{1.75}{7} = 0.25$$
$$a = \pm 0.5$$

17. If α, β are the roots of the equation $ax^2 + bx + b = 0$, then find the value of $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}}$.

Sol :

Given equation : $ax^2 + bx + b = 0$

$$\alpha + \beta = \frac{-b}{a},$$

$$\alpha\beta = \frac{b}{a}$$

Now, $$\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{b}{a}} = \frac{\alpha+\beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{b}{a}}$$

$$= \frac{\left(\frac{-b}{a}\right)}{\sqrt{\frac{b}{a}}} + \sqrt{\frac{b}{a}}$$

$$= -\sqrt{\frac{b}{a}} + \sqrt{\frac{b}{a}} = 0$$

18. In the matrix $A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 \\ 0 & 5 & y \end{bmatrix}$, Find

(i) the order of the matrix 'A'

(ii) the number of elements in 'A'

(iii) a_{23}, a_{31}, a_{12}

Sol :

$$A = \begin{bmatrix} a & 1 & x \\ 2 & \sqrt{3} & x^2 \\ 0 & 5 & y \end{bmatrix}$$

(i) Order of the matrix A $= 3 \times 3$

(ii) Number of elements $= 3 \times 3 = 9$

(iii) $a_{23} = x^2, a_{31} = 0, a_{12} = 1$

19. Find the values of a, b, c and d from the following equation $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Sol :

Equating the corresponding elements.

$$a - b = -1 \quad \dots(1)$$

$$2a + c = 5 \quad \dots(2)$$

$$2a - b = 0 \quad \dots(3)$$

$$3c + d = 13 \quad \dots(4)$$

Solving (1) and (3)

$$a - b = -1$$

$$2a - b = 0$$

(1) – (2) $\Rightarrow$ $-a = -1$

$a = 1$

Substituting in (1) $1 - b = -1$

$b = 2$

Substituting a = 1 in (2) $\Rightarrow 2(1) + c = 5$

$c = 3$

Substituting in (4) $\Rightarrow 3(3) + d = 13$

$d = 4$

$\therefore a = 1, b = 2, c = 3, d = 4$

20. Find A and B if $2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix}$

Sol :

$$2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix} \quad \dots(1)$$

$$A + 2B = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix} \quad \dots(2)$$

$$(2) \times (2) \Rightarrow \quad 2A + 4B = \begin{bmatrix} 10 & 0 & 6 \\ 2 & 12 & 4 \end{bmatrix} \quad \dots(3)$$

$$2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix} \quad \dots(1)$$

$$(3) - (1) \Rightarrow \quad B = \begin{bmatrix} 8 & 1 & 2 \\ -1 & 10 & -1 \end{bmatrix}$$

Substituting in (2)

$$A + 2\begin{bmatrix} 8 & 1 & 2 \\ -1 & 10 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 16 & 2 & 4 \\ -2 & 20 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & -2 & -1 \\ 3 & -14 & 4 \end{bmatrix}$$

21. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}, kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$, find the values of k, a and b.

Sol :

$$A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$$

$$kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$$

$$k\begin{bmatrix}0 & 2\\3 & -4\end{bmatrix} = \begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$$

$$\begin{bmatrix}0 & 2k\\3k & -4k\end{bmatrix} = \begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$$

Equating the corresponding elements.

$$2k = 3a, 3k = 2b, -4k = 24,$$

$$k = \frac{24}{-4} = -6.$$

$$\therefore \quad a = \frac{2k}{3} = \frac{2(-6)}{3} = -4$$

$$b = \frac{3k}{2} = \frac{3(-6)}{2} = -9$$

$$\therefore \quad k = -6, \quad a = -4, \quad b = -9$$

22. If $2\begin{bmatrix}x & 5\\7 & y-3\end{bmatrix}+\begin{bmatrix}3 & 4\\1 & 2\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$ find the values of x and y.

Sol :

$$2\begin{bmatrix}x & 5\\7 & y-3\end{bmatrix}+\begin{bmatrix}3 & 4\\1 & 2\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$$

$$\begin{bmatrix}2x+3 & 10+4\\14+1 & 2y-6+2\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$$

$$\begin{bmatrix}2x+3 & 14\\15 & 2y-4\end{bmatrix} = \begin{bmatrix}7 & 14\\15 & 14\end{bmatrix}$$

Equating the corresponding elements

$$2x+3 = 7 \qquad 2y-4 = 14$$

$$2x = 4 \qquad 2y = 18$$

$$x = \frac{4}{2} = 2 \qquad y = 9$$

$$\therefore x = 2 \qquad y = 9$$

23. If $\begin{bmatrix}4 & 1 & 2\\0 & 5 & 3\end{bmatrix}\begin{bmatrix}3 & 4 & 5\\-1 & 0 & -2\\3 & 4 & 7\end{bmatrix} = \begin{bmatrix}8x+3y & 6z & 32\\4 & 12 & 26x-5y\end{bmatrix}$, find the values of x, y and z.

Sol :

Now, LHS

$$= \begin{bmatrix}4 & 1 & 2\\0 & 5 & 3\end{bmatrix}\begin{bmatrix}3 & 4 & 5\\-1 & 0 & -2\\3 & 4 & 7\end{bmatrix}$$

$$= \begin{bmatrix}12-1+6 & 16+0+8 & 20-2+14\\0-5+9 & 0+0+12 & 0-10+21\end{bmatrix}$$

$$= \begin{bmatrix}17 & 24 & 32\\4 & 12 & 11\end{bmatrix} = \begin{bmatrix}8x+3y & 6z & 32\\4 & 12 & 26x-5y\end{bmatrix}$$

Equating corresponding elements:

$$8x+3y = 17 \qquad \ldots (1)$$

$$6z = 24 \Rightarrow z = 4 \qquad \ldots (2)$$

$$26x-5y = 11 \qquad \ldots (3)$$

Solving (1) and (3)

We get $x = 1, y = 3$

$$\therefore x = 1, y = 3, z = 4$$

24. If $A = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$, find (A – 2I) (A – 3I)

Sol :

$$A-2I = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}-2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}-\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}2 & 2\\-1 & -1\end{bmatrix}$$

$$A-3I = \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}-3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}-\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2\\-1 & -2\end{bmatrix}$$

Now, (A – 2I) (A – 3I)

$$= \begin{bmatrix}2 & 2\\-1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\-1 & -2\end{bmatrix}$$

$$= \begin{bmatrix}2-2 & 4-4\\-1+1 & -2+2\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0$$

25. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Prove that $A^3 - 4A^2 + A = 0$

Sol :

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14+12 & 21+24 \\ 8+7 & 12+14 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix}$$

Now, $A^3 - 4A^2 + A$

$$= \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix} - 4\begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 45 \\ 15 & 26 \end{bmatrix} - \begin{bmatrix} 28 & 48 \\ 16 & 28 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence proved.

IV. Long Answer Questions:

1. Solve $4x - 2y + 3z = 1$, $x + 3y - 4z = -7$, $3x + y + 2z = 5$

Sol :

$$4x - 2y + 3z = 1 \quad \dots (1)$$
$$x + 3y - 4z = -7 \quad \dots (2)$$
$$3x + y + 2z = 5 \quad \dots (3)$$

Consider (2) and (3)

$(3) \times (3) \Rightarrow$

$$9x + 3y + 6z = 15 \quad \dots (4)$$
$$x + 3y - 4z = -7 \quad \dots (2)$$
$$(-)$$

$(4) - (2) \Rightarrow \quad 8x + 10z = 22 \quad \dots (5)$

Now, consider (1) and (3)

$(3) \times 2 \Rightarrow \quad 6x + 2y + 4z = 10 \quad \dots (6)$

$$4x - 2y + 3z = 1 \quad \dots (1)$$

$(6) + (1) \Rightarrow \quad 10x + 7z = 11 \quad \dots (7)$

Consider (5) and (7)

$(5) \times 7 \Rightarrow \quad 56x + 70z = 154 \quad \dots (8)$

$(7) \times 10 \Rightarrow \quad 100x + 70z = 110 \quad \dots (9)$

$$(-)$$

$(8) - (9) \Rightarrow \quad -44x = 44$

$$x = \frac{44}{44} = -1$$

Substituting, $x = -1$ is in (7)

$$10(-1) + 7z = 11$$
$$-10 + 7z = 11$$
$$7z = 11 + 10$$
$$z = \frac{21}{7} = 3$$

Substituting, $x = -1, z = 3$ is in (3)

$$3(-1) + y + 2(3) = 5$$
$$-3 + y + 6 = 5$$
$$y = 5 - 3 = 2$$

$\therefore$ solution $x = -1, y = 2, z = 3$

2. Solve $x = 3z - 5$, $2x + 2z = y + 16$, $7x - 5z = 3y + 19$

Sol :

Standard form of the given equations

$$x - 3z = -5 \quad \dots (1)$$
$$2x - y + 2z = 16 \quad \dots (2)$$
$$7x - 3y - 5z = 19 \quad \dots (3)$$

Consider (2) and (3)

$(2) \times 3 \Rightarrow \quad 6x - 3y + 6z = 48 \quad \dots (4)$

$$7x - 3y - 5z = 19 \quad \dots (3)$$
$$(-)$$

$(4) - 3 \Rightarrow \quad -x + 11z = 29 \quad \dots (5)$

Consider (1) and (5)

$$x - 3z = -5 \quad \dots (1)$$
$$-x + 11z = 29 \quad \dots (5)$$

$(1) + (5) \Rightarrow \quad 8z = 24$

$$z = \frac{24}{8} = 3$$

Substituting, $z = 3$ in (1)

$$x = 3(3) - 5$$
$$= 9 - 5 = 4$$

Substituting, $x = 4, z = 3$ in (2)

$$2(4) - y + 2(3) = 16$$
$$8 + 6 - 16 = y$$
$$\Rightarrow y = 14 - 16 = -2$$

$\therefore$ solution is $x = 4, y = -2, z = 3$

3. Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24 m, find the sides of the two squares.

Sol :

Let the sides of two squares be 'x' and 'y' respectively.

Sum of areas $x^2 + y^2 = 468$ ... (1)

Difference of perimeters $4x - 4y = 24$ [$\because x > y$]

$$x - y = 6 \quad ... (2)$$
$$y = x - 6$$

Substituting in (1)

$$x^2 + (x-6)^2 = 468$$
$$x^2 + x^2 - 12x + 36 - 468 = 0$$
$$2x^2 - 12x - 432 = 0$$
$$x^2 - 6x - 216 = 0$$
$$(x-18)(x+12) = 0$$
$$x = 18$$

When $x = 18, y = 18 - 6 = 12$

$\therefore$ The sides of two squares are 18 m and 12 m respectively.

4. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

Sol :

Let the larger tap fill the tank in 'x' hours.

$\therefore$ Smaller tap fill the tank in (x + 10) hours (given)

Portion of tank filled by larger tap in 1 hour $= \frac{1}{x}$

Portion of tank filled by smaller tap in 1 hour $= \frac{1}{x+10}$

Given, Both the taps can fill the tank in $9\frac{3}{8} = \frac{75}{8}$ hours.

[Time and work done are reciprocals of each other]

$$\frac{1}{x} + \frac{1}{x+10} = \frac{8}{75}$$
$$\frac{x+10+x}{x(x+10)} = \frac{8}{75}$$
$$75(2x+10) = 8(x^2+10x)$$
$$8x^2 - 70x - 750 = 0$$
$$4x^2 - 35x - 375 = 0$$
$$4x^2 - 60x + 25x - 375 = 0$$
$$4x(x-15) + 25(x-15) = 0$$
$$(4x+25)(x-15) = 0$$
$$x = 15, \quad x = \frac{-25}{7} \text{ is not possible.}$$

$\therefore$ Time required to fill the tank by the 1st tap is 15 hrs and by the 2nd tap is 25 hours.

5. A plane left 30 minutes later than the scheduled time and in order to reach its destination 1500 km away in time it has to increase its speed by 250 km/hr from its usual speed. Find its usual speed.

Sol :

Let the usual speed of the plane be 'x' km/hr and

usual time to cover 1500 km $= \frac{1500}{x}$ hrs

Given that speed increased by 250 km/hr.

Time taken to cover the distance with new speed $= \frac{1500}{x+150}$

$$\therefore \frac{1500}{x+250} - \frac{1500}{x} = \frac{1}{2}$$

On Simplifying we get, $x^2 + 250x - 750000 = 0$

$$x(x+1000) - 750(x+1000) = 0$$
$$(x-750)(x+1000) = 0$$

$x = 750, x = -1000$ is not possible.

$\therefore$ The usual speed of the plane is 750 km/hr.

6. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, verify that $(A+B)^2 \neq A^2 + 2AB + B^2$

Sol :

$$A + B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$$

$$(A+B)^2 = (A+B)(A+B)$$

$$= \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+0 & 0+0 \\ 9+9 & 0+9 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 18 & 9 \end{bmatrix} \quad \dots (1)$$

Now $A^2 = A.A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1-2 & -1-3 \\ 2+6 & -2+9 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1+0 \\ 4+3 & 2+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 7 & 2 \end{bmatrix}$$

$$B^2 = B.B$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 2+0 \\ 2+0 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

Now $A^2 + 2AB + B^2$

$$= \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 14 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 24 & 12 \end{bmatrix} \quad \dots (2)$$

from (1), (2), $(A + B)^2 \neq A^2 + 2AB + B^2$.

7. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then prove that $A^2 - 8A + 7I = 0$.

Sol :

$$A^2 = A.A$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0+0 \\ -1-7 & 0+49 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

$$8A = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix}$$

$$7I = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$\therefore A^2 - 8A + 7I = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7+7 & 0 \\ 0+0 & -7+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Hence proved.

8. Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, show that (AB) C = A (BC).

Sol :

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-1 & 3+2-4 \\ 2+0+3 & 6+0+12 \\ 3+0+2 & 9-2+8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 5 & 18 \\ 5 & 15 \end{bmatrix}$$

LHS = (AB) C

$$= \begin{bmatrix} 0 & 1 \\ 5 & 18 \\ 5 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 0+0 & 0-2 & 0+1 \\ 5+36 & 10+0 & 15-36 & -20+18 \\ 5+30 & 10+0 & 15-30 & -20+15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 & 1 \\ 41 & 10 & -21 & -2 \\ 35 & 10 & -15 & -5 \end{bmatrix}$$

$$\text{Now, } BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ 1+8 & 2+0 & 3-8 & -4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 9 & 2 & -5 & 0 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 9 & 2 & -5 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7+4-9 & 2+0-2 & -3-4+5 & -1+2+0 \\ 14+0+27 & 4+0+6 & -6+0-15 & -2+0+0 \\ 21-4+18 & 6+0+4 & -9+4-10 & -3-2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 & 1 \\ 41 & 10 & -21 & -2 \\ 35 & 10 & -15 & -5 \end{bmatrix}$$

LHS = RHS

$\therefore (AB)C = A(BC)$

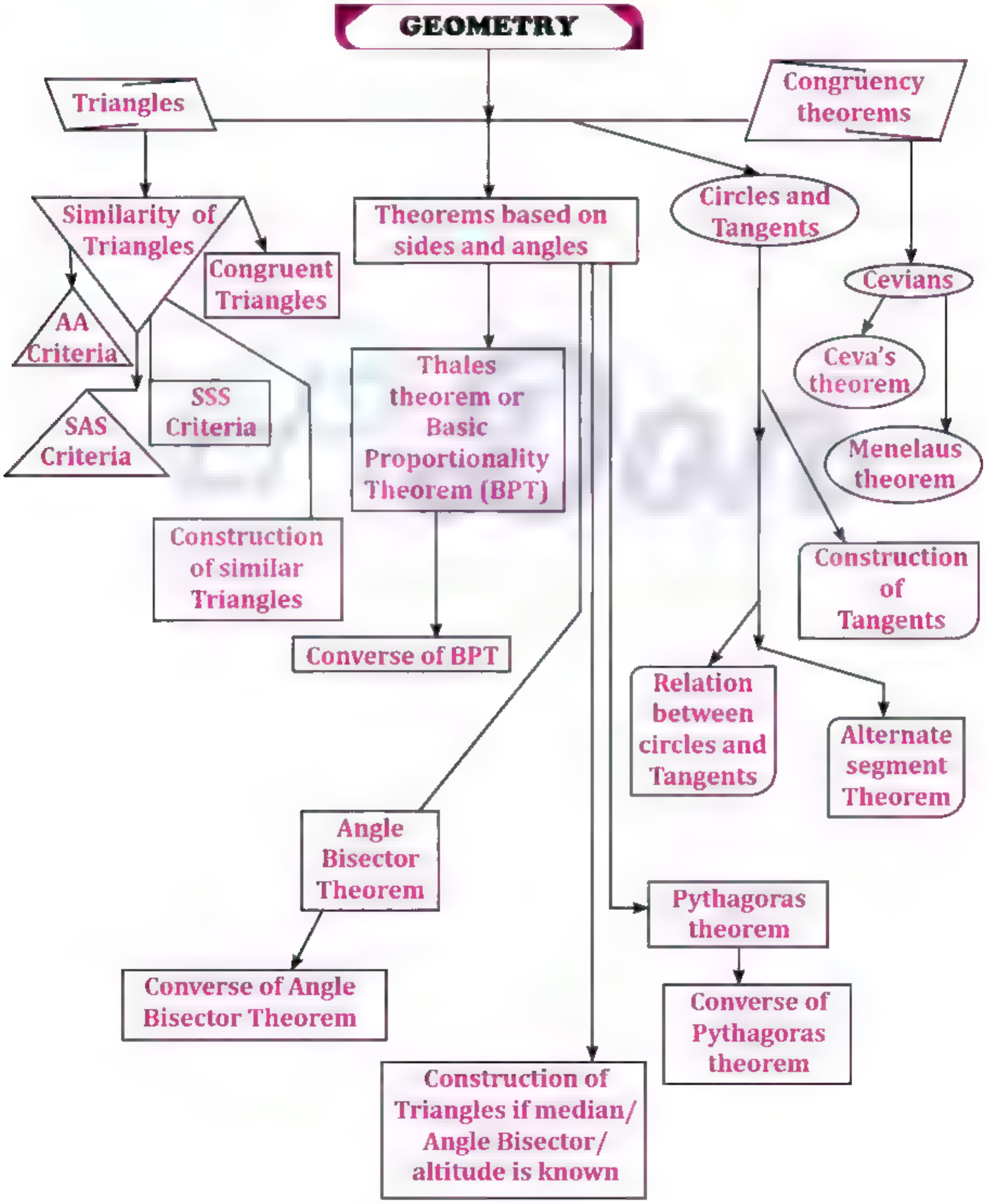
Hence proved.

❀❀❀❀❀

UNIT 4

GEOMETRY

MIND MAP



SIMILARITY

Key Points

SIMILARITY

- The ratio of the corresponding measurements of two **similar** objects must be proportional.
- Two geometrical figures are **congruent,** if they have same size and shape.

SIMILAR TRIANGLES

- If two triangles are similar then their corresponding angles are **equal** and their corresponding sides are proportional.
- The triangles are equiangular if the corresponding angles are equal
- If triangle ABC and ΔPQR are similar, they can be written as $\Delta ABC \sim \Delta PQR$.

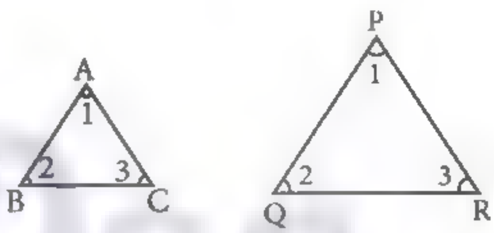
If $\Delta ABC \sim \Delta PQR$ then

$\angle A = \angle P$; $\angle B = \angle Q$; $\angle C = \angle R$

Also AB ≠ PQ; BC ≠ QR; CA ≠ RP

But $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} > 1$ or < 1.

Shapes are same but not the sizes.



CRITERIA OF SIMILARITY

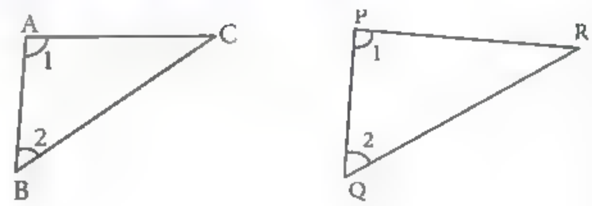
1. AA criterion of similarity

If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.

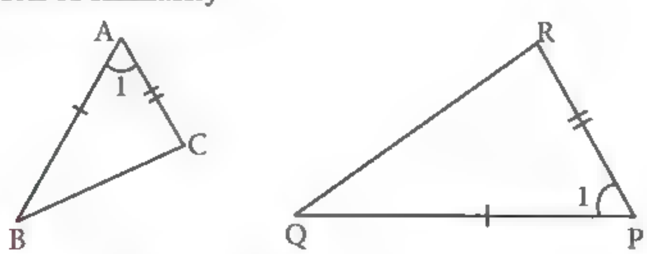
i.e., If $\angle A = \angle P = 1$

$\angle B = \angle Q = 2$ then $\Delta ABC \sim \Delta PQR$ then $\angle C$ must be equal to $\angle R$ [By angle sum property of triangles]

∴ This is AA similarity or AAA similarity.



2. SAS criterion of similarity



If one angle of a triangle is equal to one angle of another triangle and if the sides including them are proportional then the two triangles are similar.

If $\angle A = \angle P = 1$ and

$\frac{AB}{PQ} = \frac{AC}{PR}$ then

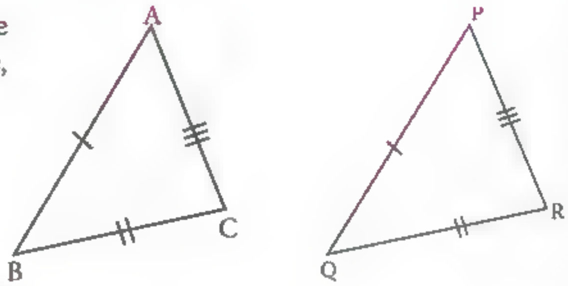
$\Delta ABC \sim \Delta PQR$

3. **SSS criterion of similarity**

If three sides of a triangle are proportional to the three corresponding sides of another triangle, then two triangles are similar.

So if $\frac{AB}{PQ} = \frac{AC}{PR} = \frac{BC}{QR}$ then

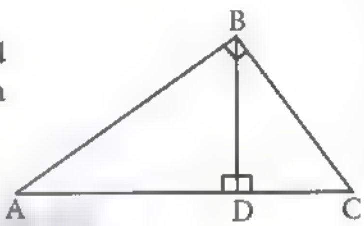
$\Delta ABC \sim \Delta PQR$



SOME USEFUL RESULTS ON SIMILAR TRIANGLES

1. A perpendicular line drawn from the vertex of a right angled triangle divides the triangle into two triangles similar to each other and also to original triangle.

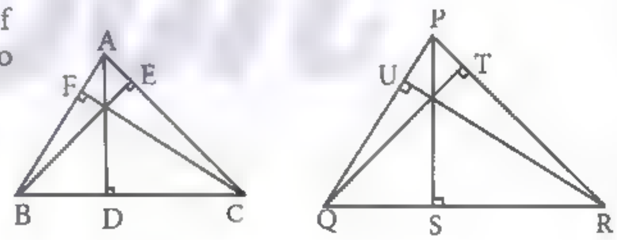
$\Delta ADB \sim \Delta BDC$, $\Delta ABC \sim \Delta ADB$, $\Delta ABC \sim \Delta BDC$



2. If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of their corresponding altitudes

i.e., if $\Delta ABC \sim \Delta PQR$ then

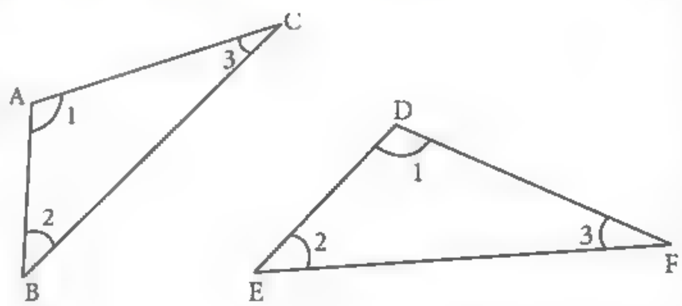
$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{PR} = \frac{AD}{PS} = \frac{BE}{QT} = \frac{CF}{RU}$$



3. If two triangles are similar, then the ratio of the corresponding sides are equal to the ratio of the corresponding perimeters.

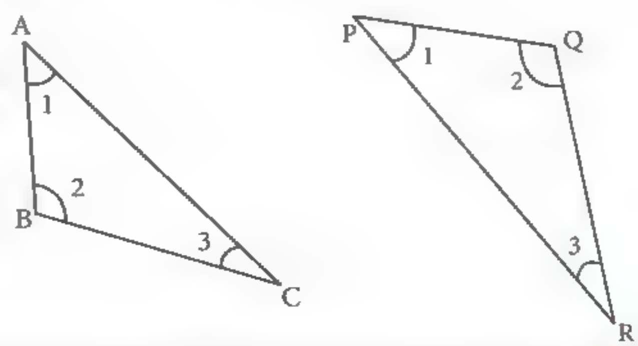
$\Delta ABC \sim \Delta DEF$ then

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{AB + BC + CA}{DE + EF + FD}$$

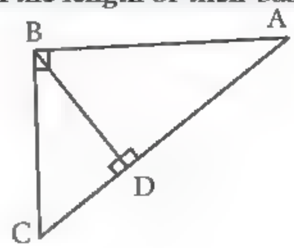


4. The ratio of the area of two similar triangles are equal to the ratio of the squares of their corresponding sides

$$\frac{area(\Delta ABC)}{area(\Delta PQR)} = \frac{AB^2}{PQ^2} = \frac{BC^2}{QR^2} = \frac{AC^2}{PR^2}$$



5. If two triangles have common vertex and their bases are on the same straight line, the ratio between their areas is equal to the ratio between the length of their bases.



Here, $\dfrac{area(\Delta ABD)}{area(\Delta BDC)} = \dfrac{AD}{DC}$

Note:

i) A pair of equiangular triangles are **similar.**

ii) If two triangles are similar, then they are equiangular.

CONSTRUCTION OF SIMILAR TRIANGLES

"Scale Factor" measures the ratio of the sides of a triangle to be constructed with the corresponding sides of the given triangle.

CONGRUENCY OF TRIANGLES

- Congruency is a particular case of similarity. Two triangles are said to be congruent if
  - (a) their corresponding angles are equal.
  - (b) their corresponding sides are also equal i.e., they have the same shape and size, if they are congruent

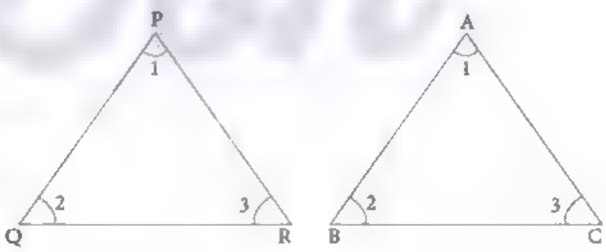
$\Delta PQR \cong \Delta ABC$

If $\angle P = \angle A$; $\angle Q = \angle B$; $\angle R = \angle C$

and AB = PQ; BC = QR; CA = RP

$\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{CA}{RP} = 1$

Same shape and same size.

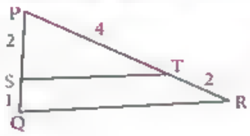


- Two polygons are similar if
  - (i) The lengths of their corresponding sides are proportional.
  - (ii) Their corresponding angles are equal.
- Two congruent figures are always similar, but two similar figures need not be congruent.
  - ☆ All line segments are similar.
  - ☆ All equilateral triangles are similar.
  - ☆ All squares are similar.
  - ☆ All circles are similar.

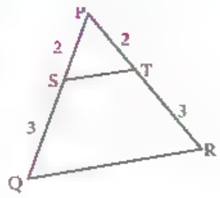
Worked Examples

4.1 Show that $\Delta PST \sim \Delta PQR$.

(i)



(ii)



Sol :

(i) In ΔPST and ΔPQR,

$$\frac{PS}{PQ} = \frac{2}{2+1} = \frac{2}{3},\quad \frac{PT}{PR} = \frac{4}{4+2} = \frac{2}{3}$$

Thus, $\frac{PS}{PQ} = \frac{PT}{PR}$ and $\angle P$ is common.

Therefore, by SAS similarity,

$\Delta PST \sim \Delta PQR$

(ii) In ΔPST and ΔPQR,

$$\frac{PS}{PQ} = \frac{2}{2+3} = \frac{2}{5},\quad \frac{PT}{PR} = \frac{2}{2+3} = \frac{2}{5}$$

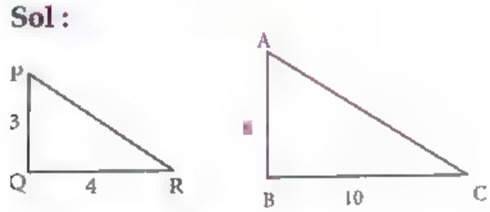
Thus, $\frac{PS}{PQ} = \frac{PT}{PR}$ and $\angle P$ is common

Therefore, by SAS similarity,

$\Delta PST \sim \Delta PQR$

4.2 Is $\Delta ABC \sim \Delta PQR$?

Sol :



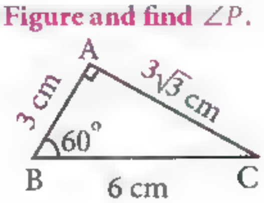
In ΔABC and ΔPQR,

$$\frac{PQ}{AB} = \frac{3}{6} = \frac{1}{2};\quad \frac{QR}{BC} = \frac{4}{10} = \frac{2}{5}$$

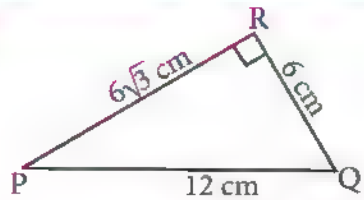
Since $\frac{1}{2} \neq \frac{2}{5}$, $\frac{PQ}{AB} \neq \frac{QR}{BC}$.

The corresponding sides are not proportional. Therefore ΔABC is not similar to ΔPQR.

4.3 Observe Figure and find $\angle P$.



Sol :



In ΔBAC and ΔPQR, $\frac{AB}{RQ} = \frac{3}{6} = \frac{1}{2}$;

$$\frac{BC}{QP} = \frac{6}{12} = \frac{1}{2};\quad \frac{CA}{PR} = \frac{3\sqrt{3}}{6\sqrt{3}} = \frac{1}{2}$$

Therefore, $\frac{AB}{RQ} = \frac{BC}{QP} = \frac{CA}{PR}$

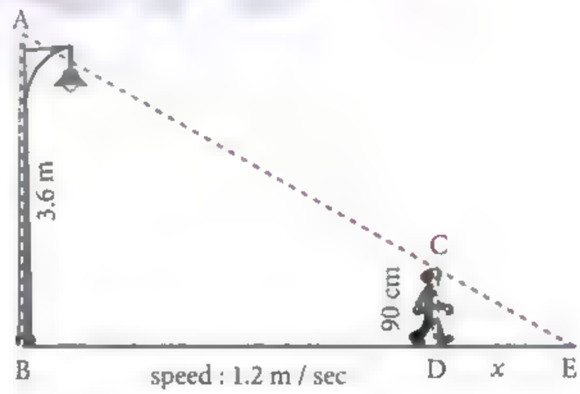
By SSS similarity, we have $\Delta BAC \sim \Delta QRP$

$\angle P = \angle C$ (since the corresponding parts of similar triangle)

$$\angle P = \angle C = 180^\circ - (\angle A + \angle B)$$
$$= 180^\circ - (90^\circ + 60^\circ)$$
$$\angle P = 180^\circ - 150^\circ = 30^\circ$$

4.4 A boy of height 90 cm is walking away from the base of a lamppost at a speed of 1.2 m/sec. If the lamppost is 3.6 m above the ground, find the length of his shadow cast after 4 seconds.

Sol :



Given Speed = 1.2 m/s

time = 4 seconds

Distance = speed × time

= 1.2 × 4 = 4.8 m

Let x be the length of the shadow after 4 seconds

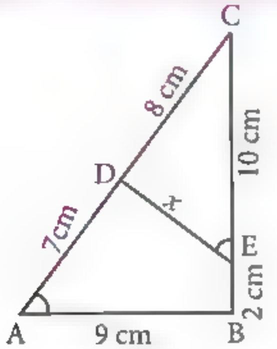
Since, $\Delta ABE \sim \Delta CDE$, $\frac{BE}{DE} = \frac{AB}{CD}$

$$\Rightarrow \frac{4.8 + x}{x} = \frac{3.6}{0.9} = 4. \text{ (since 90 cm = 0.9 m)}$$

$= 4.8 + x = 4x \Rightarrow 3x = 4.8 \Rightarrow x = 1.6$ m

The length of his shadow DE = 1.6 m

4.5 **In Figure $\angle A = \angle CED$ prove that $\Delta CAB \sim \Delta CED$. Also find the value of x.**



Sol :

In ΔCAB and ΔCED, $\angle C$ is common,

$\angle A = \angle CED$

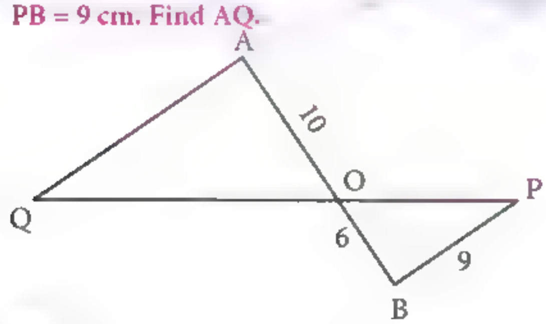
Therefore, $\Delta CAB \sim \Delta CED$ (By AA similarity)

Hence, $\dfrac{CA}{CE} = \dfrac{AB}{DE} = \dfrac{CB}{CD}$

$$\frac{AB}{DE} = \frac{CB}{CD} \Rightarrow \frac{9}{x} = \frac{10+2}{8}$$

$$\Rightarrow x = \frac{8\times 9}{12} = 6 \text{ cm.}$$

4.6 **In Figure QA and PB are perpendiculars to AB. If AO = 10 cm, BO = 6 cm and PB = 9 cm. Find AQ.**



Sol :

In ΔAOQ and ΔBOP, $\angle OAQ = \angle OBP = 90^\circ$

$\angle AOQ = \angle BOP$ (Vertically opposite angles)

Therefore, by AA criterion of similarity,

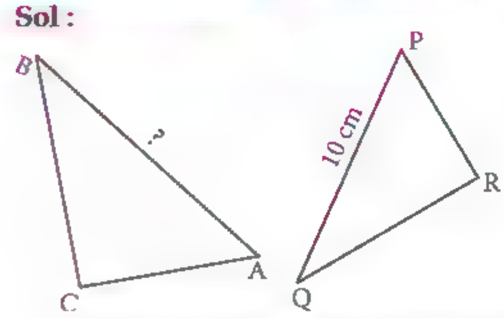
$\Delta AOQ \sim \Delta BOP$

$$\frac{AO}{BO} = \frac{OQ}{OP} = \frac{AQ}{BP}$$

$$\frac{10}{6} = \frac{AQ}{9} \Rightarrow AQ = \frac{10\times 9}{6} = 15 \text{ cm}$$

4.7 **The perimeters of two similar triangles ABC and PQR are respectively 36 cm and 24 cm. If PQ = 10 cm, find AB.**

Sol :



The ratio of the corresponding sides of similar triangles is same as the ratio of their perimeters.

Since $\Delta ABC \sim \Delta PQR$,

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR} = \frac{36}{24}$$

$$\frac{AB}{PQ} = \frac{36}{24} \Rightarrow \frac{AB}{10} = \frac{36}{24}$$

$$AB = \frac{36\times 10}{24} = 15 \text{ cm}$$

4.8 **If ΔABC is similar to ΔDEF such that BC = 3 cm, EF = 4 cm and area of ΔABC = 54 cm^2. Find the area of ΔDEF.**

Sol :

Since the ratio of area of two similar triangles is equal to the ratio of the squares on any two corresponding sides, we have

$$\frac{Area(\Delta ABC)}{Area(\Delta DEF)} = \frac{BC^2}{EF^2}$$

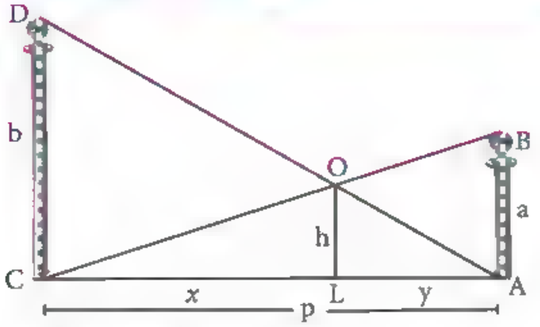
$$\Rightarrow \frac{54}{Area(\Delta DEF)} = \frac{3^2}{4^2}$$

$$\text{Area } (\Delta DEF) = \frac{16\times 54}{9} = 96 \text{ cm}^2$$

4.9 **Two poles of height 'a' metres and 'b' metres are 'p' metres apart. Prove that the height of the point of intersection of the lines joining the top of each pole to the foot of the opposite pole is given by $\dfrac{ab}{a+b}$ metres.**

Sol :

Let AB and CD be two poles of height 'a' metres and 'b' metres respectively such that the poles are 'P' metres apart. That is AC = p metres. Suppose the lines AD and BC meet at O, such that OL = h metres.



Let CL = x and LA = y.

Then, x + y = p

In ΔABC and ΔLOC,

we have $\angle CAB = \angle CLO$ [each equal to 90°]

$\angle C = \angle C$ [C is common]

$\Delta CAB \sim \Delta CLO$ [By AA similarity]

$$\frac{CA}{CL} = \frac{AB}{LO} \Rightarrow \frac{p}{x} = \frac{a}{h} \Rightarrow x = \frac{ph}{a} \quad ...(1)$$

In ΔALO and ΔACD, we have

$\angle ALO = \angle ACD$ [each equal to 90°]

$\angle A = \angle A$ [A is common]

$\Delta ALO \sim \Delta ACD$ [by AA similarity]

$$\frac{AL}{AC} = \frac{OL}{DC} \Rightarrow \frac{y}{p} = \frac{h}{b} \Rightarrow y = \frac{ph}{b} \quad ...(2)$$

$$(1) + (2) \Rightarrow x + y = \frac{ph}{a} + \frac{ph}{b}$$

$$p = ph\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{(Since } x + y = p\text{)}$$

$$1 = h\left(\frac{a+b}{ab}\right)$$

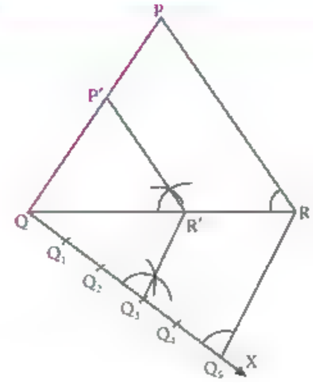
Therefore, $h = \dfrac{ab}{a+b}$

Hence, the height of the intersection of the lines joining the top of each pole to the foot of the opposite pole is $\dfrac{ab}{a+b}$ metres.

4.10 Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{3}{5}$ of the corresponding sides of the triangle PQR (scale factor $\frac{3}{5} < 1$).

Sol :

Given a triangle PQR we are required to construct another triangle whose sides are $\frac{3}{5}$ of the corresponding sides of the triangle PQR



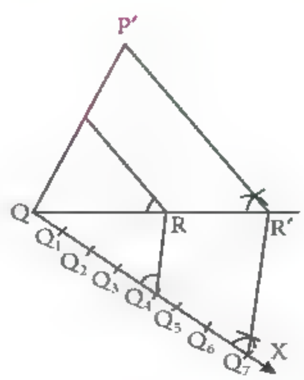
Steps of construction:

1. Construct a ΔPQR with any measurement.
2. Draw a ray QX making an acute angle with QR on the side opposite to the vertex P.
3. Locate 5 (the greater of 3 and 5 in $\frac{3}{5}$) Points. Q_1, Q_2, Q_3, Q_4 and Q_5 on QX so that $QQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4Q_5$
4. Join Q_5R and draw a line through Q_3 (the third point, 3 being smaller of 3 and 5 in $\frac{3}{5}$) parallel to Q_5R to intersect QR at R'.
5. Draw line through R' parallel to the line RP to intersect QP at P'.

   Then, $\Delta P'QR'$ is the required triangle each of whose sides is three-fifths of the corresponding sides of ΔPQR.

4.11 Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{7}{4}$ of the corresponding sides of the triangle PQR (scale factor $\frac{7}{4} > 1$)

Sol :



Given a triangle PQR, we are required to construct another triangle whose sides are $\frac{7}{4}$ of the corresponding sides of the triangle PQR.

Steps of construction

1. Construct a ΔPQR with any measurement.
2. Draw a ray QX making an acute angle with QR on the side opposite to the vertex P.
3. Locate 7 points (the greater of 7 and 4 in $\frac{7}{4}$) $Q_1, Q_2, Q_3, Q_4, Q_5, Q_6$ and Q_7 on QX so that $QQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4$
$= Q_4Q_5 = Q_5Q_6 = Q_6Q_7$.
4. Join Q_4 (the fourth point, 4 being smaller of 4 and 7 in $\frac{7}{4}$) to R and draw a line through Q_7 parallel to Q_4R, intersecting the extended line segment QR at R'.
5. Draw a line through R' parallel to RP inter-secting the extended line segment QP at P'.

Then $\Delta P'QR'$ is the required triangle each of whose sides is seven-fourths of the corresponding sides of ΔPQR.

Progress Check

1. All circles are _____ (congruent/similar).

Ans : similar

2. All squares are _____ (similar/congruent).

Ans : similar

3. Two triangles are similar, if their corresponding angles are _____ and their corresponding sides are _____.

Ans : equal, proportional

4. (a) All similar triangles are congruent True/ False.

Ans : False

(b) All congruent triangles are similar True/ False.

Ans : True

5. Give two different examples of pair of non-similar figures.

Ans : (i) A circle and a triangle are non similar figures.

(ii) An isosceles triangle and a scalene triangle are non-similar figures.

Thinking Corner

1. Are square and a rhombus similar or congruent. Discuss.

Ans :

For a rhombus

☆ All sides are equal.

☆ The diagonals of a rhombus are perpendicular bisectors of one another.

☆ Opposite angles are equal.

For a square

☆ All sides are equal.

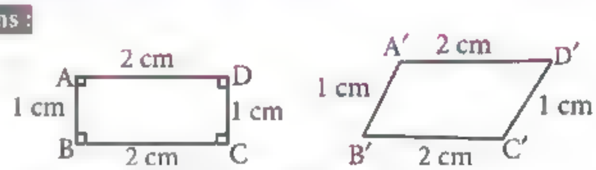
☆ All angles are 90°.

☆ Diagonals are equal and perpendicular bisectors of each other.

For both the shapes the angles are not equal, but corresponding sides are proportional. so they are neither similar nor congruent.

2. Are a rectangle and a parallelogram similar. Discuss.

Ans :



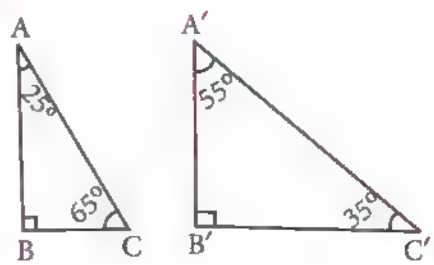
On observing the given figures.

Their corresponding sides are proportional but their corresponding angles are not equal.

∴ The shapes parallelogram and rectangle are not similar.

3. Are any two right angled triangles similar? Why?

Ans :



For any two right angled triangles.

☆ The angles other than right angle need

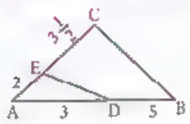
not be equal.

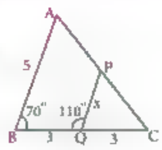
☆ The corresponding sides need not be proportional.

± ∴ Any two right angled triangles need not be similar.

Exercise 4.1

1. Check whether the given triangles are similar and find the value of x.

(i) 

(ii) 

Sol :

(i) In ΔABC and ΔADE $\angle A$ is common

$$\frac{AE}{EC} = \frac{2}{3\frac{1}{2}} = \frac{2}{\frac{7}{2}} = \frac{2\times 2}{7} = \frac{4}{7}$$

$$\frac{AD}{DB} = \frac{3}{5}$$

Here $\frac{4}{7} \pm \frac{3}{5}$

$$\frac{AE}{EC} \neq \frac{AD}{DB}$$

The corresponding sides are not proportional.

∴ ΔABC and ΔADE are not similar.

(ii) In ΔCPQ and ΔCAB $\angle C$ is common.

$\angle PQC = 180° - 110° = 70°$

[∵ $\angle PQC$ and $\angle PQB$ are liner pair of angles]

$\angle ABC = 70°$

∴ $\angle BAC = \angle QPC$

[∵ sum of three angles of a triangle are 180°]

$\angle PQC = 180° - (\angle QPC + 70°)$

∴ $\angle ABC = \angle PQC = 70°$

$\angle C$ common and $\angle BAC = \angle QPC$

By AAA similarity criteria, $\Delta ABC \sim \Delta PQC$

∴ Corresponding sides are proportional

$$\frac{AB}{PQ} = \frac{BC}{QC}$$

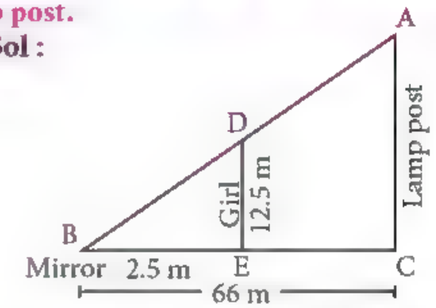
$$\frac{5}{x} = \frac{6}{3}$$

[∵ BC = BQ + QC = 3 + 3 = 6]

$$x = \frac{5}{6}\times 3 = 2.5 \qquad \therefore x = 2.5$$

2. A girl looks the reflection of the top of the lamp post on the mirror which is 66 m away from the foot of the lamp post. The girl whose height is 12.5 m is standing 2.5 m away from the mirror. Assuming the mirror is placed on the ground facing the sky and the girl, mirror and the lamp post are in a same line, find the height of the lamp post.

Sol :



Let AC is the lamp post and ED is the girl.

From the triangles ΔABC and ΔDBE

$\angle B$ is common. $\angle DEB = \angle ACB = 90°$

By AA criteria

$\Delta ABC \sim \Delta DBE$

∴ Their sides are proportional

i.e., $$\frac{AC}{DE} = \frac{BC}{BE}$$

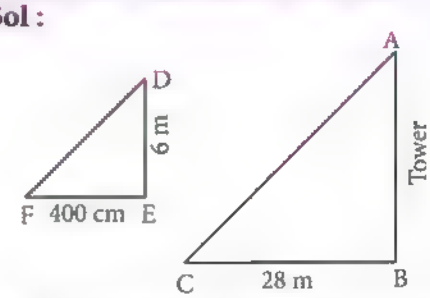
$$\frac{AC}{12.5} = \frac{66}{2.5}$$

$$AC = \frac{66\times 12.5}{2.5} = \frac{66\times \cancel{12.5}^{5}}{\cancel{2.5}_{1}} = 330 \text{ m}$$

∴ Height of the lamp post = 330 m

3. A vertical stick of length 6 m casts a shadow 400 cm long on the ground and at the same time a tower casts a shadow 28 m long. Using similarity, find the height of the tower.

Sol :



Let DE is the vertical stick and AB is the tower.

DE = 6 m, EF = 400 cm = 4 m, BC = 28 m

From ΔDFE and ΔACB

Using similarity criteria

$$\frac{AB}{DE} = \frac{BC}{EF}$$

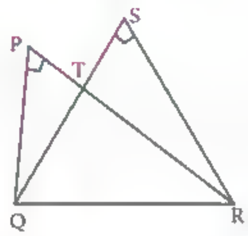
$$\frac{AB}{6} = \frac{28}{4}$$

$$AB = \frac{28\times 6}{4} = 42 \text{ m}$$

$\therefore$ Height of the tower = 42 m

4. Two triangles QPR and QSR, right angled at P and S respectively are drawn on the same base QR and on the same side of QR. If PR and SQ intersect at T, prove that PT × TR = ST × TQ.

Sol :



Given $\angle QSR = \angle QPR = 90^\circ$

PR and SQ intersect at T.

In ΔQPT and ΔRST

$\angle QSR = \angle QPR = 90^\circ$ Given

$\angle PTQ = \angle STR$

[$\because$ Vertically opposite angles]

$\therefore$ By AA similarity criteria

$\Delta PQT \sim \Delta SRT$

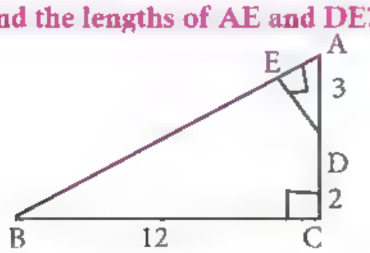
$\therefore$ Their corresponding sides are proportional

$$\therefore \frac{PT}{ST} = \frac{TQ}{TR}$$

$$PT \times TR = ST \times TQ$$

Hence proved.

5. In the adjacent figure ΔABC is right angled at C and DE ⊥ AB. Prove that $\Delta ABC \sim \Delta ADE$ and hence find the lengths of AE and DE?



Sol :

Given $\angle C = 90^\circ = \angle DEA$

$\angle A$ is common to both the triangles ΔABC and ΔADE

$\therefore$ By AA criteria for similarity

$\Delta ABC \sim \Delta ADE$

$\therefore$ Their corresponding sides are proportional.

$$\therefore \frac{BC}{DE} = \frac{AC}{AE} = \frac{AB}{AD}$$

$$\frac{12}{DE} = \frac{3+2}{AE} = \frac{AB}{3} \quad \text{... (1)}$$

Also in right ΔABC, $\angle C = 90^\circ$

$\therefore$ using Pythagoras theorem, we have

$$AB^2 = BC^2 + AC^2 = 12^2 + 5^2$$

$$= 144 + 25 = 169$$

$$\therefore AB = \sqrt{169}$$

$$AB = 13 \text{ cm}$$

Now from (1), we get

$$\frac{12}{DE} = \frac{13}{3}$$

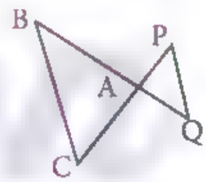
$$DE = \frac{12\times 3}{13} = \frac{36}{13}$$

$$DE = 2.77 \text{ cm}$$

Also from (1), $\frac{5}{AE} = \frac{13}{3}$

$$AE = \frac{5\times 3}{13} = \frac{15}{13} = 1.15 \text{ cm}$$

6. If figure $\Delta ACB \sim \Delta APQ$. If BC = 8 cm, PQ = 4 cm, BA = 6.5 cm and AP = 2.8 cm, find CA and AQ.



Sol : Given $\Delta ACB \sim \Delta APQ$

$\therefore$ Their corresponding sides are proportional

$$\frac{AC}{AP} = \frac{CB}{PQ} = \frac{AB}{AQ}$$

$$\frac{AC}{2.8} = \frac{8}{4} = \frac{6.5}{AQ}$$

Taking $\frac{AC}{2.8} = \frac{8}{4}$

$$AC = \frac{8\times 2.8}{4} = 5.6 \text{ cm}$$

$$AC = 5.6 \text{ cm}$$

Also taking $\frac{8}{4} = \frac{6.5}{AQ}$

$$AQ = \frac{6.5}{8}\times 4 = 3.25 \text{ cm}$$

$$AQ = 3.25 \text{ cm}$$

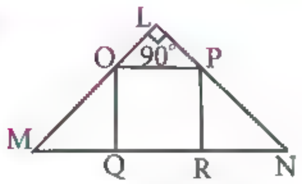
7. In figure OPQR is a square and $\angle MLN = 90^\circ$. Prove that

(i) $\Delta LOP \sim \Delta QMO$ **(ii)** $\Delta LOP \sim \Delta RPN$

(iii) $\Delta QMO \sim \Delta RPN$

(iv) $QR^2 = MQ \times RN$

Sol :



(i) In ΔLOP and ΔQMO

$\angle OLP = \angle MQO = 90°$

$\angle LOP = \angle QMO$ (corresponding angles)

$\therefore$ By AA criterion of similarity.

$\Delta LOP \sim \Delta QMO$

(ii) In ΔLOP and ΔRPN,

we have $\angle PLO = \angle NRP = 90°$

and $\angle LPO = \angle RNP$

(corresponding angles)

$\therefore$ By AA criterion of similarity

$\Delta LOP \sim \Delta RPN$

(iii) Also in ΔQMO and ΔRPN

$\angle OQM = \angle NRP = 90°$

we have $\Delta LOP \sim \Delta QMO$ and $\Delta LOP \sim \Delta RPN$

$\Delta QMO \sim \Delta RPN$

(iv) We have $\Delta QMO \sim \Delta RPN$

$$\therefore \frac{MQ}{PR} = \frac{QO}{RN}$$

$$\frac{MQ}{QR} = \frac{QR}{RN}$$

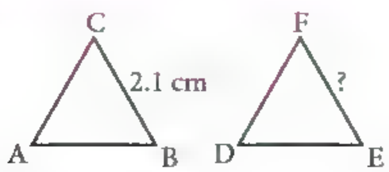
[$\because$ OQRP is a square PR = QR and QO = QR]

$\therefore QR \times QR = MQ \times RN$

$QR^2 = MQ \times RN$

8. If $\Delta ABC \sim \Delta DEF$ such that area of ΔABC is 9 cm² and the area of ΔDEF is 16 cm² and BC = 2.1 cm. Find the length of EF.

Sol :



Given $\Delta ABC \sim \Delta DEF$

then we have

$$\frac{Area(\Delta ABC)}{Area(\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{AC^2}{DF^2}$$

$$\frac{9}{16} = \frac{BC^2}{EF^2}$$

$$\frac{9}{16} = \frac{2.1 \times 2.1}{EF^2}$$

$$EF^2 = \frac{2.1 \times 2.1 \times 16}{9} = \frac{2.1 \times 2.1 \times 4 \times 4}{3 \times 3}$$

$$EF^2 = \left(\frac{2.1 \times 4}{3}\right)^2$$

$$EF = \frac{2.1 \times 4}{3} = 2.8$$

EF = 2.8 cm

Another method:

$$\frac{9}{16} = \frac{BC^2}{EF^2}$$

Taking square root

$$\frac{3}{4} = \frac{BC}{EF}$$

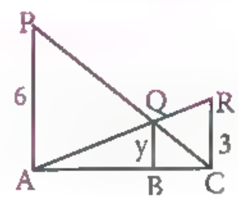
$$\frac{3}{4} = \frac{2.1}{EF}$$

$$3 \times EF = 2.1 \times 4$$

$$EF = \frac{2.1 \times 4}{3}$$

$= 2.8$ cm

9. Two vertical poles of heights 6 m and 3 m are erected above a horizontal ground AC. Find the value of y.



Sol :

Let AP and CR be the vertical poles of height 6 m and 3 m respectively

Let BQ = y m

In ΔCRA and ΔBQA

$\angle A = \angle A$ common

$\angle ACR = \angle ABQ = 90°$

$\therefore$ By AA criterion of similarity

$\Delta CRA \sim \Delta BQA$

$\therefore$ Their corresponding sides are proportional.

$$\frac{AC}{AB} = \frac{CR}{BQ}$$

$$\frac{AC}{AB} = \frac{3}{y}$$

$$AB = \frac{AC \times y}{3} \quad \text{... (1)}$$

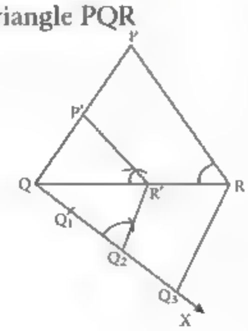
In ΔCBQ and ΔCAP

$$\angle CBQ = \angle CAP = 90^o$$
$$\angle C = \angle C \quad \text{[common]}$$
$$\therefore \Delta CBQ \sim \Delta CAP$$
$$\frac{CB}{CA} = \frac{BQ}{AP}$$
$$\frac{CB}{CA} = \frac{y}{6}$$
$$BC = \frac{y \times CA}{6} \quad \text{... (2)}$$
$$(1) + (2) \Rightarrow AB + BC = \frac{AC \times y}{3} + \frac{y \times CA}{6}$$
$$AC = y \times AC\left(\frac{1}{3} + \frac{1}{6}\right)$$
$$\frac{AC}{AC} = y\left(\frac{2+1}{6}\right)$$
$$1 = y\left(\frac{3}{6}\right)$$
$$1 = \frac{1}{2}y$$
$$y = 2 \text{ m.}$$

10. Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{2}{3}$ of the corresponding sides of the triangle PQR (scale factor $\frac{2}{3}$)

Sol :

Given a triangle PQR we are required to construct another triangle whose sides are $\frac{2}{3}$ of the corresponding sides of the triangle PQR



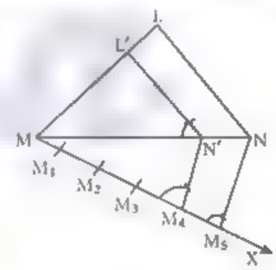
Steps of construction:

1. Constructed a ΔPQR with any measurement.
2. Drawn a ray QX making an acute angle with QR on the side opposite to the vertex P.
3. Located 3 points Q_1, Q_2 and Q_3 on QX so that $QQ_1 = Q_1Q_2 = Q_2Q_3$.
4. Joined Q_3R and drawn a line through Q_2 parallel to Q_3R to intersect QR at R'
5. Drawn a line through R' parallel to the line RP to intersect QP at P'. Then $\Delta P'QR'$ is the required triangle each of whoses sides is two third of the corresponding sides of ΔPQR

11. Construct a triangle similar to a given triangle LMN with its sides equal to $\frac{4}{5}$ of the corresponding sides of the triangle LMN (scale factor $\frac{4}{5}$)

Sol :

Given a triangle LMN. We are required to construct another triangle whose sides are $\frac{4}{5}$ of the corresponding sides of the ΔLMN.



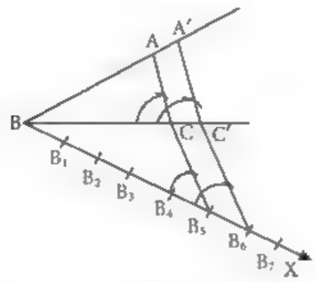
Steps of construction:

1. Constructed a ΔLMN with any measurement.
2. Drawn a ray MX making an acute angle with MN on the side opposite to the vertex L.
3. Located 5 points M_1, M_2, M_3, M_4, M_5 on MX so that
   $MM_1 = M_1M_2 = M_2M_3 = M_3M_4 = M_4M_5$
4. Joined M_5N and drawn a line through M_4 parallel to M_5N to intersect MN at N'..
5. Drawn a line through N' parallel to the line NL to intersect ML at L'.
   Then $\Delta L'MN'$ is the required triangle each of whose sides is four fifth of the corresponding sides of ΔLMN ..

12. Construct a triangle similar to a given triangle ABC with its sides equal to 6/5 of the corresponding sides of the triangle ABC (scale factor 6/5)

Sol :

Given a triangle ABC, we are required to construct another triangle whose sides are $\frac{6}{5}$ of the corresponding sides of the triangle ABC



Steps of construction:

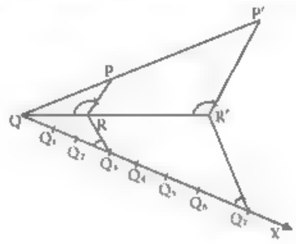
1. Constructed a $\Delta\, ABC$ with any measurement.
2. Drawn a ray BX making an acute angle with BC on the side opposite to the vertex A.
3. Located 6 points B_1, B_2, B_3, B_4, B_5 and B_6 so that $BB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B_5 = B_5B_6$.
4. Joined B_5 to C and drawn a line through B_6 parallel to B_5C intersecting the extended line segment BC at C'
5. Drawn a line through C' parallel to CA intersecting the extended BA at A'.

   Then $\Delta A'BC'$ is the required triangle each of whose sides is six fifth of the corresponding sides of ΔABC.

13. Construct a triangle similar to a given triangle PQR with its sides equal to $\frac{7}{3}$ of the corresponding sides of the triangle PQR (scale factor $\frac{7}{3}$)

Sol :

Given a triangle PQR, we are required to construct another triangle whose sides are $\frac{7}{3}$ of the corresponding sides of the triangle PQR



Steps of construction:

1. Constructed a ΔPQR with any measurement.
2. Drawn a ray QX making an acute angle with QR on the side opposite to the vertex P.
3. Located 7 points Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 and Q_7 on QX so that $QQ_1 = Q_1Q_2 = Q_2Q_3 = Q_3Q_4 = Q_4Q_5 = Q_5Q_6 = Q_6Q_7$.
4. Joined Q_3 to R and drawn a line through Q_7 parallel to Q_3R, intersecting the extended line segment QR at R'
5. Drawn a line through R' parallel to RP intersecting the extended line segment QP at P'.

Then $\Delta P'QR'$ is the required triangle each of whose sides is seven-thirds of the corresponding sides of ΔPQR.

THALES THEOREM AND ANGLE BISECTOR THEOREM

Key Points

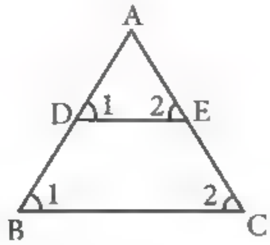
Thales theorem and angle bisector theorem

- Thales was the first man to announce that any idea that emerged should be tested scientifically and only then it can be accepted.
- Thales was credited for providing first proof in mathematics called "Basic proportionality theorem" or otherwise "Thales theorem".

Basic proportionality theorem or thales theorem

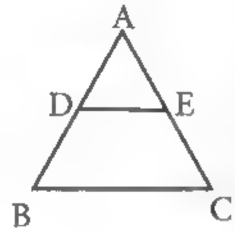
"A straight line drawn parallel to a side of triangle, divides the other two sides proportionally".

$$\frac{AD}{DB} = \frac{AE}{EC}$$



Corollary

If in a ΔABC, a straight line DE parallel to BC, intersects AB at D and AC at E, then



(i) $\frac{AB}{AD} = \frac{AC}{AE}$

(ii) $\frac{AB}{DB} = \frac{AC}{EC}$

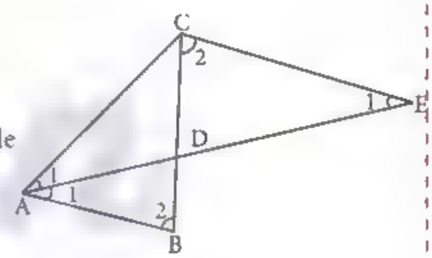
Converse of Thales theorem

If a straight line divides two sides of a triangle proportionally then the straight line is parallel to the third side.

Angle bisector theorem

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the corresponding sides containing the angle.

i.e., $\frac{AB}{AC} = \frac{BD}{CD}$



Converse of Angle bisector theorem

"If a straight line through one vertex of a triangle divides the opposite side internally in the ratio of the other two sides, then the line bisects the angle internally at the vertex'.

Theorem 1:

Basic proportionality theorem or Thales theorem.

Statement

"A straight line drawn parallel to a side of triangle intersecting the other two sides, divides the sides in the same ratio.

Proof

Given : In ΔABC, D is a point on AB and E is a point on AC.

To prove : $\frac{AD}{DB} = \frac{AE}{EC}$

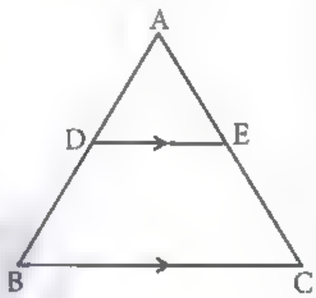
Construction : Draw a line DE || BC

Sl. No.	Statement	Reason
1	$\angle ABC = \angle ADE = \angle 1$	Corresponding angles are equal because $DE \parallel BC$
2	$\angle ACB = \angle AED = \angle 2$	Corresponding angles are equal because $DE \parallel BC$
3	$\angle DAE = \angle BAC = \angle 3$	Both triangles have a common angle

4	$\Delta ABC \sim \Delta ADE$ $\frac{AB}{AD} = \frac{AC}{AE}$ $\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$ $1 + \frac{DB}{AD} = 1 + \frac{EC}{AE}$ $\frac{DB}{AD} = \frac{EC}{AE}$ $\frac{AD}{DB} = \frac{AE}{EC}$	By AAA similarity Corresponding sides are proportional Split AB and AC Simplification Cancelling 1 on both sides Taking reciprocals

Hence proved.

Corollary



If in a ΔABC, a straight line DE parallel to BC, intersects AB at D and AC at E, then

(i) $\frac{AB}{AD} = \frac{AC}{AE}$ **(ii)** $\frac{AB}{DB} = \frac{AC}{EC}$.

Proof

In ΔABC, DE || BC,

therefore, $\frac{AD}{DB} = \frac{AE}{EC}$ (by B.P.T. theorem)

(i) Taking reciprocals, we get $\frac{DB}{AD} = \frac{EC}{AE}$

Add 1 to both in the sides $\frac{DB}{AD} + 1 = \frac{EC}{AE} + 1$

$$\Rightarrow \frac{DB + AD}{AD} = \frac{EC + AE}{AE} \Rightarrow \frac{AB}{AD} = \frac{AC}{AE}$$

(ii) Add 1 to both the sides

$$\frac{AD}{DB} + 1 = \frac{AE}{EC} + 1$$

Therefore, $\frac{AB}{DB} = \frac{AC}{EC}$

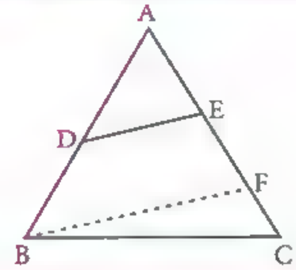
Theorem 2:

Converse of Basic Proportionality Theorem (or) Thales Theorem.

Statement:

"If a straight line divides any two sides of a triangle in the same ratio, then the line must be parallel to the third side."

Proof:



Given : In ΔABC, $\frac{AD}{DB} = \frac{AE}{EC}$

To prove : DE || BC

Construction : Draw BF || DE

Sl. No.	Statement	Reason
1	In ΔABC, $BF \parallel DE$	Construction
2	$\frac{AD}{DB} = \frac{AE}{EC}$ ... (1)	Thales theorem (In ΔABC taking D in AB and E in AC)
3	$\frac{AD}{DB} = \frac{AF}{FC}$ ... (2)	Thales theorem (In ΔABC taking F in AC)
4	$\frac{AE}{EC} = \frac{AF}{FC}$ $\frac{AE}{EC} + 1 = \frac{AF}{FC} + 1$ $\frac{AE + EC}{EC} = \frac{AF + FC}{FC}$ $\frac{AC}{EC} = \frac{AC}{FC}$ EC = FC Therefore, F = C Thus $DE \parallel BC$	From (1) and (2) Adding 1 to both sides Cancelling AC on both sides F lies between E and C. Hence proved

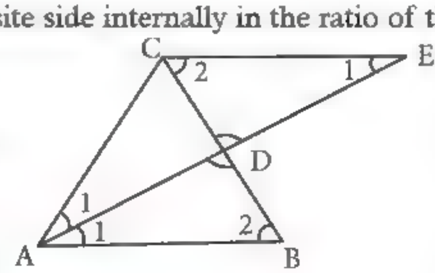
Theorem 3:

Angle Bisector Theorem

Statement:

"The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the corresponding sides containing the angle"

Proof:



Given : In ΔABC, AD is the internal bisector

To prove : $\frac{AB}{AC} = \frac{BD}{CD}$

Construction : Draw a line through C parallel to AB. Extend AD to meet line through C at E.

Sl. No.	Statement	Reason
1	$\angle AEC = \angle BAE = \angle 1$	Two parallel lines cut by a transversal make alternate equal angles

2	ΔACE is isosceles AC = CE ... (1)	In $\Delta ACE, \angle CAE = \angle CEA$
3	$\Delta ABD \sim \Delta ECD$ $\frac{AB}{CE} = \frac{BD}{CD}$	By AA Similarity
4	$\frac{AB}{AC} = \frac{BD}{CD}$	From (1) AC = CE Hence proved.

Theorem 4:

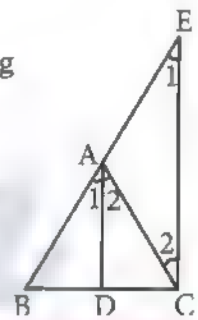
Converse of Angle Bisector Theorem

Statement:

"If a straight line through one vertex of a triangle divides the opposite side internally in the ratio of the other two sides, then the line bisects the angle internally at the vertex."

Proof:

Given, ABC is a triangle. AD divides BC in the ratio of the sides containing the angles $\angle A$ to meet BC at D.



That is $\frac{AB}{AC} = \frac{BD}{DC}$ ... (1)

To prove: AD bisects $\angle A$ i.e., $\angle 1 = \angle 2$

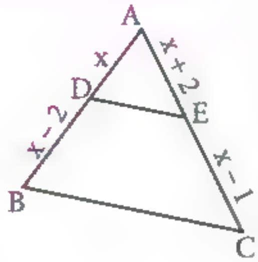
Construction: Draw CE ∥ DA. Extend BA to meet at E.

Sl. No.	Statement	Reason
1	Let $\angle BAD = \angle 1$ and $\angle DAC = \angle 2$	Assumption
2	$\angle BAD = \angle AEC = \angle 1$ ... (1)	Since $DA \parallel CE$, corresponding angles are equal
3	$\angle DAC = \angle ACE = \angle 2$	Since $DA \parallel CE$, Alternate angles are equal
4	$\frac{BA}{AE} = \frac{BD}{DC}$ ... (2)	In ΔBCE by Thales theorem
5	$\frac{AB}{AC} = \frac{BD}{DC}$	From (1)
6	$\frac{AB}{AC} = \frac{BA}{AE}$	From (1) and (2)
7	AC = AE ... (3)	Cancelling AB
8	$\angle 1 = \angle 2$	ΔACE is isosceles by (3)
9	AD bisects $\angle A$	Since, $\angle 1 = \angle BAD = \angle 2 = \angle DAC$ Hence proved.

Worked Examples

4.12 **In ΔABC DE || BC, if AD = x, DB = x – 2, and EC = x – 1 then find the lengths of the sides AB and AC.**

Sol :



In ΔABC

we have DE || BC.

By Thales theorem, we have $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

$$\frac{x}{x-2} = \frac{x+2}{x-1}$$

$$\Rightarrow x(x-1) = (x-2)(x+2)$$

$$\Rightarrow x^2 - x = x^2 - 4 \Rightarrow x = 4$$

When x = 4, AD = 4, DB = x – 2 = 2,

AE = x + 2 = 6; EC = x – 1 = 3.

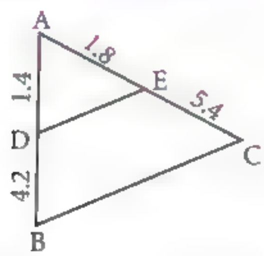
Hence, AB = AD + DB = 4 + 2 = 6,

AC = AE + EC = 6 + 3 = 9.

Therefore, AB = 6, AC = 9.

4.13 **D and E are respectively the points on the sides AB and AC of a ΔABC such that AB = 5.6 cm, AD = 1.4 cm, AC = 7.2 cm and AE = 1.8 cm, show that DE || BC.**

Sol :



We have AB = 5.6 cm, AD = 1.4 cm, AC = 7.2 cm and AE = 1.8 cm.

BD = AB – AD = 5.6 – 1.4 = 4.2 cm

and EC = AC – AE = 7.2 – 1.8 = 5.4 cm.

$$\frac{AD}{DB} = \frac{1.4}{4.2} = \frac{1}{3} \text{ and } \frac{AE}{EC} = \frac{1.8}{5.4} = \frac{1}{3}$$

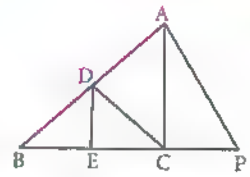
$$\frac{AD}{DB} = \frac{AE}{EC}$$

Therefore, by converse of Basic Proportionality theorem, we have DE is parallel to BC.

Hence proved.

4.14 **In the Figure, DE || AC and DC || AP. Prove that $\dfrac{BE}{EC} = \dfrac{BC}{CP}$.**

Sol :



In ΔBPA we have DC || AP. By Basic proportionality Theorem,

we have $\dfrac{BC}{CP} = \dfrac{BD}{DA}$ ... (1)

In ΔBCA, we have DE || AC. By Basic Proportionality Theorem, we have

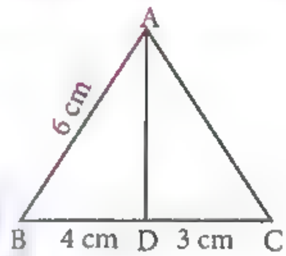
$$\frac{BE}{EC} = \frac{BD}{DA} \quad \text{... (2)}$$

From (1) and (2) we get, $\dfrac{BE}{EC} = \dfrac{BC}{CP}$

Hence proved.

4.15 **In the Figure AD is the bisector of $\angle A$. If BD = 4 cm, DC = 3 cm and AB = 6 cm, find AC.**

Sol :



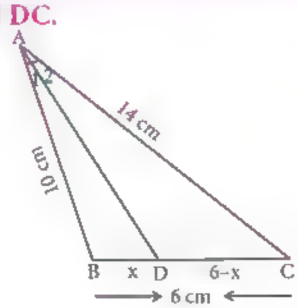
In ΔABC, AD is the bisector of $\angle A$

Therefore by Angle Bisector Theorem

$$\frac{BD}{DC} = \frac{AB}{AC}$$

$$\frac{4}{3} = \frac{6}{AC} \Rightarrow 4\,AC = 18 \Rightarrow \text{Hence } AC = \frac{9}{2} = 4.5 \text{ cm}$$

4.16 **In the Figure AD is the bisector of $\angle BAC$ if AB = 10 cm, AC = 14 cm and BC = 6 cm. Find BD and DC.**



Sol :

Let BD = x cm, then DC = (6 – x) cm

AD is the bisector of $\angle A$

Therefore by Angle Bisector Theorem

$$\frac{AB}{AC} = \frac{BD}{DC}$$

$$\frac{10}{14} = \frac{x}{6-x} \Rightarrow \frac{5}{7} = \frac{x}{6-x}$$

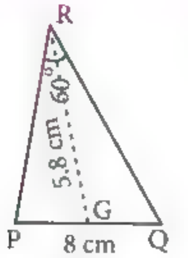
$$\Rightarrow 12x = 30 \Rightarrow x = \frac{30}{12} = 2.5 \text{ cm}$$

Therefore, BD = 2.5 cm,

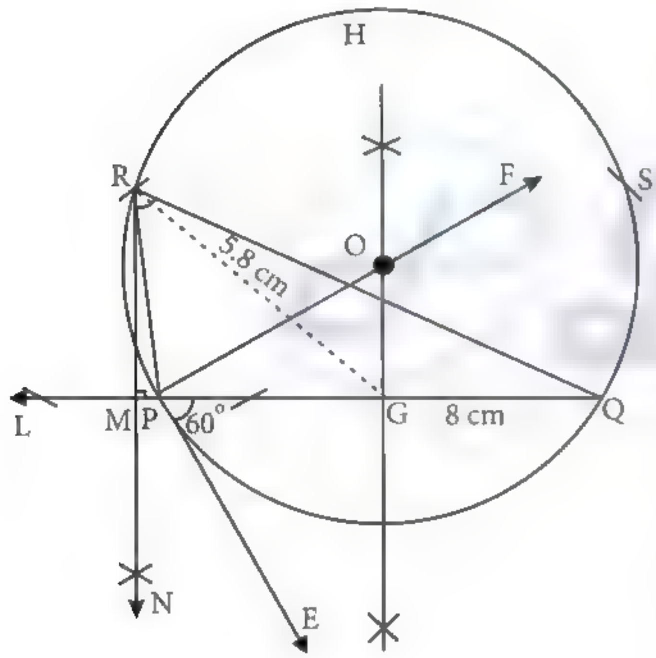
DC = 6 – x = 6 – 2.5 = 3.5 cm

4.17 **Construct a ΔPQR in which PQ = 8 cm, $\angle R = 60^\circ$ and the median RG from R to PQ is 5.8 cm. Find the length of the altitude from R to PQ.**

Sol :



Rough diagram



Construction :

Step 1: Draw a line segment PQ = 8 cm.

Step 2: At P, draw PE such that $\angle QPE = 60^\circ$.

Step 3: At P, draw PF such that $\angle EPF = 90^\circ$.

Step 4: Draw the perpendicular bisector to PQ, which intersects PF at O and PQ at G.

Step 5: With O as centre and OP as radius draw a circle.

Step 6: From G mark arcs of radius 5.8 cm on the circle. Mark them as R and S.

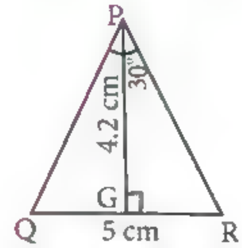
Step 7: Join PR and RQ. Then ΔPQR is the required triangle.

Step 8: From R drawn a line RN which perpendicular to LQ. LQ meets RN at M.

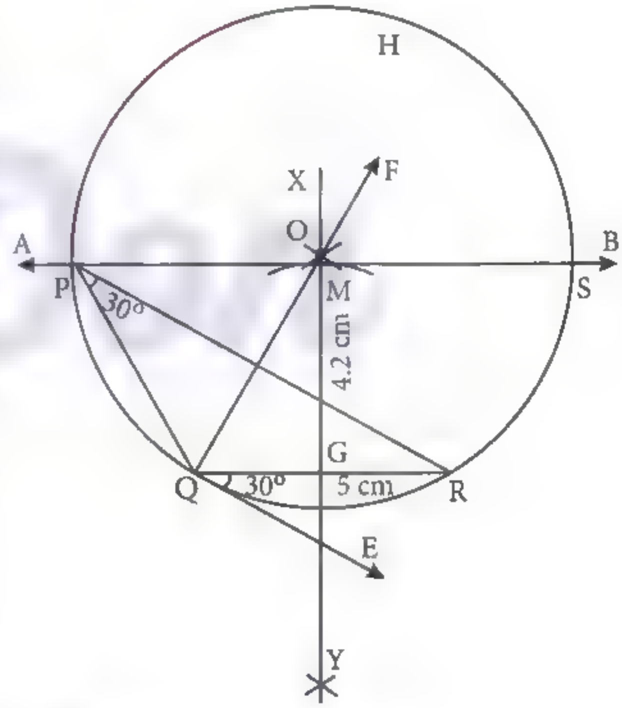
Step 9: The length of the altitude is RM = 3.5 cm.

4.18 **Construct a triangle ΔPQR such that QR = 5 cm, $\angle P = 30^\circ$ and the altitude from P to QR is of length 4.2 cm.**

Sol :



Rough diagram



Construction:

Step 1: Draw a line segment QR = 5 cm.

Step 2: At Q draw QE such that $\angle RQE = 30^\circ$.

Step 3: At Q draw QF such that $\angle EQF = 90^\circ$.

Step 4: Draw the perpendicular bisector XY to QR which intersects QF at O and QR at G.

Step 5: With O as centre and OQ as radius draw a circle.

Step 6: XY intersects QR at G. On XY, from G mark an arc at M, such that GM = 4.2 cm.

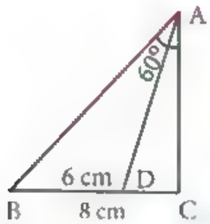
Step 7: Draw AB through M which is parallel to QR.

Step 8: AB meets the circle at P and S.

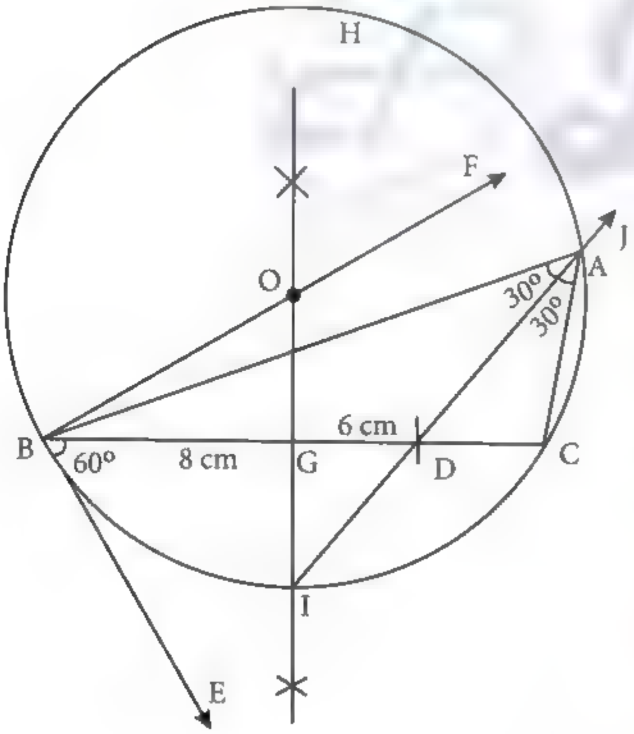
Step 9: Join QP and RP. Then ΔPQR is the required triangle.

4.19 Draw a triangle ABC of base BC = 8 cm, $\angle A = 60^\circ$ and the bisector of $\angle A$ meets BC at D such that BD = 6 cm.

Sol :



Rough diagram



Construction:

Step 1: Draw a line segment BC = 8 cm.

Step 2: At B, draw BE such that $\angle CBE = 60^\circ$.

Step 3: At B, draw BF such that $\angle EBF = 90^\circ$.

Step 4: Draw the perpendicular bisector to BC, which intersects BF at O and BC at G.

Step 5: With O as centre and OB as radius draw a circle.

Step 6: From B, mark an arc of 6 cm on BC at D.

Step 7: The perpendicular bisector intersects the circle at I. Joint ID.

Step 8: ID produced meets the circle at A. Now join AB and AC.

Then ΔABC is the required triangle.

Progress Check

1. A straight line drawn _____ to a side of a triangle divides the other two sides proportionally.
   Ans : Parallel

2. Basic Proportionality Theorem is also known as _____.
   Ans : Thales Theorem

3. Let ΔABC be equilateral. Using Angle Bisector Theorem, $\frac{BD}{DE}$ is _____ where D is a point on BC and AD is the internal bisector of $\angle A$.
   Ans : 1

4. The _____ of an angle of a triangle divides the opposite side internally in the ratio of the corresponding sides containing the angle.
   Ans : Internal bisector

5. If the median AD to the side BC of a ΔABC is also an angle bisector of $\angle A$ then $\frac{AB}{AC}$ is _____.
   Ans : 1

Exercise 4. 2

1. In ΔABC, D and E are points on the sides AB and AC respectively such that DE || BC

(i) If $\frac{AD}{DB} = \frac{3}{4}$ and AC = 15 cm find AE.

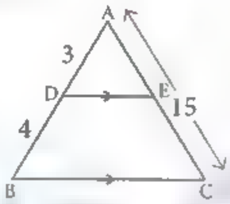
(ii) If AD = 8x – 7, DB = 5x – 3, AE = 4x – 3 and EC = 3x – 1, find the value of x.

Sol :

(i) Given $\frac{AD}{DB} = \frac{3}{4}$

$$AC = 15 \text{ cm}$$
$$EC = AC - AE$$
$$= 15 - AE$$

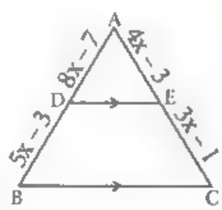
By Basic proportionality theorem.



We have $\frac{AD}{DB} = \frac{AE}{EC}$

$$\frac{3}{4} = \frac{AE}{AC - AE}$$
$$\frac{3}{4} = \frac{AE}{15 - AE}$$
$$3(15 - AE) = 4AE$$
$$45 - 3AE = 4AE$$
$$45 = 4AE + 3AE$$
$$7AE = 45$$
$$AE = \frac{45}{7} = 6.43 \text{ cm}$$

(ii) $\frac{AD}{DB} = \frac{AE}{EC}$



Given $AD = 8x - 7$

$DB = 5x - 3$

$AE = 4x - 3$

$EC = 3x - 1$

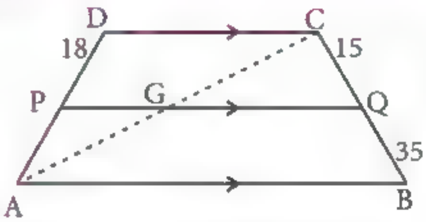
$$\frac{8x-7}{5x-3} = \frac{4x-3}{3x-1}$$
$$(8x - 7)(3x - 1) = (4x - 3)(5x - 3)$$
$$24x^2 - 21x - 8x + 7 = 20x^2 - 15x - 12x + 9$$
$$24x^2 - 20x^2 - 29x + 27x + 7 - 9 = 0$$
$$4x^2 - 2x - 2 = 0$$
$$\div \text{ by } 2 \Rightarrow 2x^2 - x - 1 = 0$$
$$2x^2 - 2x + x - 1 = 0$$
$$2x(x - 1) + 1(x - 1) = 0$$
$$(x - 1)(2x + 1) = 0$$
$$x - 1 = 0 \text{ or } 2x + 1 = 0$$
$$x = 1 \quad \text{or} \quad x = -\frac{1}{2}$$

x cannot be negative

$\therefore$ x = 1

2. ABCD is a trapezium in which AB || DC and P, Q are points on AD and BC respectively, such that PQ || DC if PD = 18 cm, BQ = 35 cm and QC = 15 cm. Find AD.

Sol :



Join AC such that it meets PQ at G.

$\therefore$ AB || DC and PQ || DC

PQ || AB

In ΔADC

PG || DC

$$\frac{AP}{PD} = \frac{AG}{GC} \quad \text{... (1)}$$

In ΔCAB

$$\frac{AG}{GC} = \frac{BQ}{QC} \quad \text{... (2)}$$

From (1) and (2) we get

$$\frac{AP}{PD} = \frac{BQ}{QC}$$
$$\frac{AP}{18} = \frac{35}{15}$$
$$AP = \frac{35 \times 18}{15}$$
$$AP = 42 \text{ cm}$$

Now AD = AP + PD

= 42 + 18 = 60 cm

AD = 60 cm

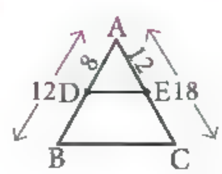
3. In a ΔABC D and E are points on the sides AB and AC respectively. For each of the following cases show that DE || BC.

(i) AB = 12 cm, AD = 8 cm, AE = 12 cm and AC = 18 cm.

(ii) AB = 5.6 cm, AD = 1.4 cm, AC = 7.2 cm and AE = 1.8 cm.

Sol :

(i)



Given AB = 12 cm, AD = 8 cm, AE = 12 cm, AC = 18 cm

By basic proportionality theorem $\frac{AD}{DB} = \frac{AE}{EC}$

$$\frac{AD}{DB} = \frac{8}{AB-AD} = \frac{8}{12-8}$$

$$\frac{AD}{DB} = \frac{8}{4} = 2 \quad \text{... (1)}$$

$$\frac{AE}{EC} = \frac{12}{AC-AE}$$

$$\frac{AE}{EC} = \frac{12}{18-12} = \frac{12}{6} = 2 \quad \text{... (2)}$$

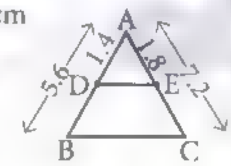
From (1) and (2)

$$\frac{AD}{DB} = \frac{AE}{EC}$$

∴ DE || BC

(ii) Given AB = 5.6 cm; AD = 1.4 cm; AC = 7.2 cm; AE = 1.8 cm

DB = AB – AD
= 5.6 – 1.4 = 4.2 cm
EC = AC – AE
= 7.2 – 1.8 = 5.4 cm



Now $\frac{AD}{DB} = \frac{1.4}{4.2}$

$$\frac{AD}{DB} = \frac{1}{3} \quad \text{... (1)}$$

$$\frac{AE}{EC} = \frac{1.8}{5.4}$$

$$\frac{AE}{EC} = \frac{1}{3} \quad \text{... (2)}$$

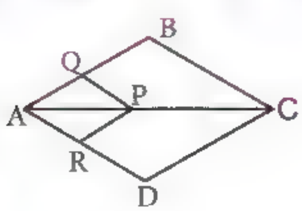
From (1) and (2)

$$\frac{AD}{DB} = \frac{AE}{EC}$$

∴ DE || BC

4. In figure if PQ || BC and PR || CD prove that

(i) $\frac{AR}{AD} = \frac{AQ}{AB}$ **(ii)** $\frac{QB}{AQ} = \frac{DR}{AR}$



Sol :

(i) In ΔABC

PQ || CB

Using Basic Proportionality theorem, we have

$$\frac{AQ}{AB} = \frac{AP}{AC} \quad \text{... (1)}$$

Again in ΔACD PR || CD

Using Basic Proportionality theorem

$$\frac{AP}{AC} = \frac{AR}{AD} \quad \text{... (2)}$$

From (1) and (2)

$$\frac{AQ}{AB} = \frac{AP}{AC} = \frac{AR}{AD}$$

Thus we have $\frac{AR}{AD} = \frac{AQ}{AB}$

(ii) From (1) and (2) we have

$$\frac{AQ}{AB} = \frac{AR}{AD}$$

$$\frac{AB}{AQ} = \frac{AD}{AR}$$

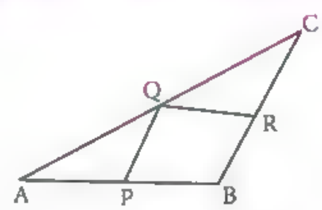
$$\frac{AQ+QB}{AQ} = \frac{AR+RD}{AR}$$

$$1 + \frac{QB}{AQ} = 1 + \frac{RD}{AR}$$

$$\Rightarrow \frac{QB}{AQ} = \frac{DR}{AR}$$

5. Rhombus PQRB is inscribed in ΔABC such that $\angle B$ is one of its angle. P, Q and R lie on AB, AC and BC respectively. If AB = 12 cm and BC = 6 cm, find the sides PQ, RB of the rhombus.

Sol :



Given AB = 12 cm, BC = 6 cm

In ΔAPQ and ΔQRC

$\angle QAP = \angle CQR$

(∵ PB || QR, corresponding angles)

$\angle PQA = \angle RCQ$

(∵ PQ || BR, corresponding angles)

∴ By AA criterion of similarity, we have

$\Delta APQ \sim \Delta QRC$

$$\Rightarrow \frac{AP}{QR} = \frac{PQ}{RC} = \frac{AQ}{QC}$$

$$\frac{AP}{QR} = \frac{PQ}{RC}$$

$$\frac{PQ}{AP} = \frac{RC}{QR} \quad \text{... (1)}$$

Now in ΔAPQ and ΔABC, we have

$$\angle CAB = \angle QAP \quad \text{(common)}$$

$$\angle AQP = \angle ACB$$

$\therefore$ By AA criterion of similarity,

we have $\Delta APQ \sim \Delta ABC$

$$\frac{AP}{AB} = \frac{PQ}{BC} = \frac{AQ}{AC}$$

$$\frac{AP}{AB} = \frac{PQ}{BC}$$

$$\frac{PQ}{AP} = \frac{BC}{AB} \quad \text{... (2)}$$

$$\frac{PQ}{AP} = \frac{6}{12}$$

Since PQRB is a rhombus, PQ = QR = RB = PB

$$\frac{PQ}{AB-PB} = \frac{6}{12}$$

$$\frac{PQ}{AB-PQ} = \frac{6}{12}$$

$$\frac{PQ}{12-PQ} = \frac{6}{12}$$

$$12\,PQ = 6(12 - PQ)$$

$$12\,PQ = 72 - 6\,PQ$$

$$12\,PQ + 6\,PQ = 72$$

$$18\,PQ = 72$$

$$PQ = \frac{72}{18} = 4$$

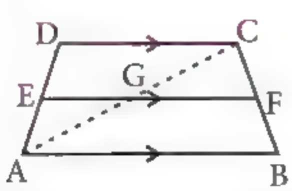
PQ = 4 cm

Since PQ = RB we have

PQ = RB = 4 cm

6. In trapezium ABCD, AB || DC, E and F are points on non-parallel sides AD and BC respectively, such that EF || AB. Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

Sol :



Given: ABCD is a trapezium in which DC || AB and EF || AB

To prove : $\frac{AE}{ED} = \frac{BF}{FC}$

Construction: Join AC meeting EF at G

Proof:

In ΔADC, we have

EG || DC

$$\Rightarrow \frac{AE}{ED} = \frac{AG}{GC} \quad \text{[By thales theorem] ..... (1)}$$

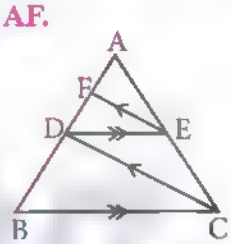
In ΔABC, we have

$$\frac{AG}{GC} = \frac{BF}{FC} \quad \text{[By thales theorem] ... (2)}$$

From (1) and (2), we get

$$\frac{AE}{ED} = \frac{BF}{FC}$$

7. In figure DE || BC and CD || EF. Prove that $AD^2 = AB \times AF$.



Sol :

In ΔABC, we have DE || BC

$$\frac{AB}{AD} = \frac{AC}{AE} \quad \text{[By Thales Theorem] ... (1)}$$

In ΔADC, we have

$$\frac{AD}{AF} = \frac{AC}{AE} \quad \text{[By Thales Theorem] ... (2)}$$

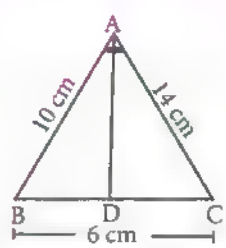
From (1) and (2) we get

$$\frac{AB}{AD} = \frac{AD}{AF}$$

$$AD^2 = AB \times AF$$

8. In a ΔABC, AD is the bisector of $\angle A$ meeting side BC at D, if AB = 10 cm, AC = 14 cm and BC = 6 cm, find BD and DC.

Sol :



By angle bisector theorem, we have

$$\frac{AB}{AC} = \frac{BD}{CD} \quad \text{... (1)}$$

$$\frac{10}{14} = \frac{BD}{CD}$$

[$\because$ BC = BD + CD = 6; DC = 6 – BD]

$$\frac{10}{14} = \frac{BD}{6-BD}$$

$$\frac{5}{7} = \frac{BD}{6-BD}$$

$$5(6-BD) = 7\,BD$$

$$30 - 5\,BD = 7\,BD$$

$$30 = 7\,BD + 5\,BD$$

$$30 = 12\,BD$$

$$BD = \frac{30}{12} = \frac{5}{2}$$

$$BD = 2.5 \text{ cm}$$

Also from (1)

$$\frac{10}{14} = \frac{6-CD}{CD}$$

$$\frac{5}{7} = \frac{6-CD}{CD}$$

$$5\,CD = 42 - 7\,CD$$

$$5\,CD + 7\,CD = 42$$

$$12\,CD = 42$$

$$CD = \frac{42}{12} = \frac{7}{2} = 3.5 \text{ cm}$$

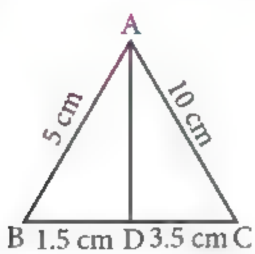
$$CD = 3.5 \text{ cm}$$

9. Check whether AD is bisector of $\angle A$ of ΔABC in each of the following

(i) AB = 5 cm, AC = 10 cm, BD = 1.5 cm and CD = 3.5 cm.

(ii) AB = 4 cm, AC = 6 cm, BD = 1.6 cm and CD = 2.4 cm.

Sol :



(i) We have AB = 5 cm, AC = 10 cm
BD = 1.5 cm, CD = 3.5 cm

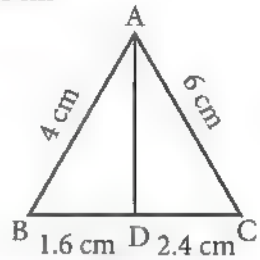
$$\frac{AB}{AC} = \frac{5}{10} = \frac{1}{2}$$

$$\frac{BD}{DC} = \frac{1.5}{3.5} = \frac{3}{7}$$

$$\frac{AB}{AC} \neq \frac{BD}{DC}$$

$\therefore$ By the converse of the angle bisector theorem, AD is not a bisector of $\angle A$

(ii) We have AB = 4 cm, AC = 6 cm,
BD = 1.6 cm, CD = 2.4 cm



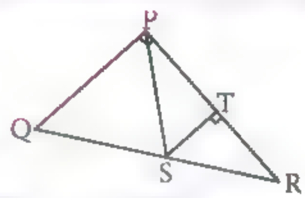
$$\frac{AB}{AC} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{BD}{DC} = \frac{1.6}{2.4} = \frac{2}{3}$$

$$\frac{AB}{AC} = \frac{BD}{DC}$$

$\therefore$ By the converse of the Angle Bisector theorem AD is the bisector of $\angle A$

10. In figure $\angle QPR = 90°$, PS is its bisector. If ST $\perp$ PR, prove that ST × (PQ + PR) = PQ × PR.



Sol :

Given that PS is the bisector of $\angle P$ of ΔPQR

$$\frac{PQ}{PR} = \frac{QS}{SR} \quad \text{[By Angle Bisector Theorem]}$$

Adding 1 on both the sides

$$\frac{PQ}{PR} + 1 = \frac{QS}{SR} + 1$$

$$\frac{PQ + PR}{PR} = \frac{QS + SR}{SR}$$

$$\frac{PQ + PR}{PR} = \frac{QR}{SR} \quad \text{... (1)}$$

In ΔRST and ΔRQP we have

$$\angle SRT = \angle QRP = \angle R \quad \text{[common]}$$

$$\angle QPR = \angle STR = 90°$$

By AA criterion for similarity, we have
$\Delta RST \sim \Delta RQP$

$$\frac{RS}{RQ} = \frac{ST}{QP}$$

$$\frac{QP}{ST} = \frac{QR}{RS} \quad \text{... (2)}$$

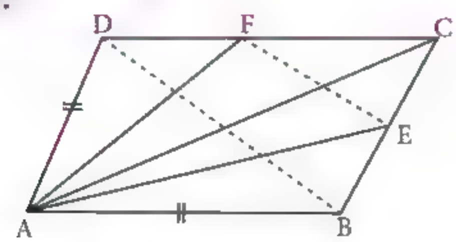
From (1) and (2)

$$\frac{QP}{ST} = \frac{PQ + PR}{PR}$$

$$PQ \times PR = ST\,(PQ + PR)$$

11. ABCD is a quadrilateral in which AB = AD, the bisector of $\angle BAC$ and $\angle CAD$ intersect the sides BC and CD at the point E and F respectively. Prove that EF || BD.

Sol :



Given: A quadrilateral ABCD in which AB = AD and the bisectors of $\angle BAC$ and $\angle CAD$ meet the sides BC and CD at E and F respectively.

To prove: EF || BD

Construction: Join AC, BD and EF.

Proof: In ΔCAB, AE is the bisector of $\angle BAC$

$$\therefore \quad \frac{AC}{AB} = \frac{CE}{BE} \quad \text{... (1)}$$

In ΔACD, AF is the bisector of $\angle CAD$

$$\frac{AC}{AD} = \frac{CF}{DF}$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{CF}{DF} \quad \text{... (2)} \quad [\because AD = AB]$$

From (1) and (2) we get

$$\frac{CE}{BE} = \frac{CF}{DF}$$

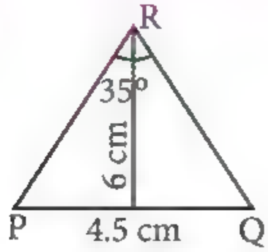
$$\Rightarrow \quad \frac{CE}{EB} = \frac{CF}{FD}$$

Thus in ΔCBD, E and F divide the sides CB and CD respectively in the same ratio.

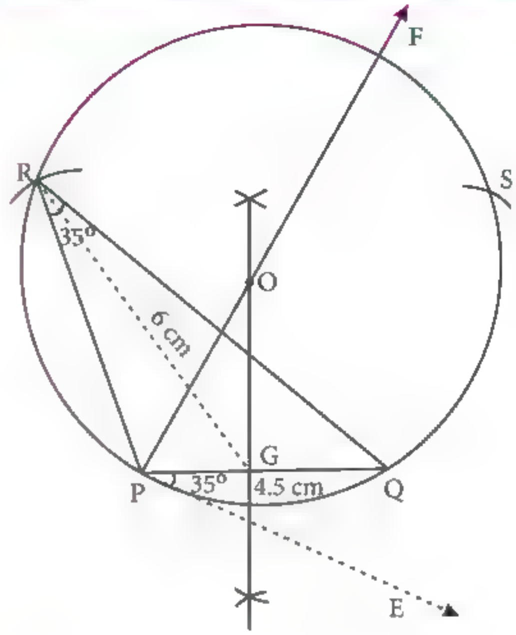
$\therefore$ By the converse of Thales theorem, we have EF || BD.

12. Construct a ΔPQR which the base PQ = 4.5 cm, $\angle R = 35°$ and the median from R to PQ is 6 cm.

Sol :



Rough Diagram



Construction:

Step 1: Drawn a line segment PQ = 4.5 cm

Step 2: At P, drawn PE such that $\angle QPE = 35°$

Step 3: At P, drawn PF such that $\angle EPF = 90°$

Step 4: Drawn the perpendicular bisector to PQ, which intersects PF at O and PQ at G

Step 5: With O as center and OP as radius drawn a circle

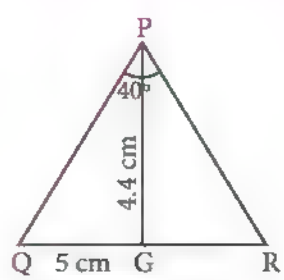
Step 6: From G, marked arcs of radius 6 cm on the circle marked them as R and S.

Step 7: Joined PR and RQ. Then ΔPQR is the required triangle.

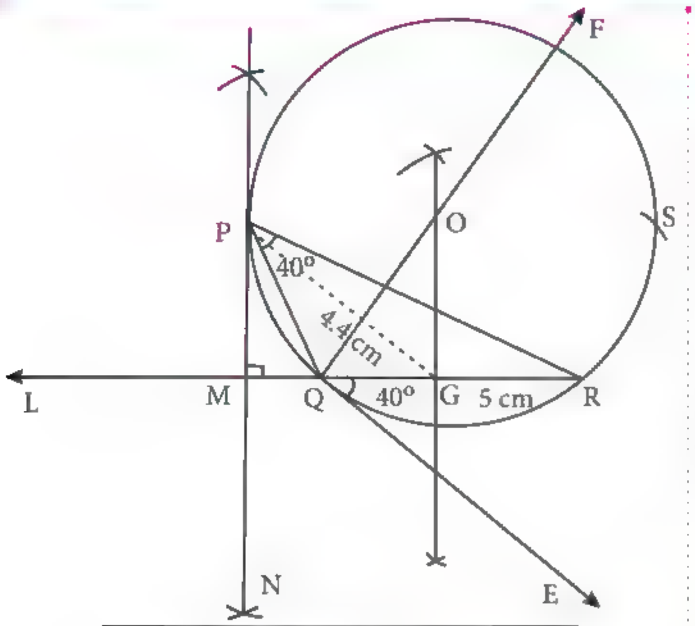
Step 8: ΔPQS is also another required triangle.

13. Construct a ΔPQR in which QR = 5 cm, $\angle P = 40°$ and the median PG from P to QR is 4.4 cm. Find the length of the altitude from P to QR.

Sol :



Rough Diagram



Length of the altitude = PM = 2.1 cm.

Construction:

Step 1: Drawn a line segment QR = 5 cm

Step 2: At Q, drawn QE such that $\angle RQE = 40°$

Step 3: At Q, drawn AF such that $\angle EQF = 90°$

Step 4: Drawn a perpendicular bisector to QR, which intersects QF at 'O' and QR at G.

Step 5: With 'O' as center and OQ as radius drawn a circle.

Step 6: From G marked arcs of radius 4.4 cm on the circle. Marked them as P and S.

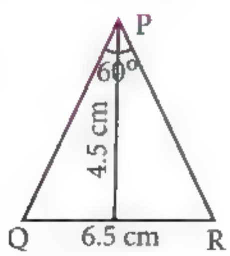
Step 7: Joined QP and PR. Now ΔPQR is the required triangle.

Step 8: From P drawn a line PN which is perpendicular to LR. LR meets PN at M.

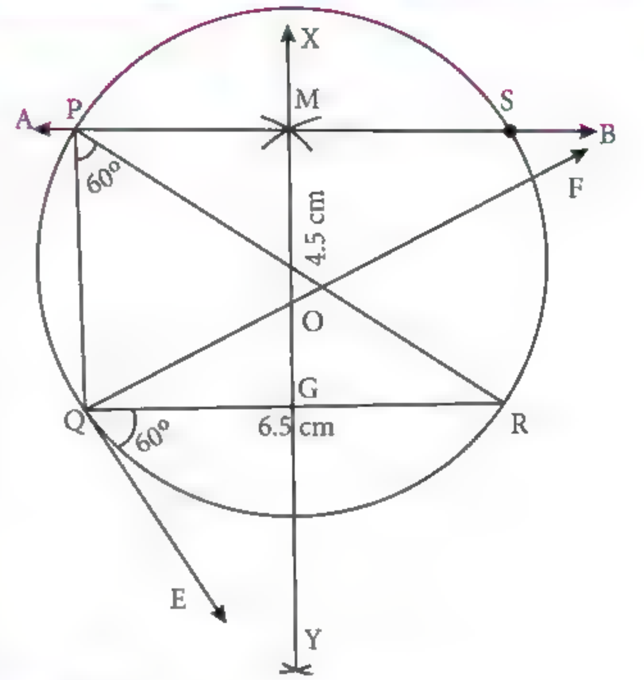
Step 9: The length of the altitude is PM = 2.1 cm.

14. Construct a ΔPQR such that QR = 6.5 cm, $\angle P = 60°$ and the altitude from P to QR is of length 4.5 cm.

Sol :



Rough diagram



Construction:

Step 1: Drawn a line segment QR = 6.5 cm

Step 2: At Q drawn QE such that $\angle RQE = 60°$

Step 3: At Q drawn QF such that $\angle EQF = 90°$

Step 4: Drawn the perpendicular bisector XY to QR which intersects QF at O and QR at G.

Step 5: With O as center and OQ as radius drawn a circle

Step 6: XY intersects QR at G. On XY, from G marked an arc at M, such that GM = 4 .5 cm

Step 7: Drawn AB through M which is parallel to QR

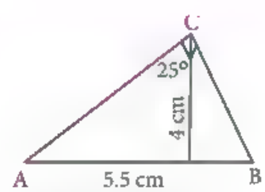
Step 8: AB meets the circle at P and S

Step 9: Joined QP and RP . Then ΔPQR is the required triangle.

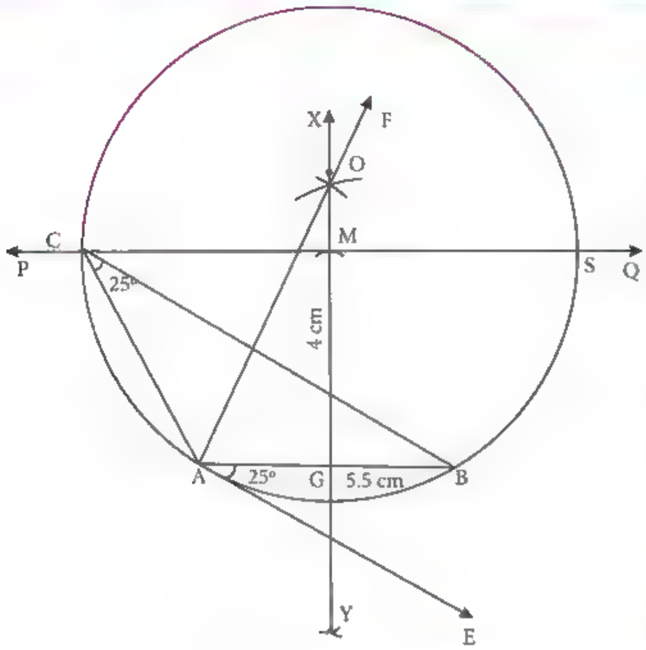
Step 10: Here ΔSQR is also another required triangle.

15. Construct a ΔABC such that AB = 5.5 cm, $\angle C = 25°$ and the altitude from C to AB is 4 cm.

Sol :



Rough diagram



Construction:

Step 1: Drawn a line segment AB = 5.5 cm

Step 2: At A drawn AE such that $\angle BAE$ = 25°

Step 3: At A drawn AF such that $\angle FAE$ = 90°

Step 4: Drawn the perpendicular bisector XY to AB which intersects AF at O and AB at G.

Step 5: With O as center and OA as radius drawn a circle

Step 6: XY intersects AB at G. On XY from G marked an arc at M such that GM = 4 cm

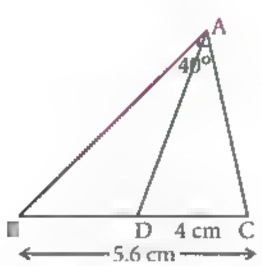
Step 7: Drawn PQ through M which is parallel to AB.

Step 8: PQ meets the circle at C and S.

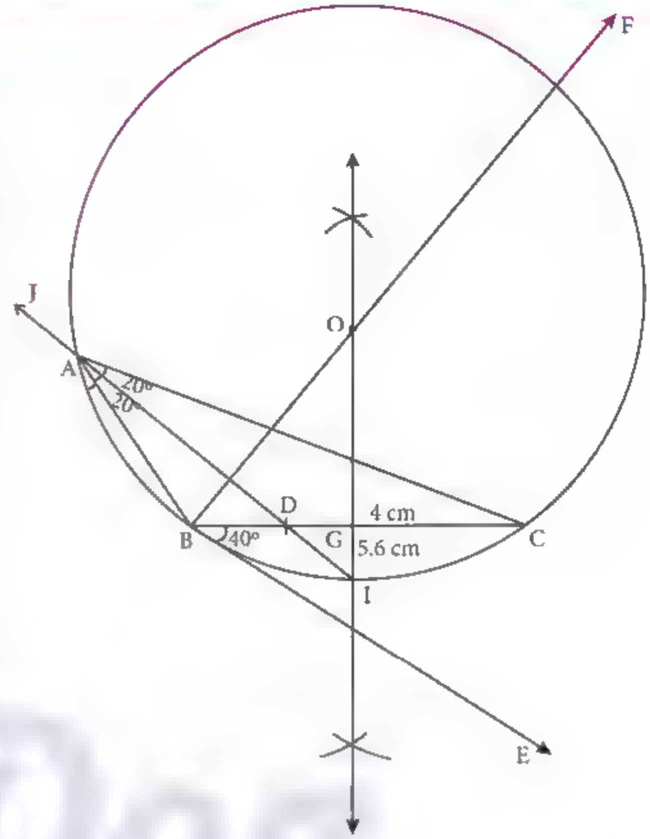
Step 9: Joined AC and BC. Now ΔABC is the required triangle.

16. Draw ■ triangle ABC of base BC = 5.6 cm, $\angle A$ = 40° and the bisector of $\angle A$ meets BC at D such that CD = 4 cm.

Sol :



Rough Diagram



Construction:

Step 1: Drawn a line segment BC = 5.6 cm

Step 2: At B, drawn BE such that $\angle CBE$ = 40°

Step 3: At B, drawn BF such that $\angle EBF$ = 90°

Step 4: Drawn the perpendicular bisector to BC, which intersects BF at O and BC at G.

Step 5: With O as center and OB as radius drawn a circle.

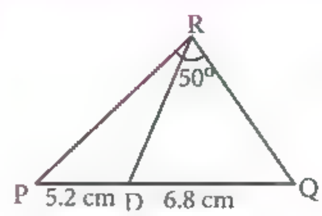
Step 6: From B, marked an arc of 4 cm on BC at D.

Step 7: The perpendicular bisector intersects the circle at I. Joined ID.

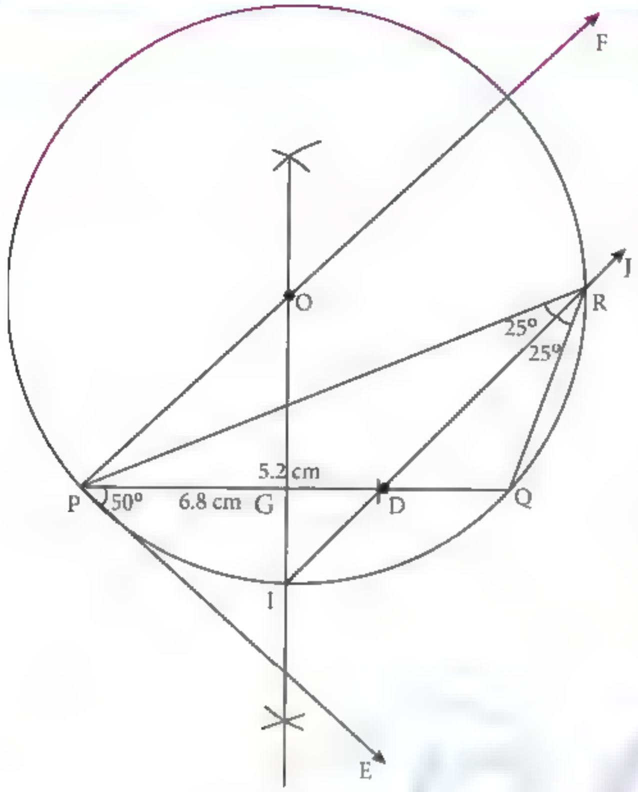
Step 8: ID produced meets the circle at A. Now joined AB and AC. Then ΔABC is the required triangle.

17. Draw ΔPQR such that PQ = 6.8 cm, vertical angle is 50° and the bisector of the vertical angle meets the base at D where PD = 5.2 cm.

Sol :



Rough Diagram



Construction:

Step 1: Drawn a line segment PQ = 6.8 cm

Step 2: At P, drawn PE such that $\angle QPE = 50°$

Step 3: At P, drawn PF such that $\angle EPF = 90°$

Step 4: Drawn the perpendicular bisector to PQ, which intersects PF at 'O' and PQ at G.

Step 5: With O as center and OP as radius drawn a circle.

Step 6: From P, marked an arc of 5.2 cm on PQ at D

Step 7: The perpendicular bisector intersects the circle at I. Joined ID

Step 8: ID produced meets the circle at R now joining PR and QR, we get the required ΔPQR

Pythagoras Theorem

Key Points

Pythagoras theorem

- Pythagoras theorem has the maximum number of proofs.
- Three numbers (a, b, c) are said to form Pythagorean Triple, if they form sides of a right triangle.
- (a, b, c) is a Pythagorean Triplet if and only if $c^2 = a^2 + b^2$.
- In a right angled triangle, the side opposite to 90° (the right angle) is called the **hypotenuse**.
- The hypotenuse will be the **longest** side of the triangle.
- Pythagoras theorem states that "In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides".
- In India pythagoras theorem is also referred as **"Baudhyana Theorem".**
- If the square of the longest side of a triangle is equal to the sums of squares of other two sides, then the triangle is a right angle triangle.

Circles and tangents

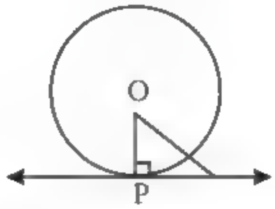
- If a line and a circle are in the same plane

  (i) the line may touch the circle at a point on it.

  (ii) the line intersects the circle at two points.

  (iii) the line may not touch the circle.

- If the line touches the circle at a point then we say the line is the **tangent** to the circle. In this case the number of point of intersection is **one.**
- If the line intersects the circle at two points then we say that the line is the secant of the circle.
- The chord of a circle is a subsection of a **secant**.
- A tangent at any point on a circle and the radius through the point are **perpendicular** to each other.



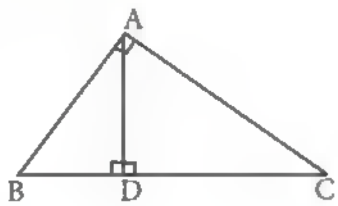
- **No tangents** can be drawn from an interior point of the circle.
- **Only one** tangent can be drawn at any point on the circle.
- **Two tangents** can be drawn from any exterior point of a circle.
- The lengths of the two tangents drawn from an exterior point to a circle are **equal.**
- If two circles touch externally, the distance between their centres is equal to the **sum of their radii.**
- It two circles touch internally, the distance between their centres is equal to the **difference of their radii.**
- The two direct common tangents drawn to the circles are **equal.**

Theorem 5: Pythagoras Theorem

Statement:

"In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides."

Proof:



Given: In ΔABC , $\angle A = 90^\circ$

To prove: $AB^2 + AC^2 = BC^2$

Construction: Draw $AD \perp BC$

Sl. No.	Statement	Reason
1	Compare $\Delta\ ABC$ and $\Delta\ ABD$ $\angle B$ is common $\angle BAC = \angle BDA = 90^\circ$ Therefore $\Delta\ ABC \sim \Delta\ ABD$ $\Rightarrow \frac{AB}{BD} = \frac{BC}{AB}$ $\Rightarrow AB^2 = BC \times BD$ ... (1)	Given $\angle BAC = 90^\circ$ and by construction $\angle BDA = 90^\circ$ BA AA similarity

2	Compare $\Delta\, ABC$ and $\Delta\, ADC$ $\angle C$ is common $\angle BAC = \angle ADC = 90^\circ$ Therefore, $\Delta\, ABC \sim \Delta\, ADC$ $\frac{BC}{AC} = \frac{AC}{DC}$ $AC^2 = BC \times DC$ ...(2)	Given $\angle BAC = 90^\circ$ and by construction $\angle CDA = 90^\circ$ By AA similarity

Adding (1) and (2) we get

$$AB^2 + AC^2 = BC \times BD + BC \times DC$$
$$= BC\,[BD + DC] = BC \times BC$$
$$AB^2 + AC^2 = BC^2$$

Hence the theorem is proved.

Converse of pythagoras Theorem

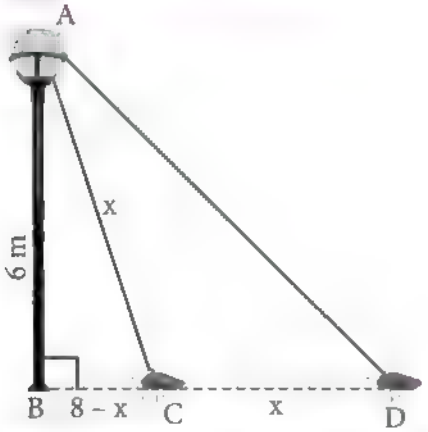
Statement

"If the square of the longest side of a triangle is equal to sums of squares of other two sides, then the triangle is a right angled triangle."

Worked Examples

4.20 **An insect 8 m away initially from the foot of a lamp post which is 6 m tall, crawls towards it moving through a distance. If its distance from the top of the lamp post is equal to the distance it has moved, how far is the insect away from the foot of the lamp post?**

Sol :



Distance between the insect and the foot of the lamp post = BD = 8 m

The height of the lamp post AB = 6 m

After moving a distance of x m, let the insect be at C

Let, AC = CD = x.

Then BC = BD – CD = 8 – x

In ΔABC, $\angle B = 90^\circ$

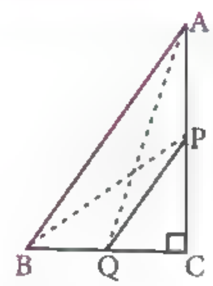
$$AC^2 = AB^2 + BC^2 \Rightarrow x^2 = 6^2 + (8 - x)^2$$
$$x^2 = 36 + 64 - 16x + x^2$$
$$16x = 100 \Rightarrow x = 6.25$$

Then, BC = 8 – x = 8 – 6.25 = 1.75 m

Therefore the insect is 1.75 m away from the foot of the lamp post.

4.21 **P and Q are the mid-points of the sides CA and CB respectively of a ΔABC, right angled at C. Prove that $4(AQ^2 + BP^2) = 5AB^2$.**

Sol :



Since, ΔAQC is a right triangle at C,

$$AQ^2 = AC^2 + QC^2 \quad \text{... (1)}$$

Since, ΔBPC is a right triangle at C,

$$BP^2 = BC^2 + CP^2 \quad \text{... (2)}$$

From (1) and (2),

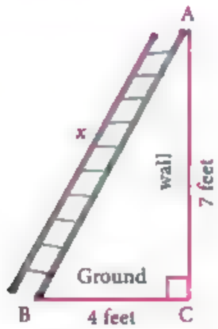
$$AQ^2 + BP^2 = AC^2 + QC^2 + BC^2 + CP^2$$
$$4(AQ^2 + BP^2) = 4AC^2 + 4QC^2 + 4BC^2 + 4CP^2$$
$$= 4AC^2 + (2QC)^2 + 4BC^2 + (2CP)^2$$
$$= 4AC^2 + BC^2 + 4BC^2 + AC^2$$

(Since P and Q are mid points)

$$= 5(AC^2 + BC^2)$$
$$4(AQ^2 + BP^2) = 5AB^2$$
(By Pythagoras Theorem)

4.22 **What length of ladder is needed to reach a height of 7 ft along the wall when the base of the ladder is 4 ft from the wall? Round off your answer to the next tenth place.**

Sol :



Let x be the length of the ladder, BC = 4 ft,

AC = 7 ft.

By Pythagoras theorem we have,

$$AB^2 = AC^2 + BC^2$$
$$x^2 = 7^2 + 4^2 \Rightarrow x^2 = 49 + 16$$
$$x^2 = 65 \Rightarrow x = \sqrt{65}$$

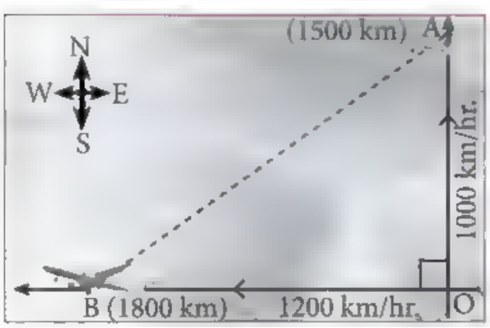
The number $\sqrt{65}$ is between 8 and 8.1.

$$8^2 = 64 < 65 < 65.61 = 8.1^2$$

Therefore, the length of the ladder is approximately 8.1 ft.

4.23 **An Aeroplane leaves an airport and flies due north at a speed of 1000 km/hr. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km/hr. How far apart will be the two planes after $1\frac{1}{2}$ hours?**

Sol :



Let the first aeroplane starts from O and goes upto A towards north,

(Distance = Speed × time)

$$\text{where OA} = \left(1000 \times \frac{3}{2}\right) \text{km} = 1500 \text{ km}$$

Let the second aeroplane starts from O at the same time and goes upto B towards west,

$$\text{where OB} = \left(1200 \times \frac{3}{2}\right) \text{km} = 1800 \text{ km}$$

The required distance = BA. In a right angled triangle AOB, $AB^2 = OA^2 + OB^2$

$$AB^2 = (1500)^2 + (1800)^2 = 100^2 (15^2 + 18^2)$$
$$= 100^2 \times 549 = 100^2 \times 9 \times 61$$
$$AB = 100 \times 3 \times \sqrt{61} = 300\sqrt{61} \text{ kms.}$$

Progress Check

1. **_____ is the longest side of the right angled triangle.**

   Ans : Hypotenuse

2. **In India, Pythagoras theorem is called as _____**

   Ans : Baudhayana Theorem

3. **The first theorem in mathematics is _____**

   Ans : Thales Theorem

4. **If the square of the longest side of a triangle is equal to sums of squares of other two sides, then the triangle is _____**

   Ans : Right angled Triangle

5. **State True or False**

   (i) Pythagoras Theorem is applicable to all triangles.

   Ans : False

   (ii) One side of a right angled triangle must be a multiple of 4.

   Ans : True

Thinking Corner

1. **Write down any five Pythagorean triplets?**

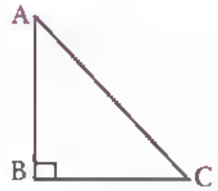
   Ans : (3, 4, 5), (9, 12, 15), (12, 16, 20), (8, 15, 17) (15, 20, 25)

2. **In a right angle triangle the sum of other two angles is _____**

   Ans : 90^o.

3. Can all the three sides of a right angled triangle be odd numbers? Why?

Ans :



No, All the three sides of a right angled triangle cannot be odd numbers.

If ΔABC is right angled triangle with $\angle B = 90°$, then $AC^2 = AB^2 + BC^2$

If AB and BC are odd, then their squares AB^2 and BC^2 are odd numbers.

Their sum $AB^2 + BC^2$ is an even number.

Square root of $AB^2 + BC^2$ is also an even number.

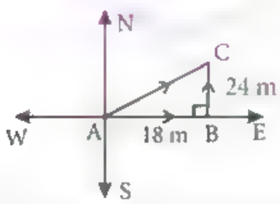
$\therefore$ AC is even.

So three measures cannot be odd.

Exercise 4.3

1. A man goes 18 m due east and then 24 m due north. Find the distance of his current position from the starting point?

Sol :



Let the initial position of the man be A and his final position be C

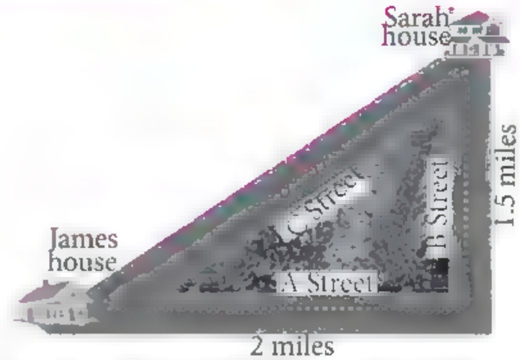
Since the man goes 18 m east and then 24 m north, ΔABC is a right angled triangle with $\angle B = 90°$; AB = 18 m and BC = 24 m

By pythagoras theorem, we have

$$AC^2 = AB^2 + BC^2$$
$$AC^2 = 18^2 + 24^2$$
$$AC^2 = 324 + 576$$
$$AC^2 = 900 = 30 \times 30$$
$$AC = 30 \text{ m}$$

$\therefore$ His current distance from starting point = 30 m

2. There are two paths that one can choose to go from Sarah's house to James house. One way is to take C street and the other way requires to take A street and then B street. How much shorter is the direct path along C street? (Using figure).



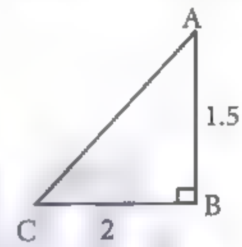
Sol :

Let Sarah's house is at 'A' and James's house is at 'B' from the picture.

Distance between Sarah's house to James house through Street B and C

= 1.5 miles + 2 miles = 3.5 miles

Distance through street C is $AC^2 = AB^2 + BC^2$



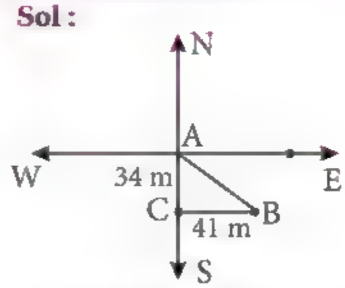
$$AC^2 = (1.5)^2 + (2)^2$$
$$= 2.25 + 4$$
$$AC^2 = 6.25$$
$$AC = \sqrt{6.25} = 2.5$$
$$AC = 2.5 \text{ miles}$$

Difference between two paths = 3.5 – 2.5 = 1 mile

$\therefore$ Direct path along C street is 1 mile shorter.

3. To get from point A to point B you must avoid walking through a pond. You must walk 34 m south and 41 m east. To the nearest meter, how many meters would be saved if it were possible to walk through the pond?

Sol :



Let A be the starting position and 'B' be the final position. C be the position south of A at 34 m distance.

Clearly $\angle C = 90°$ in ΔACB

$$AC^2 + CB^2 = AB^2$$

[By pythagoras theorem]

$$34^2 + 41^2 = AB^2$$
$$1156 + 1681 = AB^2$$
$$2837 = AB^2$$
$$AB = 53.26 \text{ m}$$

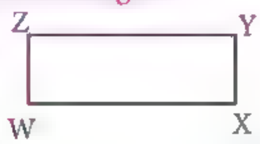
Distance from A to B through the pond = 53.26 m

Distance through C = 34 m + 41 m = 75 m

Difference = 75 – 53.26 = 21.74 m

21.74 m would be saved if it is possible to walk through the pond.

4. In the rectangle WXYZ, XY + YZ = 17 cm, and XZ + YW = 26 cm. Calculate the length and breadth of the rectangle?



Sol :

Given XY + YZ = 17 cm

XZ + YW = 26 cm

We know that diagonals if a rectangle bisect each other and the diagonals have equal length.

$$\therefore \text{Each diagonal} = \frac{26}{2} = 13 \text{ cm}$$

i.e., XZ = 13 cm and YW = 13 cm

Also given XY + YZ = 17 cm

Squaring on both sides $(XY + YZ)^2 = 17^2$

$$(XY)^2 + (YZ)^2 + 2 \times (XY) \times (YZ) = 289$$

By Pythagoras theorem $(XY)^2 + (YZ)^2 = XZ^2$

$$\therefore [XZ]^2 + 2\,(XY) \times (YZ) = 289$$
$$13^2 + 2 \times \text{length} \times \text{breadth} = 289$$
$$2 \times \text{Area} = 289 - 169$$
$$\text{Area} = \frac{289 - 169}{2} = \frac{120}{2}$$
$$\text{Area} = 60 \text{ cm}^2$$

∴ The possible length and breadth are (1, 60) (2, 30) (3, 20) (4, 15), (5, 12) (6, 10).

In this pair the length and breadth should satisfy pythagoras theorem for diagonal.

∴ 5, 12 is the possible length and breadth.

5. The hypotenuse of a right triangle is 6 m more than twice of the shortest side. If the third side is 2 m less than the hypotenuse, find the sides of the triangle?

Sol :

Let ΔABC be the right triangle with $\angle B = 90°$.

Let the shortest side of the right triangle be x m.

Then Hypotenuse AC = (2x + 6) m

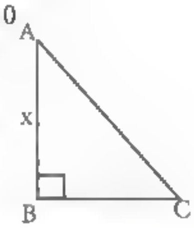
Third side BC = [(2x + 6) – 2] m

= (2x + 6 – 2) m

= (2x + 4) m

By pythagoras theorem

$$AC^2 = AB^2 + BC^2$$
$$(2x + 6)^2 = x^2 + [2x + 4]^2$$
$$4x^2 + 36 + 24x = x^2 + 4x^2 + 16 + 16x$$
$$4x^2 + 24x + 36 = 5x^2 + 16x + 16$$
$$5x^2 + 16x + 16 - 4x^2 - 24x - 36 = 0$$
$$x^2 - 8x - 20 = 0$$
$$(x + 2)(x - 10) = 0$$
$$x = -2 \text{ or } x = 10$$



Length cannot be negative.

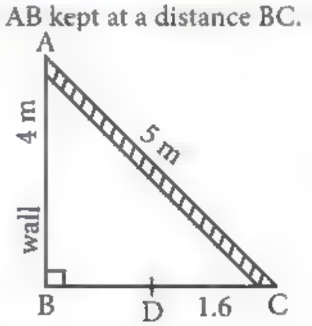
$$\therefore x = 10 \text{ m}$$
$$\therefore AB = 10 \text{ m}$$
$$BC = 2x + 4 = 2(10) + 4$$
$$= 24 \text{ m}$$
$$AC = 2x + 6 = 2(10) + 6 = 26 \text{ m}$$

∴ The sides of the triangle are 10 m, 24 m, 26 m.

6. 5 m long ladder is placed learning towards a vertical wall such that it reaches the wall at a point 4 m high. If the foot of the ladder is moved 1.6 m towards the wall, then find the distance by which the top of the ladder would slide upwards on the wall.

Sol :

Clearly the ladder AC make a right triangle with the wall AB kept at a distance BC. $\angle B = 90°$



∴ By Pythagoras theorem

$$AC^2 = AB^2 + BC^2$$
$$5^2 = 4^2 + BC^2$$
$$BC^2 = 25 - 16$$
$$BC^2 = 9$$
$$BC = 3 \text{ m}$$

If C moves 1.6 m towards the wall BC becomes

$$3\text{m} - 1.6 \text{ m} = 1.4 \text{ m}$$

Now in ΔABC

$$AC^2 = AB^2 + BC^2$$
$$5^2 = AB^2 + (1.4)^2$$
$$25 - 1.96 = AB^2$$

$$AB^2 = 23.04$$
$$AB = 4.8 \text{ m}$$

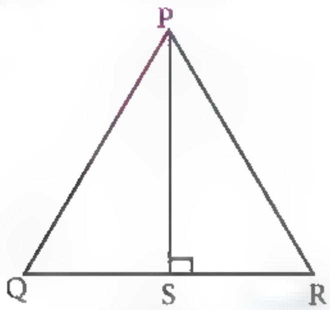
The new height of the wall = 4.8 m

Difference = 4.8 – 4 = 0.8 m

∴ The ladder would be placed 0.8 m upward the wall.

7. The perpendicular PS on the base QR of a ΔPQR intersects QR at S, such that QS = 3 SR. Prove that $2PQ^2 = 2PR^2 + QR^2$.

Sol :



We have QS = 3 SR

$$QR = QS + SR = 3SR + SR$$
$$QR = 4SR$$
$$SR = \frac{1}{4} QR \quad \text{... (a)}$$
$$SR = \frac{1}{4} QR \text{ and}$$
$$QS = 3SR = 3 \times \frac{1}{4} QR$$
$$QS = \frac{3}{4} QR \quad \text{... (1)}$$

Since ΔPSQ is a right triangle with $\angle S = 90^\circ$

$$PQ^2 = PS^2 + SQ^2 \quad \text{... (2)}$$

Similarly in ΔPSR, $\angle S = 90^\circ$

$$\therefore \quad PR^2 = PS^2 + SR^2 \quad \text{... (3)}$$

(2) – (3) $\Rightarrow$ $PQ^2 - PR^2 = SQ^2 - SR^2$

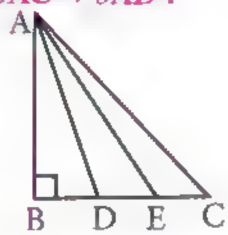
$$PQ^2 - PR^2 = \left(\frac{3}{4}QR\right)^2 - \left(\frac{1}{4}QR\right)^2$$

($\because$ from (a) and (1))

$$= \frac{9}{16}QR^2 - \frac{QR^2}{16}$$
$$= \frac{1}{16}(9QR^2 - QR^2) = \frac{8\,QR^2}{16}$$
$$PQ^2 - PR^2 = \frac{1}{2} QR^2$$
$$2(PQ^2 - PR^2) = QR^2$$
$$2PQ^2 - 2PR^2 = QR^2$$
$$2PQ^2 = 2PR^2 + QR^2$$

Hence proved.

8. In figure, ABC is a right angled triangle with right angle at [redacted] and points D, E trisect BC. Prove that $8AE^2 = 3AC^2 + 5AD^2$.



Sol :

Since D and E are the points of trisection of BC.

$$\therefore \quad BD = DE = CE$$

Let BD = DE = CE = x,

then BE = 2x and BC = 3x

In right triangles ABD, ABE and ABC, we have

$$\Rightarrow \quad AD^2 = AB^2 + BD^2$$
$$\Rightarrow \quad AD^2 = AB^2 + x^2 \quad \text{... (1)}$$

In rtΔ ABE, $AE^2 = AB^2 + BE^2$

$$\Rightarrow \quad AE^2 = AB^2 + (2x)^2$$
$$\Rightarrow \quad AE^2 = AB^2 + 4x^2 \quad \text{... (2)}$$

In rtΔ ABC, $AC^2 = AB^2 + BC^2$

$$\Rightarrow \quad AC^2 = AB^2 + (3x)^2$$
$$\Rightarrow \quad AC^2 = AB^2 + 9x^2 \quad \text{... (3)}$$

Now $8AE^2 - 3AC^2 - 5AD^2 = 8(AB^2 + 4x^2) - 3(AB^2 + 9x^2) - 5(AB^2 + x^2)$

$$= 8AB^2 + 32x^2 - 3AB^2 - 27x^2 - 5AB^2 - 5x^2$$
$$\Rightarrow 8AE^2 - 3AC^2 - 5AD^2 = 0$$
$$8AE^2 = 3AC^2 + 5AD^2$$

Hence proved.

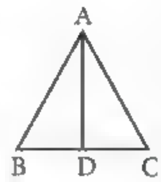
ALTERNATE SEGMENT

Key Points

Alternate segment theorem

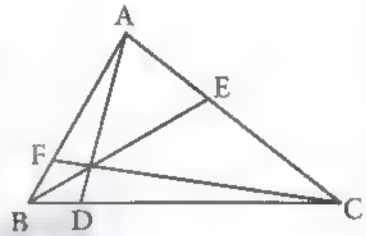
- If a line touches a circle and from the point of contact a chord is drawn, the angles between the tangent and the chord are respectively equal to the angles in the corresponding alternate segments.

Concurrency



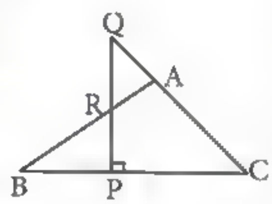
- A **cevian** is a line segment that extends from one vertex of a triangle to the opposite side. Here AD is a cevian.
- A **median** is a cevian that divides the opposite side into two congruent (equal) lengths.
- An **altitude** is a cevian that is perpendicular to the opposite side.
- An **angle bisector** is a cevian that bisects the corresponding angle.

Ceva's theorem



- Let ABC be a triangle and D, E, F be points on lines BC, CA and AB respectively.
- Then the cevians AD, BE, CF are concurrent if and only if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$ where the lengths are directed.
- The reciprocal also true in this case.
- The cevians do not necessarily lie within the triangle.

Manelaus theorem

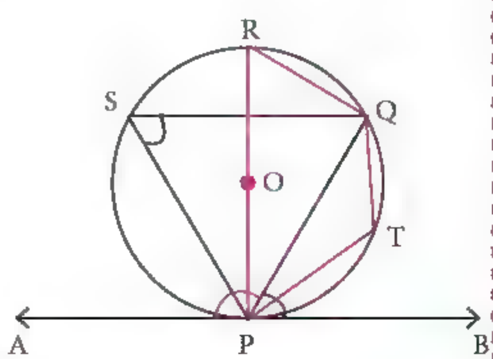


- A necessary and sufficient condition for points P, Q, R on the respective sides BC, CA, AB (or their extension) of a triangle ABC to be collinear is that $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = -1$, where all the segments are directed.
- It can also be given as BP × CQ × AR = – PC × QA × RB.
- **Centroid** is the point of concurrence of the **medians.**

Theorem 6: Alternate segment theorem

Statement

"If a line touches a circle and from the point of contact a chord is drawn, the angles between the tangent and the chord are respectively equal to the angles in the corresponding alternate segments."



Proof

Given : A circle with centre at O, tangent AB touches the circle at P and PQ is a chord. S and T are two points on the circle in the opposite sides of chord PQ.

To prove : (i) $\angle QPB = \angle PSQ$ and (ii) $\angle QPA = \angle PTQ$

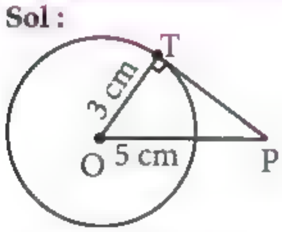
Construction : Draw the diameter POR. Draw QR, QS and PS.

Sl. No.	Statement	Reason
1	$\angle RPB = 90°$ Now, $\angle RPQ + \angle QPB = 90°$ ...(1)	Diameter RP is perpendicular to tangent AB.
2	In ΔRPQ, $\angle PQR = 90°$ ...(2)	Angle in a semicircle is 90°.
3	$\angle QRP + \angle RPQ = 90°$ ...(3)	In a right angled triangle, sum of the two acute angles is 90°.
4	$\angle RPQ + \angle QPB = \angle QRP + \angle RPQ$ $\angle QPB = \angle QRP$ ...(4)	From (1) and (3).
5	$\angle QRP = \angle PSQ$ ... (5)	Angles in the same segment are equal.
6	$\angle QPB = \angle PSQ$ ...(6)	From (4) and (5); Hence (i) is proved.
7	$\angle QPB + \angle QPA = 180°$ ...(7)	Linear pair.
8	$\angle PSQ + \angle PTQ = 180°$ ... (8)	Sum of opposite angles of a cyclic quadrilateral is 180°.
9	$\angle QPB + \angle QPA = \angle PSQ + \angle PTQ$	From (7) and (8).
10	$\angle QPB + \angle QPA = \angle QPB + \angle PTQ$	$\angle QPB = \angle PSQ$ from (6)
11	$\angle QPA = \angle PTQ$	Hence (ii) is proved.

Worked Examples

4.24 **Find the length of the tangent drawn from a point whose distance from the centre of a circle is 5 cm and radius of the circle is 3 cm.**

Sol :



Given OP = 5 cm, radius r = 3 cm

To find tangent PT = ?

In right angled ΔOTP, $\angle T = 90°$

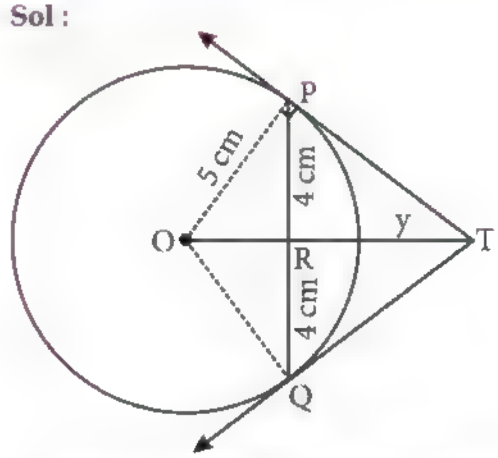
$$OP^2 = OT^2 + PT^2 \text{ (by Pythagoras theorem)}$$

$$5^2 = 3^2 + PT^2 \Rightarrow PT^2 = 25 - 9 = 16$$

Length of the tangent PT = 4 cm

4.25 **PQ is a chord of length 8 cm to a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length of the tangent TP.**

Sol :



Let TR = y. Since, OT is perpendicular bisector of PQ.

$PR = QR = 4$ cm

In ΔORP, $OP^2 = OR^2 + PR^2$

$OR^2 = OP^2 - PR^2$

$OR^2 = 5^2 - 4^2 = 25 - 16 = 9$

$\Rightarrow OR = 3$ cm

$OT = OR + RT = 3 + y$ ... (1)

In ΔPRT, $TP^2 = TR^2 + PR^2$ ... (2)

and ΔOPT we have,

$$OT^2 = TP^2 + OP^2$$

$$OT^2 = (TR^2 + PR^2) + OP^2$$

(substitute for TP^2 from (2))

$$(3 + y)^2 = y^2 + 4^2 + 5^2$$

(substitute for OT from (1))

$$9 + 6y + y^2 = y^2 + 16 + 25$$

Therefore $y = TR = \frac{16}{3}$

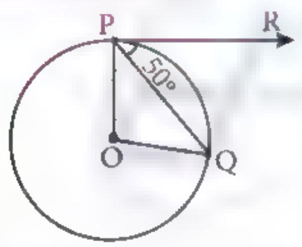
$$6y = 41 - 9 \Rightarrow y = \frac{16}{3}$$

From (2), $TP^2 = TR^2 + PR^2$

$$TP^2 = \left(\frac{16}{3}\right)^2 + 4^2 = \frac{256}{9} + 16 = \frac{400}{9}$$

$$\Rightarrow TP = \frac{20}{3} \text{ cm}$$

4.26 **In Figure, O is the centre of a circle. PQ is a chord and the tangent PR at P makes an angle of 50° with PQ. Find $\angle POQ$.**



Sol :

$\angle OPQ = 90° - 50° = 40°$

(angle between the radius and tangent is 90°)

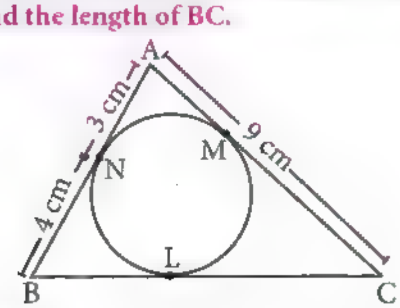
OP = OQ (Radii of a circle are equal)

$\angle OPQ = \angle OQP = 40°$ (ΔOPQ is isosceles)

$\angle POQ = 180° - \angle OPQ - \angle OQP$

$\angle POQ = 180° - 40° - 40° = 100°$

4.27 **In Figure, ΔABC is circumscribing a circle. Find the length of BC.**



Sol :

AN = AM = 3 cm (Tangents drawn from same external point are equal)

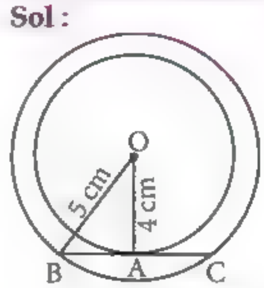
BN = BL = 4 cm;

CL = CM = AC − AM = 9 − 3 = 6 cm

$\Rightarrow$ BC = BL + CL = 4 + 6 = 10 cm

4.28 **If radii of two concentric circles are 4 cm and 5 cm then find the length of the chord of one circle which tangent to the other circle.**

Sol :



OA = 4 cm, OB = 5 cm; also $OA \perp BC$.

$OB^2 = OA^2 + AB^2$

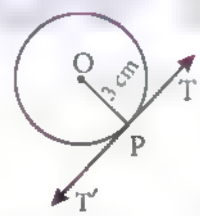
$5^2 = 4^2 + AB^2 \Rightarrow AB^2 = 9$

Therefore AB = 3 cm

$BC = 2AB \Rightarrow BC = 2 \times 3 = 6$ cm

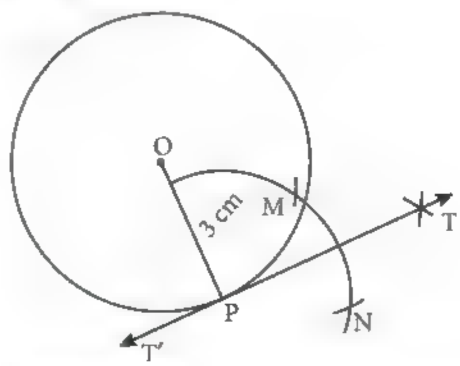
4.29 **Draw a circle of radius 3 cm. Take a point P on this circle and draw a tangent at P.**

Sol :



Rough diagram

Radius r = 3 cm



Construction:

Step 1: Draw a circle with centre at O of radius 3 cm.

Step 2: Take a point P on the circle. Join OP.

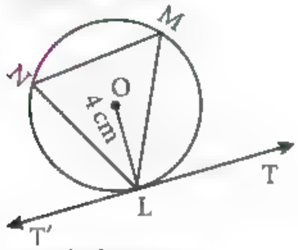
Step 3: Draw perpendicular line TT' to OP which passes through P.

Step 4: TT' is the required tangent.

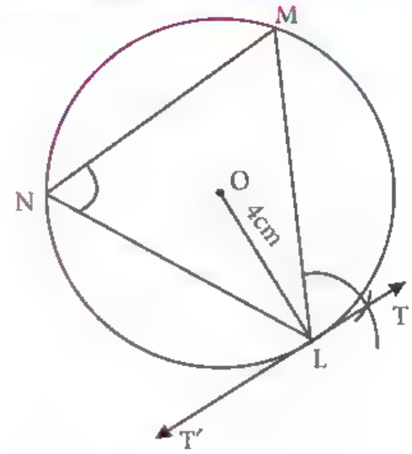
4.30 **Draw a circle of radius 4 cm. At a point L on it draw a tangent to the circle using the alternate segment.**

Sol :

Radius = 4 cm



Rough diagram



Construction:

Step 1: With O as the centre, draw a circle of radius 4 cm.

Step 2: Take a point L on the circle. Through L draw any chord LM.

Step 3: Take a point M distinct from L and N on the circle, so that L, M and N are in anti-clockwise direction. Join LN and NM.

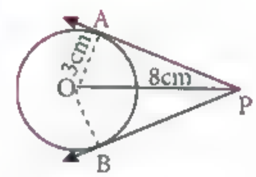
Step 4: Through L draw a tangent TT' such that $\angle TLM = \angle MNL$.

Step 5: TT' is the required tangent.

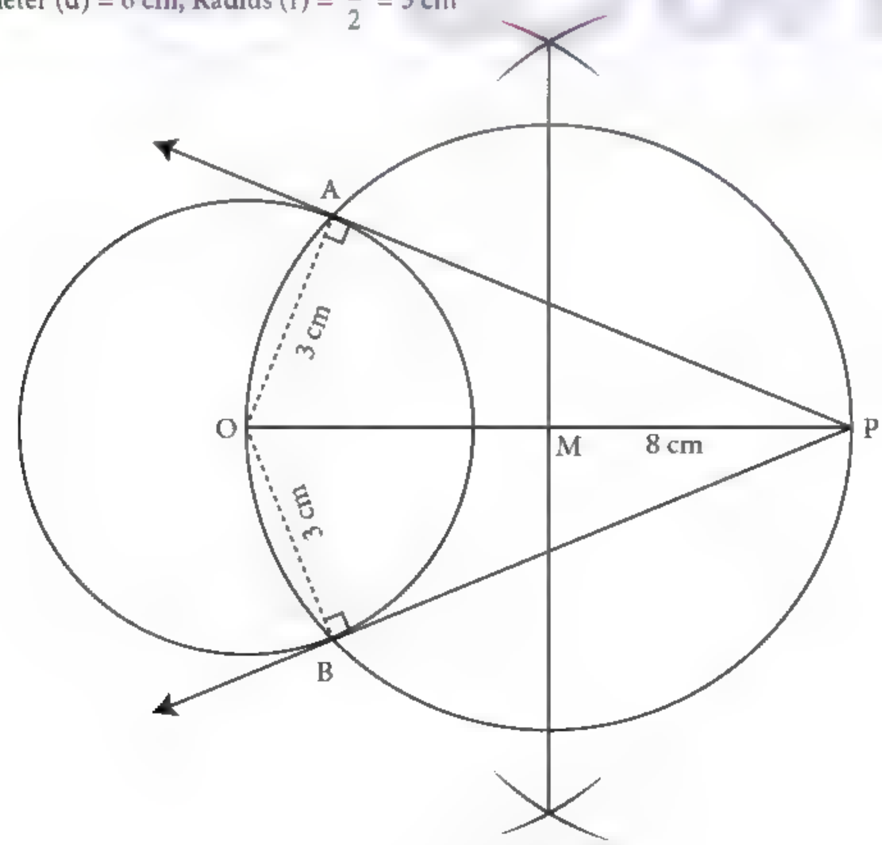
4.31 **Draw a circle of diameter 6 cm from a point P, which is 8 cm away from its centre. Draw the two tangents PA and PB to the circle and measure their lengths.**

Sol :

Diameter (d) = 6 cm, Radius (r) = $\frac{6}{2}$ = 3 cm



Rough diagram



PA = PB = 7.4 cm

Construction:

Step 1: With centre at O, draw a circle of radius 3 cm .

Step 2: Draw a line OP = 8 cm.

Step 3: Draw a perpendicular bisector of OP, which cuts OP at M.

Step 4: With M as centre and MO as radius, draw a circle which cuts the previous circle at A and B.

Step 5: Join AP and BP. AP and BP are the required tangents. Thus length of the tangents are PA = PB = 7.4 cm.

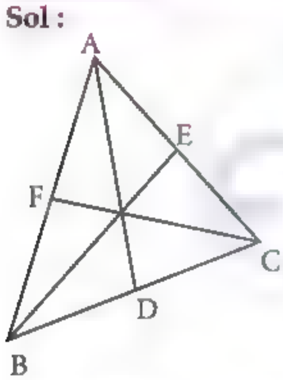
Verification: In the right angle triangle OAP,

$PA^2 = OP^2 - OA^2 = 64 - 9 = 55$

$PA = \sqrt{55} = 7.4$ cm (approximately).

4.32 **Show that in a triangle, the medians are concurrent.**

Sol :



Medians are line segments joining each vertex to the midpoint of the corresponding opposite sides.

Thus medians are the cevians where D, E, F are midpoints of BC, CA and AB respectively.

Since D is a midpoint of BC,

$$BD = DC \Rightarrow \frac{BD}{DC} = 1 \qquad \ldots (1)$$

Since, E is a midpoint of CA, CE = EA

$$\Rightarrow \frac{CE}{EA} = 1 \qquad \ldots (2)$$

Since, F is a midpoint of AB,

$$AF = FB \Rightarrow \frac{AF}{FB} = 1 \qquad \ldots (3)$$

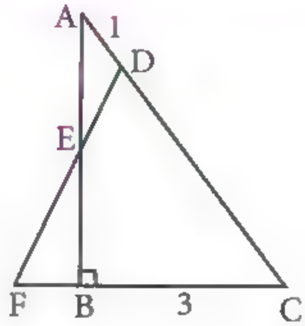
Thus, multiplying (1), (2) and (3) we get,

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1 \times 1 \times 1 = 1$$

And so, Ceva's theorem is satisfied.

Hence the Medians are concurrent.

4.33 **In Figure, ABC is a triangle with $\angle B = 90°$, BC = 3 cm and AB = 4 cm. D is point on AC such that AD = 1 cm and E is the midpoint of AB. Join D and E and extend DE to meet CB at F. Find BF.**



Sol :

Consider ΔABC . Then D, E and F are respective points on the sides CA, AB and BC. By construction D, E, F are collinear.

By Menelaus' theorem

$$\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CD}{DA} = 1 \qquad \ldots (1)$$

By assumption, AE = EB = 2, DA = 1 and

FC = FB + BC

= BF + 3

By Pythagoras theorem,

$AC^2 = AB^2 + BC^2$

$= 16 + 9 = 25.$

Therefore AC = 5

and so, CD = AC – AD = 5 – 1 = 4.

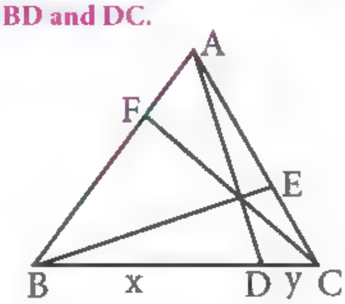
Substituting the values of FC, AE, EB, DA, CD in (1),

we get, $\frac{2}{5} \times \frac{BF}{BF+3} \times \frac{4}{1} = 1$

4BF = BF + 3

4BF – BF = 3 $\Rightarrow$ BF = 1

4.34 **Suppose AB, AC and BC have lengths 13, 14 and 15 respectively. If $\frac{AF}{FB} = \frac{2}{5}$ and $\frac{CE}{EA} = \frac{5}{8}$. Find BD and DC.**



Sol :

Given that AB = 13, AC = 14 and BC = 15.

Let BD = x and DC = y

Using Ceva's theorem,

we have, $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$ ... (1)

Substitute the value of $\frac{AF}{FB}$ and $\frac{CE}{EA}$ in (1),

we have $\frac{BD}{DC} \times \frac{5}{8} \times \frac{2}{5} = 1$

$$\frac{x}{y} \times \frac{10}{40} = 1 \Rightarrow \frac{x}{y} \times \frac{1}{4} = 1$$

$\Rightarrow$ $x = 4y$ ... (2)

$BC = 15 \Rightarrow BD + DC = 15$

$\Rightarrow$ $x + y = 15$ ... (3)

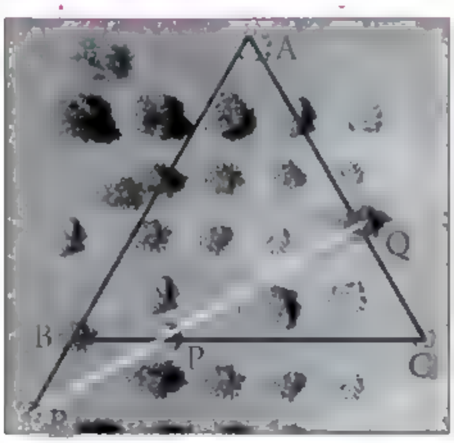
From (2), using $x = 4y$ in (3) we get,

$4y + y = 15 \Rightarrow 5y = 15 \Rightarrow y = 3$

Substitute y = 3 in (3) we get, x = 12.

Hence BD = 12, DC = 3.

4.35 In a garden containing several trees, three particular trees P, Q, R at located in the following way, BP = 2 m, CQ = 3 m, RA = 10 m, PC = 6 m, QA = 5m, RB = 2 m, where A, B, C are points such that P lies on BC, Q lies on AC and R lies on AB. Check whether the trees P, Q, R lie on a same straight line.



Sol :

By Menelaus' theorem, the trees P, Q, R will be collinear (lie on same straight line)

if $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{RA}{RB} = 1$ ... (1)

Given BP = 2 m, CQ = 3 m, RA = 10 m, PC = 6m, QA = 5 m and RB = 2 m

Substituting these values in (1) we get,

$$\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{RA}{RB} = \frac{2}{6} \times \frac{3}{5} \times \frac{10}{2} = \frac{60}{60} = 1$$

Hence the trees P, Q, R lie on a same straight line.

Progress Check

1. A straight line that touches a circle at a common point is called a _____

   Ans : tangent

2. A chord is a subsection of _____

   Ans : a secant

3. The lengths of the two tangents drawn from _____ point to a circle are equal.

   Ans : exterior

4. No tangent can be drawn from _____ of the circle.

   Ans : an interior point

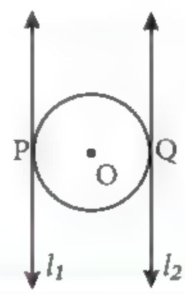
5. An _____ is a cevian that divides the angle, into two equal halves.

   Ans : Angle Bisector

Thinking Corner

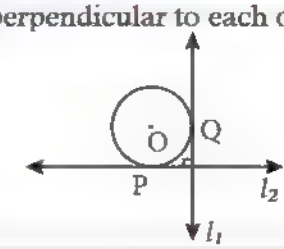
1. Can we draw two tangents parallel to each other on a circle?

   Ans : Yes. We can draw two tangents parallel to each other on a circle in two different points on the circle.



2. Can we draw two tangents perpendicular to each other on a circle?

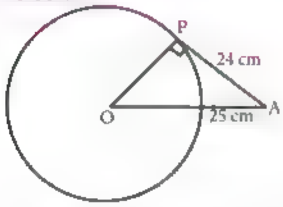
   Ans : Yes. We can draw two tangents perpendicular to each other.



Exercise 4.4

1. The length of the tangent to a circle from a point P, which is 25 cm away from the centre is 24 cm. What is the radius of the circle?

Sol :



Let 'O' be the center of the circle AP be the tangent.

Given OA = 25 cm; AP = 24 cm. OP is the radius.

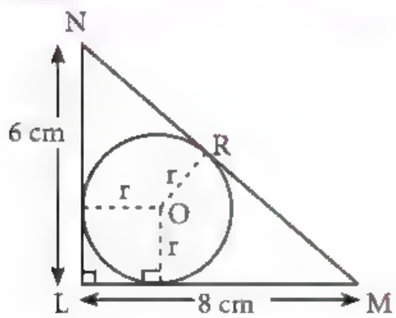
Tangent and radius through the point are perpendicular to each other.

$\therefore$ In the right triangle ΔOPA,

$$\begin{aligned} OA^2 &= OP^2 + PA^2 \\ 25^2 &= OP^2 + 24^2 \\ 625 &= OP^2 + 576 \\ OP^2 &= 625 - 576 = 49 \\ OP &= 7 \text{ cm} \\ \therefore \text{Radius} &= 7 \text{ cm} \end{aligned}$$

2. ΔLMN is a right angled triangle with $\angle L = 90°$. A circle is inscribed in it. The lengths of the sides containing the right angle are 6 cm and 8 cm. Find the radius of the circle.

Sol :



Given ΔLMN is a right angled triangle with $\angle L = 90°$

Since PL || OQ, PL = r = OP

we have NR = NP = NL – PL

[$\because$ NR and NP are tangents]

= (6 – r) cm

MR = MQ = ML – LQ = (8 – r) cm

[$\because$ MR and MQ are tangents]

NM = NR + RM

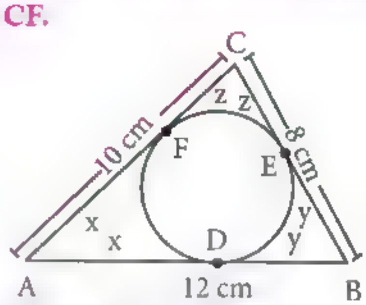
= (6 – r + 8 – r) cm

= (14 – 2r) cm

Now $NM^2 = NL^2 + LM^2$ [By Pythagoras Theorem]

$$\begin{aligned} \Rightarrow \quad (14 - 2r)^2 &= 8^2 + 6^2 \\ (14 - 2r)^2 &= 64 + 36 \\ (14 - 2r)^2 &= 100 \\ 14 - 2r &= 10 \\ -2r &= 10 - 14 \\ -2r &= -4 \\ r &= 2 \\ \therefore \text{Radius} &= 2 \text{ cm} \end{aligned}$$

3. A circle is inscribed in ΔABC having sides 8 cm, 10 cm and 12 cm as shown in figure, Find AD, BE and CF.



Sol :

We know that the tangents drawn from an external point to a circle are equal.

Let AD = AF = x

BD = BE = y

and CE = CF = z

Now AB = 12 cm, BC = 8 cm, and CA = 10 cm

$x + y = 12$; $y + z = 8$, and $z + x = 10$

$(x + y) + (y + z) + (z + x) = 12 + 8 + 10$

$2(x + y + z) = 30$

$\Rightarrow \quad x + y + z = 15$

Now $x + y = 12$ and $x + y + z = 15$

$\Rightarrow \quad 12 + z = 15$

$z = 3$

$y + z = 8$ and $x + y + z = 15$

$\Rightarrow \quad x + 8 = 15 \Rightarrow x = 7$

and $z + x = 10$ and $x + y + z = 15$

$\Rightarrow \quad y + 10 = 15$

$y = 5$

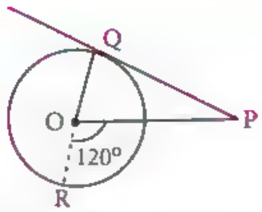
Hence AD = x = 7 cm

BE = y = 5 cm

CF = z = 3 cm

4. PQ is a tangent drawn from a point P to a circle with centre O and QOR is a diameter of the circle such that $\angle POR = 120°$. Find $\angle OPQ$.

Sol :



Given PQ is the tangent from the point P outside the circle and OQ is the radius.

We know that tangent meet the radius perpendicularly

$\therefore \angle PQO = 90°$

$\angle POQ = 180 - 120 = 60°$

[$\because \angle POQ$ and $\angle POR$ are linear pair of angles]

In ΔPQO $\angle POQ + \angle PQO + \angle OPQ = 180°$

[$\because$ sum of angles of a triangle]

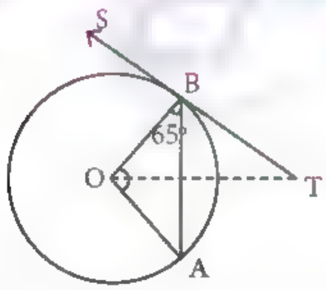
$60° + 90° + \angle OPQ = 180°$

$\angle OPQ = 180° - 150°$

$\angle OPQ = 30°$

5. A tangent ST to a circle touches it at B. AB is a chord such that $\angle ABT = 65°$. Find $\angle AOB$, where "O" is the centre of the circle.

Sol :



Given TB is the tangent from the external point T and OB-radius

$\therefore \angle OBT = 90°$

Also $\angle ABT = 65°$ (given)

$\therefore \angle OBA = \angle OBT - \angle ABT$

$= 90° - 65° = 25°$

Since $OA = OB$ [radius]

ΔABO is an isosceles triangle.

$\therefore$ Angles opposite to equal sides are equal.

$\therefore \angle OBA = \angle BAO = 25°$

Now in ΔAOB,

$\angle AOB + \angle OBA + \angle OAB = 180°$

$\angle AOB + 25° + 25° = 180°$

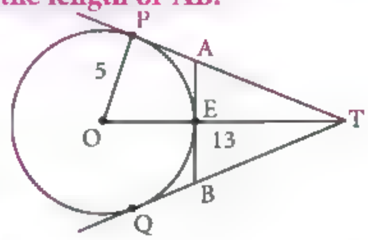
[sum of angles of a triangle]

$\angle AOB + 50° = 180°$

$\angle AOB = 180° - 50° = 130°$

$\therefore \angle AOB = 130°$

6. In figure, O is the centre of radius 5 cm. T is a point such that OT = 13 cm and OT intersects the circle E, if AB is the tangent to the circle at E, find the length of AB.



Sol :

Since OP is the radius and PT tangent.

$\angle OPT = 90°$

Applying Pythagoras theorem in ΔOPT, we have

$OT^2 = OP^2 + PT^2$

$13^2 = 5^2 + PT^2$

$PT^2 = 169 - 25$

$PT^2 = 144$

$PT = 12$ cm

Since the lengths of tangents drawn from an exterior point to a circle are equal.

$\therefore$ $AP = AE = x$ (say)

$\Rightarrow$ $AT = PT - AP = (12 - x)$ cm

Since AB is the tangent to the circle at E

$\therefore OE \perp AB$

$\Rightarrow \angle OEA = 90°$

$\Rightarrow \angle AET = 90°$

$AT^2 = AE^2 + ET^2$

[Applying pythagoras theorem in ΔAET]

$(12 - x)^2 = x^2 + (13 - 5)^2$

$144 - 24x + x^2 = x^2 + 64$

$24x = 144 - 64$

$24x = 80$

$\Rightarrow$ $3x = 10$

$x = \frac{10}{3}$ cm

Similarly $BE = \frac{10}{3}$ cm

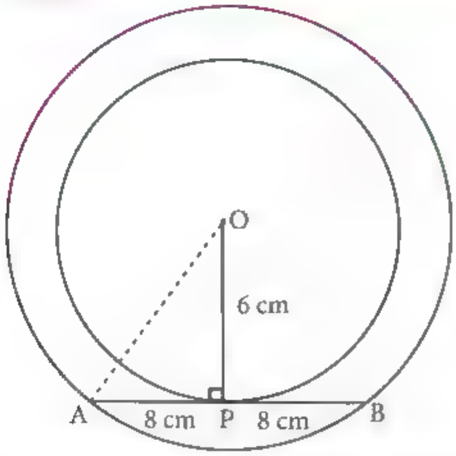
$\therefore$ $AB = AE + BE$

$= \left(\frac{10}{3} + \frac{10}{3}\right)$ cm

$AB = \frac{20}{3}$ cm

7. In two concentric circles, a chord of length 16 cm of larger circle becomes a tangent to the smaller circle whose radius is 6 cm. Find the radius of the larger circle.

Sol :



Let O be the center of concentric circles and APB be the chord of length 16 cm of the larger circle touching the smaller circle at P.

Then OP $\perp$ AB and P is the midpoint of AB.

$\therefore \quad AP = PB = 8$ cm

In ΔOPA, we have

$OA^2 = OP^2 + AP^2$ [By Pythagoras Theorem]

$OA^2 = 6^2 + 8^2$

$OA^2 = 36 + 64$

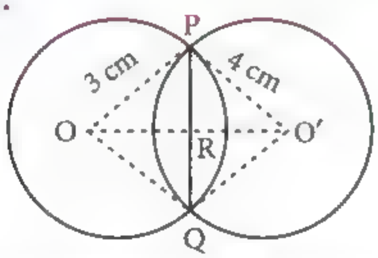
$OA^2 = 100$

$OA = 10$ cm

$\therefore$ Radius of the larger circle in 10 cm

8. Two circles with centres O and O' of radii 3 cm and 4 cm, respectively intersect at two points P and Q, such that OP and $O'P$ are tangents to the two circles. Find the length of the common chord PQ.

Sol :



Since the tangents at a point to a circle is perpendicular to the radius through the point of contact

$\therefore \angle OPO' = 90°$

In $\Delta OPO'$, we have

$OP^2 + O'P^2 = (OO')^2$

[By Pythagoras theorem]

$3^2 + 4^2 = (OO')^2$

$9 + 16 = (OO')^2$

$25 = (OO')^2$

$OO' = 5$ cm

Since the line joining the centres of two intersecting circles is perpendicular bisector of their common chord.

OR $\perp$ PQ and $O'R \perp PQ$

Also PR = QR

Let OR = x, then $O'R = 5 - x$

Also at PR = QR = y cm

In ΔORP and $\Delta O'RP$,

Applying pythagoras theorem

$OP^2 = OR^2 + RP^2$ and $O'P^2 = O'R^2 + RP^2$

$3^2 = x^2 + y^2$ and $4^2 = (5 - x)^2 + y^2$

Subtracting $\Rightarrow 4^2 - 3^2 = \{(5 - x)^2 + y^2\} - (x^2 + y^2)$

$16 - 9 = 25 - 10x + x^2 + y^2 - x^2 - y^2$

$7 = 25 - 10x$

$10x = 25 - 7$

$10x = 18$

$\Rightarrow \quad x = 1.8$ cm

$3^2 = x^2 + y^2$

$\Rightarrow \quad y = \sqrt{9 - (1.8)^2} = \sqrt{5.76}$

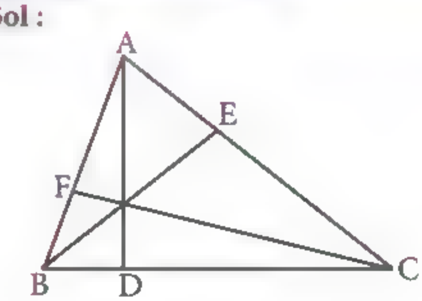
$y = 2.4$ cm

Hence PR = QR = 2.4 cm

PQ = 2y = 4.8 cm.

9. Show that the angle bisectors of a triangle are concurrent.

Sol :



Let ΔABC be B triangle points D, E, F are angular bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively. By angular bisector theorem we have

$$\frac{BD}{DC} = \frac{AB}{AC} \Rightarrow AB = \frac{BD \times AC}{DC} \quad \text{... (1)}$$

$$\frac{AC}{BC} = \frac{AF}{FB} \Rightarrow AC = \frac{AF \times BC}{FB} \quad \text{... (2)}$$

$$\frac{AE}{EC} = \frac{AB}{BC} \Rightarrow AB = \frac{AE \times BC}{EC} \quad \text{... (3)}$$

From (1) and (3), we have

$$\frac{BD \times AC}{DC} = \frac{AE \times BC}{EC} \quad \text{... (4)}$$

Now substituting (2) in (4) we have

$$\frac{BD \times \left(\frac{AF \times BC}{FB}\right)}{DC} = \frac{AE \times BC}{EC}$$

$$\frac{BD \times AF \times BC}{DC \times FB} = \frac{AE \times BC}{EC}$$

$$BD \times AF \times EC = \frac{AE \times BC \times DC \times FB}{BC}$$

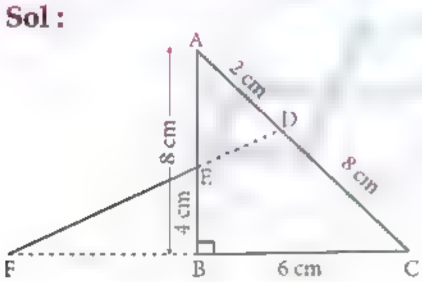
$$BD \times AF \times CE = EA \times FB \times DC$$

$$\therefore \frac{BD \times AF \times CE}{EA \times FB \times DC} = 1$$

Hence by Ceva's theorem we conclude that the angle bisectors of a triangle are concurrent.

10. In ΔABC, with $\angle B = 90°$, BC = 6 cm and AB = 8 cm, D is a point on AC such that AD = 2 cm and E is the midpoint of AB. Join D to E and extend it to meet at F. Find BF.

Sol :



Given AB = 8 cm, AD = 2 cm;

BC = 6 cm; AE = $\frac{AB}{2} = \frac{8}{2} = 4$ cm

We have ΔABC is a right triangle

$$AC^2 = AB^2 + BC^2 = 8^2 + 6^2$$
$$= 64 + 36 = 100$$
$$AC^2 = 100$$
$$AC = 10 \text{ cm}$$
$$DC = AC - AD = 10 - 2 = 8 \text{ cm}$$

In ΔABC, D, E, F are respective points on the sides CA, AB and BC.

By construction D, E, F are colinear

$\therefore$ By Menelaus' theorem $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CD}{DA} = 1$

$$\frac{4}{4} \times \frac{BF}{(FB + BC)} \times \frac{8}{2} = 1$$

$$\Rightarrow 1 \times \frac{BF}{FB + 6} \times 4 = 1$$

$$4BF = FB + 6$$

$$4FB - FB = 6$$
$$3BF = 6$$
$$BF = \frac{6}{3}$$
$$BF = 2 \text{ cm}$$

11. An artist has created a triangular stained glass window and has one strip of small length left before completing the window. She needs to figure out the length of left out portion based on the lengths of the other sides as shown in the figure.

Sol :

Clearly In ΔABC, D, E, F are points on lines BC, CA, AB respectively using Ceva's theorem, we have

$$\frac{AE}{EC} \times \frac{CD}{DB} \times \frac{BF}{FA} = 1 \quad \text{... (1)}$$

From the diagram it is clear that

AE = 3, EC = 4, CD = 10, DB = 3, FA = 5

Substituting these values in (1)

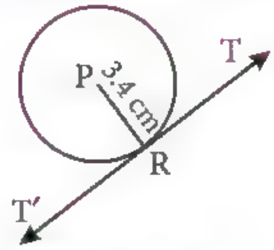
$$\frac{3}{4} \times \frac{10}{3} \times \frac{BF}{5} = 1$$

$$BF = \frac{1 \times 4 \times 3 \times 5}{3 \times 10} = 2 \text{ cm}$$

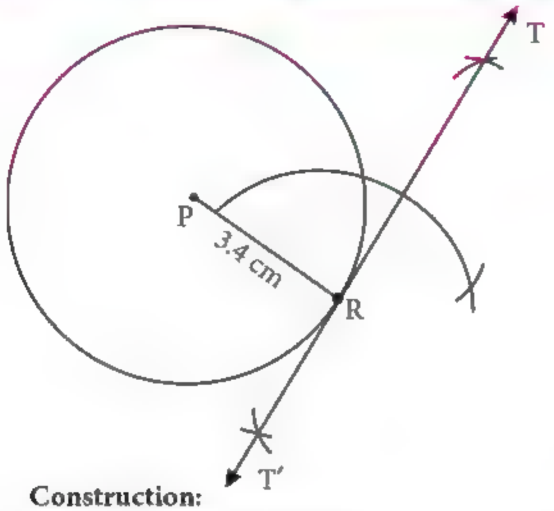
12. Draw a tangent at any point R on the circle of radius 3.4 cm and centre at P.

Sol :

Radius = 3.4 cm



Rough diagram



Construction:

Step 1: Drawn a circle with center at P of radius 3.4 cm

Step 2: Taken a point R on the circle joined PR

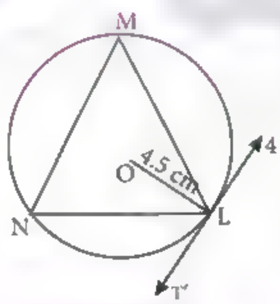
Step 3: Drawn perpendicular line TT' to PR which passes through R.

Step 4: TT' is the required tangent

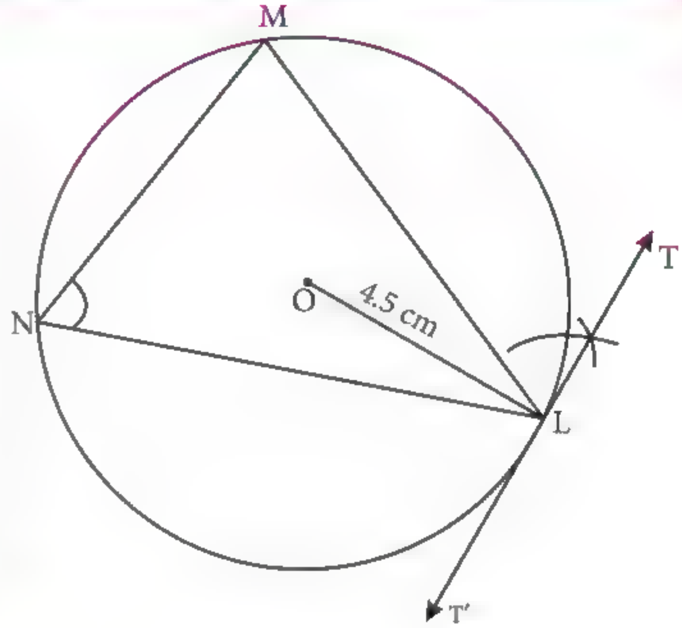
13. Draw a circle of radius 4.5 cm. Take a point on the circle. Draw the tangent at that point using the alternate-segment theorem.

Sol :

Radius = 4.5 cm



Rough diagram



Construction:

Step 1: With O as centre, drawn a circle of radius 4.5 cm

Step 2: Taken a point L on the circle through L drawn as chord LM

Step 3: Taken a point M distinct from L and N on the circle so that L, M, N are anti-clock wise direction. Joined LN and NM

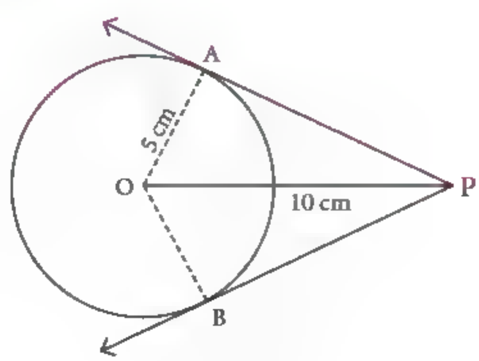
Step 4: Through 'L' drawn a tangent TT' such that $\angle TLM = \angle MNL$

Step 5: TT' is the required tangent.

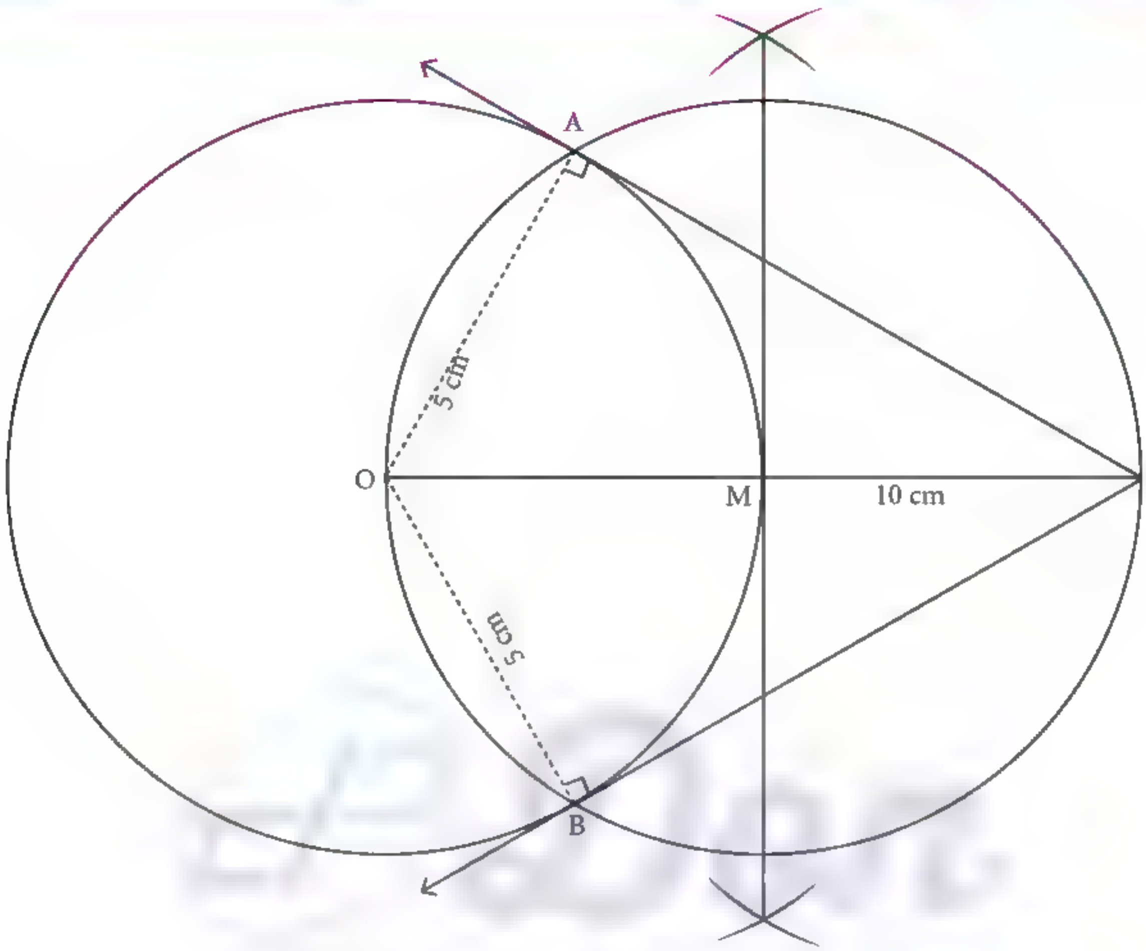
14. Draw the two tangents from a point which is 10 cm away from the centre of a circle of radius 5 cm. Also, measure the lengths of the tangents.

Sol :

Given radius r = 5 cm



Rough diagram



Length of the tangents PA = PB = 8.7 cm

Construction:

Step 1: With center at 'O', drawn a circle of radius 5 cm

Step 2: Drawn a line OP = 10 cm

Step 3: Drawn a perpendicular bisector to OP, which cuts OP at M.

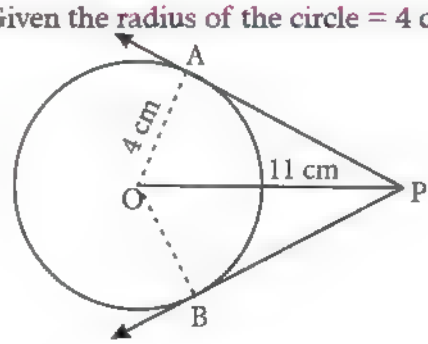
Step 4: With M as center and MO as radius, drawn a circle which cuts previous circle at A and B

Step 5: Joined AP and BP. AP and BP are the required tangents. The length of the tangents PA = PB = 8.7 cm.

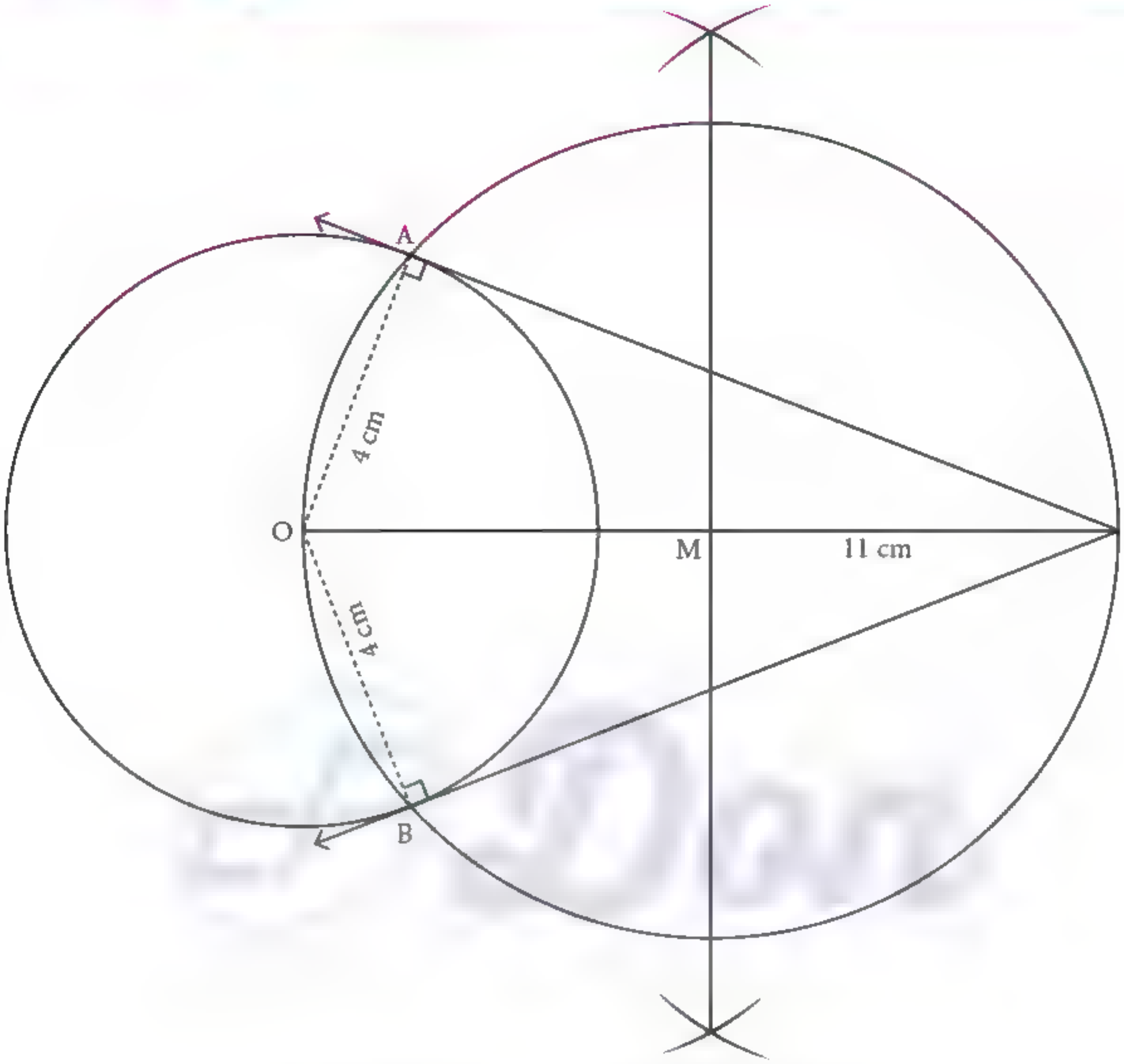
15. Take a point which is 11 cm away from the centre of a circle of radius 4 cm and draw the two tangents to the circle from that point.

Sol :

Given the radius of the circle = 4 cm



Rough diagram



Length of the tangents PA = PB = 10.2 cm

Construction:

Step 1: With center at 'O' drawn a circle of radius 4 cm

Step 2: Drawn a line OP = 11 cm

Step 3: Drawn a perpendicular bisector to OP, which cuts OP at M

Step 4: With M as center and MO as radius drawn a circle which cuts previous circle at A and B.

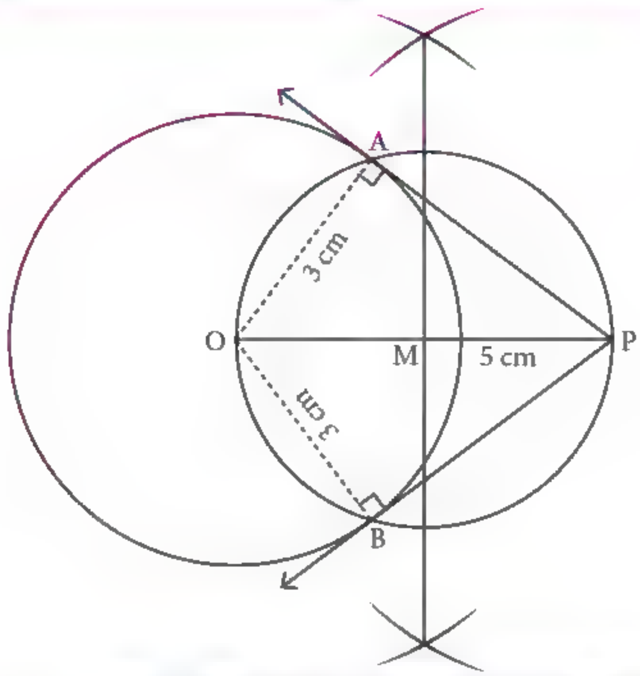
Step 5: Joined AP and BP. AP and BP are the required tangents. Thus the length of the tangents are PA = PB = 10.2 cm.

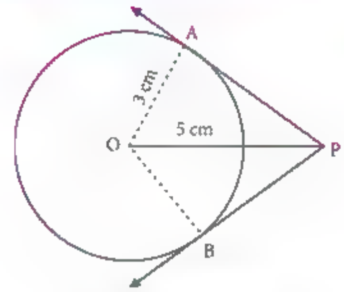
16. Draw the two tangents from a point which is 5 cm away from the centre of a circle of diameter 6 cm. Also, measure the lengths of the tangents.

Sol :

Given diameter = 6 cm

$\therefore$ radius $= \frac{6}{2} = 3$ cm





Rough diagram

Length of the tangents PA = PB = 4 cm

Construction:

Step 1: With centre at O drawn a circle of radius 3 cm

Step 2: Drawn a line OP = 5 cm

Step 3: Drawn a perpendicular bisector to OP, which cuts OP at M.

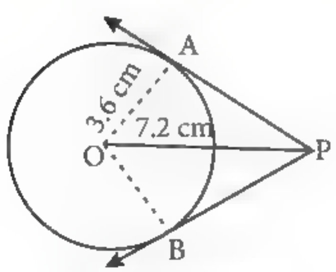
Step 4: With M as center and MO as radius drawn a circle which cuts previous circle at A and B.

Step 5: Joined AP and BP. AP and BP are the required tangents. Thus the length of the tangents are PA = PB = 4 cm

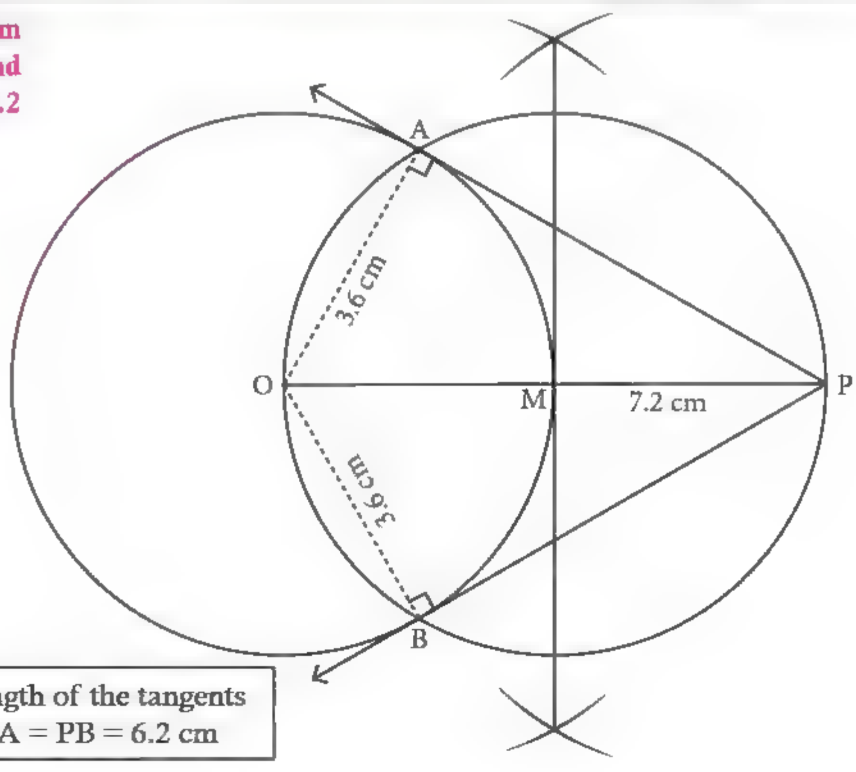
17. Draw a tangent to the circle from the point P having radius 3.6 cm and centre at O point P is at a distance 7.2 cm from the centre.

Sol :

Given radius r = 3.6 cm



Rough diagram



Length of the tangents PA = PB = 6.2 cm

Construction:

Step 1: With centre at O, drawn a circle of radius 3.6 cm.

Step 2: Drawn a line OP = 7.2 cm.

Step 3: Drawn a perpendicular bisector of OP which cuts OP at M.

Step 4: With M as center and MO as radius, drawn a circle which cuts previous circle at A and B.

Step 5: Joined AP and BP. AP and BP are the required tangents.

Length of the tengents PA = PB = 6.2 cm

Exercise 4. 5

1. If in triangles ABC and EDF, $\frac{AB}{DE} = \frac{BC}{FD}$ then they will be similar, when

(1) $\angle B = \angle E$ (2) $\angle A = \angle D$

(3) $\angle B = \angle D$ (4) $\angle A = \angle F$

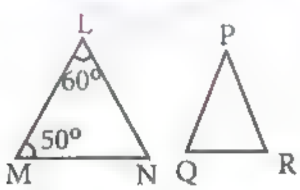
[Ans: 3]

2. In ΔLMN, $\angle L = 60°$, $\angle M = 50°$. If $\Delta LMN \sim \Delta PQR$ then the value of $\angle R$ is

(1) 40° (2) 70°

(3) 30° (4) 110° [Ans: 2]

Sol :



If $\Delta LMN \sim \Delta PQR$

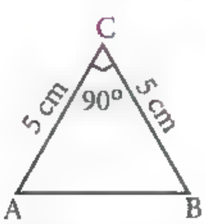
$$\angle R = \angle N = 180 - (60 + 50)$$
$$= 180 - 110 = 70°$$

3. If ΔABC is an isosceles triangle with $\angle C = 90°$ and AC = 5 cm, then AB is

(1) 2.5 cm (2) 5 cm

(3) 10 cm (4) $5\sqrt{2}$ cm [Ans: 4]

Sol :



$$AB^2 = AC^2 + BC^2$$
$$= 5^2 + 5^2 = 50$$
$$AB = \sqrt{50} = 5\sqrt{2} \text{ cm}$$

4. In a given figure ST ∥ QR, PS = 2 cm and SQ = 3 cm. Then the ratio of the area of ΔPQR to the area of ΔPST is

(1) 25 : 4 (2) 25 : 7

(3) 25 : 11 (4) 25 : 13 [Ans: 1]

Sol :

$$\left(\frac{PQ}{SP}\right)^2 = \frac{Area\ \Delta PQR}{Area\ \Delta PST}$$

$$PQ = PS + SQ = 2 + 3 = 5$$

$$\therefore \left(\frac{5}{2}\right)^2 = \frac{Area\ (\Delta PQR)}{Area\ (\Delta PST)} = \frac{25}{4} \Rightarrow 25:4$$

5. The perimeters of two similar triangles ΔABC and ΔPQR are 36 cm and 24 cm respectively. If PQ = 10 cm, then the length of AB is

(1) $6\frac{2}{3}$ cm (2) $\frac{10\sqrt{6}}{3}$ cm

(3) $66\frac{2}{3}$ cm (4) 15 cm [Ans: 4]

Sol :

If $\Delta ABC \sim \Delta PQR$

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PQ} = \frac{Perimeter\ of\ \Delta ABC}{Perimeter\ of\ \Delta PQR}$$

Given AB + BC + CA = 36 cm

PQ + QR + PR = 24 cm

$$\frac{AB}{PQ} = \frac{36}{24} \Rightarrow \frac{AB}{10} = \frac{36}{24}$$

$$AB = \frac{36 \times 10}{24}$$

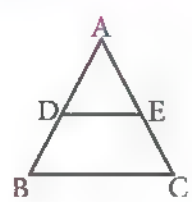
$$AB = 15 \text{ cm}$$

6. If in ΔABC, DE ∥ BC. AB = 3.6 cm, AC = 2.4 cm and AD = 2.1 cm then the length of AE is

(1) 1.4 cm (2) 1.8 cm

(3) 1.2 cm (4) 1.05 cm [Ans: 1]

Sol :



Let AE = x, then EC = 2.4 – x

AB = 3.6, AD = 2.1, $\therefore$ DB = 3.6 – 2.1 = 1.5

$$\text{Now}\quad \frac{AD}{DB} = \frac{AE}{EC}$$

$$\frac{2.1}{1.5} = \frac{x}{2.4-x}$$

$$\frac{7}{5} = \frac{x}{2.4-x}$$

$$7[2.4-x] = 5x$$

$$16.8 = 5x + 7x = 12x$$

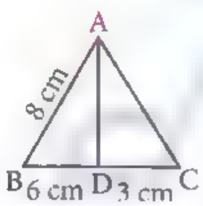
$$x = \frac{16.8}{12} = 1.4 \text{ cm}$$

7. In a ΔABC, AD is the bisector of $\angle BAC$. If AB = 8 cm, BD = 6 cm and DC = 3 cm. The length of the side AC is

(1) 6 cm (2) 4 cm

(3) 3 cm (4) 8 cm **[Ans: 2]**

Sol :

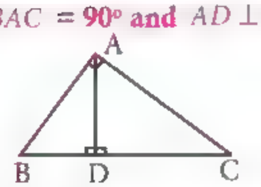


$$\frac{AB}{AC} = \frac{BD}{DC}$$

$$\frac{8}{AC} = \frac{6}{3}$$

$$AC = \frac{8}{6} \times 3 = 4 \text{ cm}$$

8. In figure $\angle BAC = 90^\circ$ and $AD \perp BC$ then



(1) BD . CD = BC^2

(2) AB . AC = BC^2

(3) BD . CD = AD^2

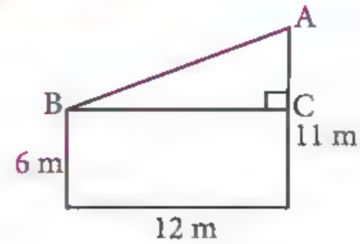
(4) AB . AC = AD^2 **[Ans: 3]**

9. Two poles of heights 6 m and 11 m stand vertically on a plane ground. If the distance between their feet is 12 m, what is the distance between their tops?

(1) 13 cm (2) 14 m

(3) 15 m (4) 12.8 m **[Ans: 1]**

Sol :



AC = 11 – 6 = 5 cm

In right ΔABC

$$AB^2 = BC^2 + AC^2 = 12^2 + 5^2$$

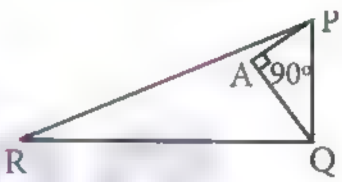
$$= 144 + 25 = 169$$

$$AB = 13 \text{ cm}$$

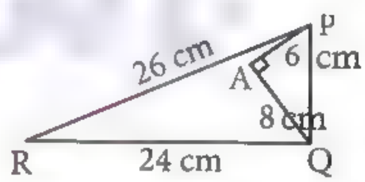
10. In the given figure PR = 26 cm, QR = 24 cm, $\angle PAQ = 90^\circ$, PA = 6 cm and QA = 8 cm. Find $\angle PQR$

(1) 80° (2) 85°

(3) 75° (4) 90° **[Ans: 4]**



Sol :



$$PQ^2 = 6^2 + 8^2 = 36 + 64 = 100$$

$$PQ = 10 \text{ cm}$$

$$\text{Now } PQ^2 + QR^2 = 10^2 + 24^2$$

$$= 100 + 576 = 676$$

$$= 26^2 = PR^2$$

$$\therefore \quad \angle PQR = 90^\circ$$

11. A tangent is perpendicular to the radius at the

(1) centre (2) point of contact

(3) infinity (4) chord **[Ans: 2]**

12. How many tangents can be drawn to the circle from an exterior point?

(1) one (2) two

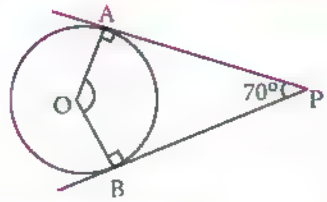
(3) infinite (4) zero **[Ans: 2]**

13. The two tangents from an external points P to a circle with centre at O are PA and PB. If $\angle APB = 70^\circ$ then the value of $\angle AOB$ is

(1) 100° (2) 110°

(3) 120° (4) 130° **[Ans: 2]**

Sol :

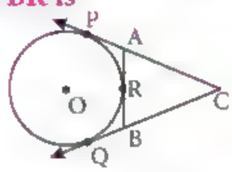


In quadrilateral APBO

$70° + 90° + 90° + \angle AOB = \quad 360°$

$\angle AOB \;=\; 360° - 250° = 110°$

14. In figure CP and CQ are tangents to a circle with centre at O. ARB is another tangent touching the circle at R. If CP = 11 cm and BC = 7 cm, then the length of BR is



(1) 6 cm (2) 5 cm

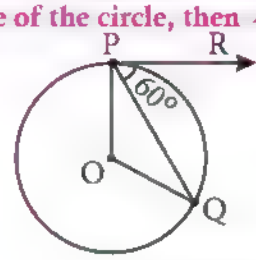
(3) 8 cm (4) 4 cm **[Ans: 4]**

Sol :

CP = 11 cm; BC = 7 cm

$\therefore$ CQ = 11 cm BQ = 11 − 7 = 4 cm = BR

15. In figure if PR is tangent to the circle at P and O is the centre of the circle, then $\angle POQ$ is



(1) 120° (2) 100°

(3) 110° (4) 90° **[Ans: 1]**

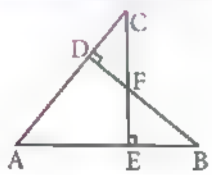
Sol :

$60 \times 2 = 120°$

UNIT EXERCISE - 4

1. In the figure, if BD ⊥ AC and CE ⊥ AB, prove that

(i) $\Delta AEC \sim \Delta ADB$ (ii) $\dfrac{CA}{AB} = \dfrac{CE}{DB}$



Sol :

(i) In ΔAEC and ΔADB

$\angle AEC = \angle ADB = 90°$

$\angle CAE = \angle BAD$

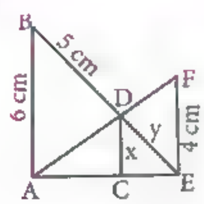
[common By AA similarity criteria]

$\Delta AEC \sim \Delta ADB$

(ii) Their corresponding sides are proportional

$\therefore \quad \dfrac{CA}{AB} = \dfrac{CE}{DB}$ Hence proved.

2. In the given figure AB || CD || EF. If AB = 6 cm, CD = x cm, EF = 4 cm , BD = 5 cm and DE = y cm. find x and y.



Sol :

Given AB || CD || EF

AB = 6 cm, BD = 5 cm, EF = 4 cm, CD = x cm, DE = y cm

In ΔECD and ΔEAB

$\angle CED = \angle AEB$ [common]

$\angle ECD = \angle EAB$ [corresponding angles]

$\Delta ECD \sim \Delta EAB$

[By AA similarity criteria] ... (1)

$\therefore \quad \dfrac{EC}{EA} = \dfrac{CD}{AB}$

[∵ Corresponding parts of similar triangles are proportional]

$\dfrac{EC}{EA} = \dfrac{x}{6}$ ... (2)

In ΔACD and ΔAEF

$\angle CAD = \angle EAF$ [Common]

$\angle ACD = \angle AEF$ [Corresponding angles]

$\Delta ACD \sim \Delta AEF$ [By AA similarity]

$\dfrac{AC}{AE} = \dfrac{CD}{EF}$

[∴ Corresponding parts of similar triangle are proportional]

$\therefore \quad \dfrac{AC}{AE} = \dfrac{x}{4}$ ... (3)

Adding (2) and (3)

$$\frac{EC}{EA}+\frac{AC}{AE} = \frac{x}{6}+\frac{x}{4}$$

$$\frac{EC+AC}{AE} = \frac{4x+6x}{24}$$

$$\frac{AE}{AE} = \frac{10x}{24}$$

$$1 = \frac{10x}{24}$$

$$x = \frac{24}{10} = \frac{12}{5} \text{ cm}$$

From (1) $\Delta ECD \sim \Delta EAB$

$$\frac{DC}{AB} = \frac{ED}{EB}$$

$$\frac{x}{6} = \frac{y}{5+y}$$

$$\because x=\frac{12}{5} \Rightarrow \frac{12/5}{6} = \frac{y}{5+y}$$

$$\frac{12}{5\times 6} = \frac{y}{5+y}$$

$$12(5+y) = 30y$$

$$60+12y = 30y$$

$$60 = 30y-12y$$

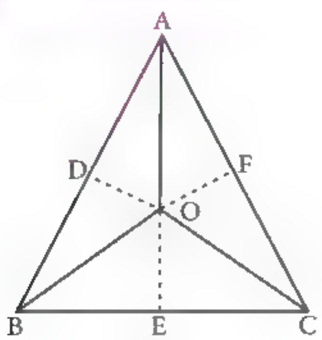
$$18y = 60$$

$$y = \frac{60}{18}$$

$$y = \frac{10}{3} \text{ cm}$$

3. O is any point inside a triangle ABC. The bisectors of $\angle AOB$, $\angle BOC$ and $\angle COA$ meet the sides AB, BC and CA in points D, E and F respectively. Show that AD × BE × CF = DB × EC × FA

Sol :



In ΔAOB, OD is the bisector of $\angle AOB$.

$$\frac{OA}{OB} = \frac{AD}{DB} \quad \text{... (1)}$$

In ΔBOC, OE is the bisector of $\angle BOC$

$$\therefore \quad \frac{OB}{OC} = \frac{BE}{EC} \quad \text{... (2)}$$

In ΔCOA, OF is the bisector of $\angle COA$

$$\therefore \quad \frac{OC}{OA} = \frac{CF}{FA} \quad \text{... (3)}$$

Multiplying the corresponding sides of (1), (2) and (3) we get

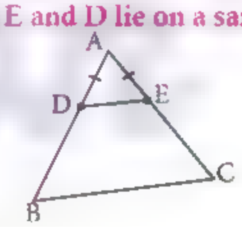
$$\frac{OA}{OB}\times\frac{OB}{OC}\times\frac{OC}{OA} = \frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CF}{FA}$$

$$\Rightarrow \quad 1 = \frac{AD}{DB}\times\frac{BE}{EC}\times\frac{CF}{FA}$$

$\Rightarrow$ DB × EC × FA = AD × BE × CF

$\Rightarrow$ AD × BE × CF = DB × EC × FA; Hence proved

4. In the figure, ABC is a triangle in which AB = AC. Points D and E are points on the sides AB and AC respectively such that AD = AE. Show that the points B, C, E and D lie on a same circle.



Sol :

To prove the points B, C, E and D lie on a same circle, it is sufficient to show that $\angle ABC + \angle CED = 180°$ and $\angle ACB + \angle BDE = 180°$.

i.e., to prove opposite angles of the quadrilateral BCED are supplementary.

In ΔABC we have

AB = AC and AD = AE

$\Rightarrow$ AB − AD = AC − AE

$\Rightarrow$ DB = EC

Thus we have

AD = AE and DB = EC

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC}$$

$\Rightarrow$ DE $\parallel$ BC [$\because$ By the converse of Thale's theorem]

$\Rightarrow$ $\angle ABC = \angle ADE$ [$\because$ Corresponding angles]

$\Rightarrow$ $\angle ABC + \angle BDE = \angle ADE + \angle BDE$

[Adding $\angle BDE$ on both sides]

$\Rightarrow \quad \angle ABC + \angle BDE = 180°$

$\Rightarrow \quad \angle ACB + \angle BDE = 180°$ ... (1)

[$\because$ AB = AC, $\angle ABC = \angle ACB$]

Again DE || BC

$\Rightarrow \quad \angle ACB = \angle AED$

$\Rightarrow \quad \angle ACB + \angle CED = \angle AED + \angle CED$

[Adding $\angle CED$ on both sides]

$\Rightarrow \quad \angle ACB + \angle CED = 180°$

[$\because$ $\angle ABC = \angle ACB$]

$\Rightarrow \quad \angle ABC + \angle CED = 180°$ ... (2)

From (1) and (2),

Thus BDEC is a quadrilateral such that

$\angle ACB + \angle BDE = 180°$

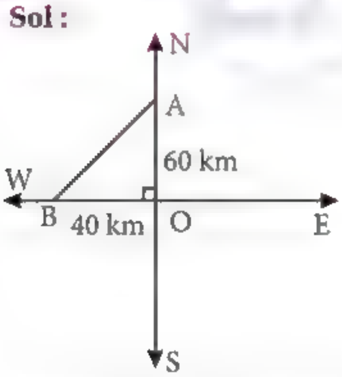
and $\angle ABC + \angle CED = 180°$

$\therefore$ BDCE is a cyclic quadrilateral.

$\therefore$ Points B, C, E and D lie on a same circle.

5. Two trains leave a railway station at the same time. The first train travels due west and the second train due north. The first train travels at a speed of 20 km/hr and the second train travels at 30 km/hr. After 2 hours, what is the distance between them?

Sol :



Distance travelled by the first train in 2 hours

$= 2 \times 20 = 40$ km

Distance travelled by the second train in 2 hours

$= 2 \times 30 = 60$ km

Let the distances are represents by OB and OA respectively

Now applying pythagoras theorem,

Distance between the trains after 2 hours is AB.

we have $AB^2 = OA^2 + OB^2 = 60^2 + 40^2$

$= 3600 + 1600$

$= 5200$

$AB = \sqrt{5200}$

$= \sqrt{2^2 \times 2^2 \times 5 \times 5 \times 13}$

$= 2^2 \times 5\sqrt{13}$

$= 20\sqrt{13}$ km

$\therefore$ Distance between the trains after 2 hrs

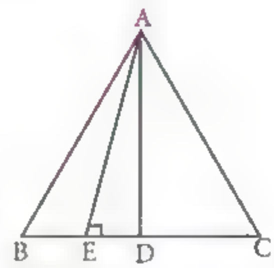
$= 20\sqrt{13}$ kms

6. D is the mid point of sides BC and AE $\perp$ BC. If BC = a, AC = b, AB = c, ED = x, AD = p and AE = h, prove that

(i) $b^2 = p^2 + ax + \frac{a^2}{4}$ (ii) $c^2 = p^2 - ax + \frac{a^2}{4}$

(iii) $b^2 + c^2 = 2p^2 + \frac{a^2}{2}$

Sol :



We have $\angle AED = 90°$ [$\because$ AE $\perp$ BC]

Given BC = a, AC = b, AB = c, ED = x, AD = p and AE = h

(i) In ΔAEC, by Pythagoras theorem

$AC^2 = AE^2 + EC^2$

$AC^2 = AE^2 + (ED + DC)^2$

$AC^2 = AE^2 + ED^2 + DC^2 + 2 . ED . DC$

$AC^2 = (AE^2 + ED^2) + DC^2 + 2 ED . DC$

$AC^2 = AD^2 + DC^2 + 2 ED . DC$

$AC^2 = AD^2 + \left(\frac{1}{2}BC\right)^2 + 2\left(\frac{1}{2}BC\right)DE$

[$\because$ D is the mid point of BC, BD = DC]

$AC^2 = AD^2 + BC . DE + \frac{1}{4}BC^2$ .... (1)

i.e., $b^2 = p^2 + ax + \frac{1}{4}a^2$

$b^2 = p^2 + ax + \frac{a^2}{4}$

(ii) Again in ΔABC, $\angle AED = \angle AEB = 90°$.

By Pythagoras theorem.

$AB^2 = AE^2 + EB^2$

[$\because$ $AE^2 = AD^2 - DE^2$]

$= AD^2 - DE^2 + (BD - DE)^2$

$= AD^2 - DE^2 + BD^2 + DE^2 - 2 BD . DE$

$= AD^2 + BD^2 - 2BD . DE$

$AB^2 = AD^2 + \left(\frac{1}{2}BC\right)^2 - 2 . \frac{1}{2} BC . DE$

$$AB^2 = AD^2 + \frac{1}{4}BC^2 - BC \,.\, DE$$

$$AB^2 = AD^2 - BC \,.\, DE + \frac{1}{4}BC^2 \quad ...(2)$$

$$c^2 = p^2 - ax + \frac{a^2}{4}$$

(iii) From (1) and (2)

$$AC^2 + AB^2 = AD^2 + BC \,.\, DE + \frac{1}{4}BC^2 + AD^2 - BC \,.\, DE + \frac{BC^2}{4}$$

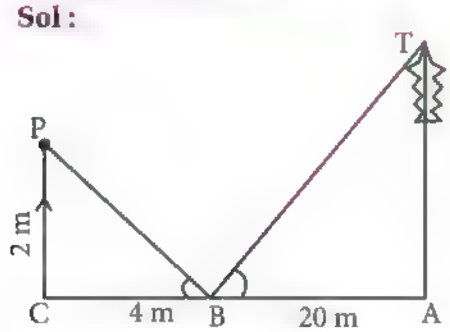
$$= 2\,AD^2 + 2\left(\frac{BC^2}{4}\right)$$

$$AC^2 + AB^2 = 2\,AD^2 + \frac{BC^2}{2}$$

$$b^2 + c^2 = 2p^2 + \frac{a^2}{2}$$

7. A clever outdoorsman whose eye-level is 2 m above the ground wishes to find the height of a tree. He places a mirror horizontally on the ground 20 m from the tree and finds that if he stands at a point C which is 4 m from the mirror B, he can see the reflection of the top of the tree. How height is the tree?

Sol :



Let CP be the man whose eye level is 2 m above the ground

AT be the height of the tree

In triangles ΔCBP and ΔABT

$$\angle C = \angle A = 90°$$

Since CA $\perp$ PC and CA $\perp$ TA

$\angle CBP = \angle ABT$ as $\angle CBP$ is the angle of reflection and $\angle ABT$ is the angle of incident.

$\therefore$ By AA similarity criteria

$$\Delta CBP \sim \Delta ABT$$

$$\therefore \quad \frac{CB}{AB} = \frac{BP}{BT} = \frac{CP}{AT}$$

$$\frac{CB}{AB} = \frac{CP}{AT}$$

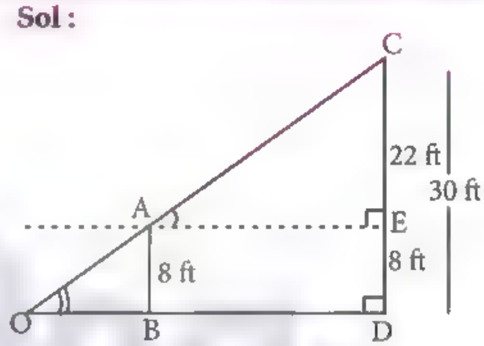
$$\frac{4}{20} = \frac{2}{AT}$$

$$AT = \frac{2 \times 20}{4} = 10 \text{ m}$$

$\therefore$ Height of the tree is 10 m.

8. Suppose an emu is 8 ft tall and is walking away from a pillar that is 30 ft tall. At height, the emu will cast a shadow on the ground. What is the relationship between the length of the shadow and the distance from the emu to the pillar?

Sol :



Let AB be the emu = 8 ft

CD be the pillar = 30 ft

OB be the shadow, BD be the distance between the pillar and the emu.

Draw AE $\parallel$ OD at a distance of 8 ft from OD.

Now, $\angle CAE = \angle AOB$ [corresponding angles]

$$\angle OBA = \angle AEC = 90°$$

$\therefore$ By AA similarity criteria

$$\Delta CEA \sim \Delta ABO$$

$$\frac{CE}{AB} = \frac{EA}{BO}$$

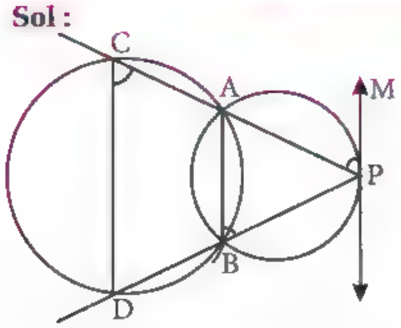
$$\frac{22}{8} = \frac{BD}{OB} \quad [\because EA = BD]$$

$$\frac{11}{4} = \frac{distance}{shadow}$$

$$\therefore \quad \text{Shadow} = \frac{4}{11} \times \text{distance}$$

9. Two circles intersect at A and B. From a point P on one of the circles lines PAC and PBD are drawn intersecting the second circle at C and D. Prove that CD is parallel to the tangent at P.

Sol :



Let MP is the tangent at P Join AB.

$\angle MPA = \angle PBA$ ... (1)

[Angles in the alternate segment]

$\angle PBA = \angle ACD$ ... (2)

[Exterior angle of a cyclic quadrilateral ABCD is equal to its interior opposite angle]

From (1) and (2), we have

$\angle MPA = \angle ACD$

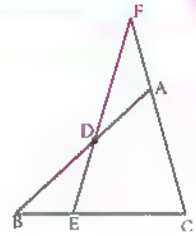
$\Rightarrow \angle MPC = \angle PCD$

But these are a pair of alternate interior angles.

$\therefore$ MP || CD

10. Let ABC be a triangle and D, E, F are points on the respective sides AB, BC, AC (or their extensions). Let AD : DB = 5 : 3, BE : EC = 3 : 2 and AC = 21. Find the length of the line segment CF.

Sol :



Let ΔABC be the given triangle and D, E, F are points on the respectives sides AB, BC AC (or their extension).

Given AD : DB = 5 : 3;

BE : EC = 3 : 2.

Let AD = $5k_1$; DB = $3k_1$; BE = $3k_2$; EC = $2k_2$

Now by Menelaus theorem, we have

$$\frac{BE}{EC} \times \frac{CF}{FA} \times \frac{AD}{DB} = -1$$

$$\frac{3k_2}{2k_2} \times \frac{CF}{FC - AC} \times \frac{5k_1}{3k_1} = -1$$

$$\frac{3}{2} \times \frac{CF}{FC - 21} \times \frac{5}{3} = -1$$

$$\frac{CF}{FC - 21} = \frac{-1 \times 2}{5}$$

$$\frac{CF}{(-CF) - 21} = \frac{-2}{5}$$

$$5\,CF = -2\,[(-CF) - 21]$$

$$5CF = 2\,CF + 42 \quad [\because FC = -CF]$$

$$5CF - 2CF = 42$$

$$3CF = 42$$

$$CF = \frac{42}{3} = 14 \text{ units.}$$

CREATIVE QUESTIONS

I. Multiple Choice Questions

Similar Triangles

1. Sides of two similar triangle are in the ratio 4 : 9. Areas of these triangles are in the ratio

(1) 2 : 3 (2) 4 : 9

(3) 81 : 16 (4) 16 : 81 **[Ans : (4)]**

Sol :

In similar triangles Ratio of corresponding Area = Square of ratio of corresponding sides

$$= \left(\frac{4}{9}\right)^2 = \left(\frac{16}{81}\right)$$

Ratio = 16 : 81

2. The areas of two similar triangles are respectively 9 cm² and 16 cm². The ratio the of their corresponding sides is

(1) 3 : 4 (2) 4 : 3

(3) 2 : 3 (4) 4 : 5 **[Ans : (1)]**

Sol :

$$\text{(Ratio of corresponding sides)}^2 = \frac{9}{16} = \frac{3^2}{4^2}$$

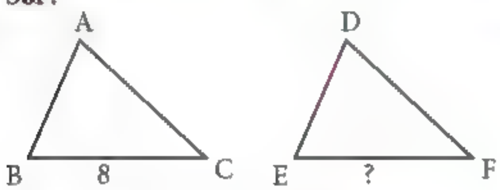
$$\therefore \text{ Ratio of corresponding sides} = \frac{3}{4}$$

Ratio $= 3 : 4$

3. If ΔABC and ΔDEF are similar such that 2AB = DE and BC = 8 cm and EF =

(1) 16 cm (2) 12 cm
(3) 8 cm (4) 4 cm **[Ans : (1)]**

Sol :



Given $\Delta ABC \sim \Delta DEF$

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Given 2AB = DE; BC = 8cm

$$\therefore \frac{AB}{2AB} = \frac{8}{EF}$$

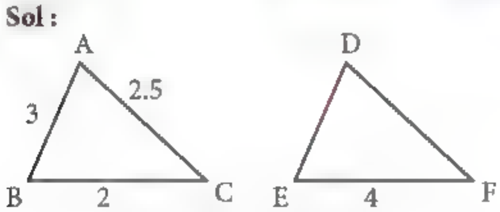
EF $= 8 \times 2$

EF = 16 cm

4. ΔABC is such that AB = 3 cm, BC = 2 cm and CA = 2.5 cm. If $\Delta DEF \sim \Delta ABC$ and EF = 4 cm, then perimeter of ΔDEF is

(1) 7.5 cm (2) 15 cm
(3) 22.5 cm (4) 30 cm **[Ans : (2)]**

Sol :



Perimeter of ΔABC = AB + BC + CA
= 3 + 2 + 2.5
= 7.5 cm

$$\frac{\textit{Perimeter of } \Delta ABC}{\textit{Perimeter of } \Delta DEF} = \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

$[\because \Delta ABC \sim \Delta DEF]$

$$\frac{7.5}{\textit{Perimeter of } \Delta DEF} = \frac{BC}{EF} = \frac{2}{4} = \frac{1}{2}$$

$\therefore$ Perimeter of $\Delta DEF = 2 \times 7.5 = 15$ cm

5. If ΔABC and ΔDEF are similar triangles such that $\angle A = 47°$ and $\angle B = 83°$, then $\angle F$ =

(1) 50° (2) 60°
(3) 70° (4) 80° **[Ans : (1)]**

Sol :

$\Delta ABC \sim \Delta DEF$

$\angle A = \angle D$; $\angle B = \angle E$; $\angle C = \angle F$

$\angle A = \angle D = 47°$ $\angle B = \angle E = 83°$

$\angle C = 180 - (\angle A + \angle B)$
$= 180° - (47° + 83°)$
$= 180° - 130°$
$= 50°$

$\therefore \angle C = \angle F = 50°$

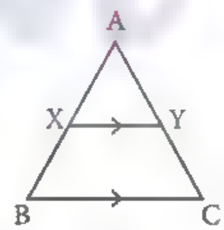
$\therefore \angle F = 50°$

Thales Theorem and Angle Bisector Theorem

6. XY is drawn parallel to the base BC of a ΔABC cutting AB at x and AC at y. If AB = 4 BX and YC = 2 cm and then AY = ____

(1) 2 cm (2) 4 cm
(3) 6 cm (4) 8 cm **[Ans : (3)]**

Sol :



$$\frac{AX}{XB} = \frac{AY}{YC} \text{ [By Thales Theorem]}$$

$$\frac{AX}{XB} = \frac{AY}{2}$$

$$\frac{AB - XB}{XB} = \frac{AY}{2}$$

$$\frac{4BX - BX}{BX} = \frac{AY}{2}$$

$$\frac{3BX}{BX} = \frac{AY}{2}$$

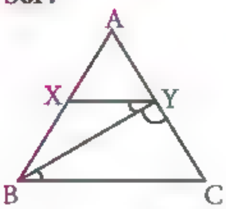
AY $= 2 \times 3$

AY = 6 cm

7. In ΔABC, a line XY parallel to BC cuts AB at X and AC at Y. If BY bisects $\angle XYC$, then

(1) BC = CY (2) BC = BY
(3) BC ≠ CY (4) BC ≠ BY **[Ans : (1)]**

Sol :



Given $XY \parallel BC$

$\angle XYB = \angle YBC$ ... (1)

[$\because$ alternate interior angles]

YB bisects $\angle XYC$.

So $\angle XYB = \angle BYC$ ... (2)

$\angle YBC = \angle BYC$ [$\because$ from (1) and (2)]

$\therefore$ BC = YC

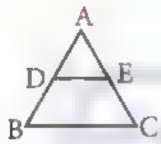
[$\because$ sides opposite to equal angles are equal]

8. In ΔABC, D and E are points on side AB and AC respectively such that $DE \parallel BC$ and AD : DB = 3 : 1. If EA = 3.3 cm then AC = _____

(1) 1.1 cm (2) 4 cm

(3) 4.4 cm (4) 5.5 cm **[Ans : (3)]**

Sol :



$DE \parallel BC$

$$\frac{AD}{DB} = \frac{AE}{EC}$$

$$\frac{3}{1} = \frac{3.3}{EC}$$

$$EC = \frac{3.3}{3}$$

$$EC = 1.1$$

$$AC = AE + EC$$

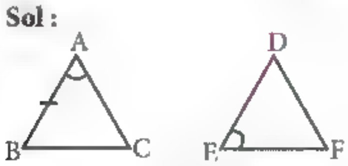
$$= 3.3 + 1.1 = 4.4 \text{ cm}$$

9. In ΔABC and $\angle A = \angle E = 40^\circ$, AB : ED = AC : EF and $\angle F = 65^\circ$, then $\angle B$ = _____

(1) 35° (2) 65°

(3) 75° (4) 85° **[Ans : (3)]**

Sol :



Given $\angle A = \angle E = 40^\circ$

$$\frac{AB}{ED} = \frac{AC}{EF}$$

$\therefore \Delta ABC \sim \Delta EDF$

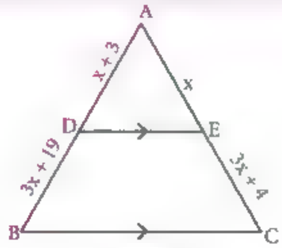
$\therefore \angle F = \angle C = 65^\circ$

$\angle B = 180^\circ - (\angle A + \angle C)$

$= 180^\circ - (\angle 40^\circ + \angle 65^\circ)$

$\angle B = \angle D = 180^\circ - 105^\circ = 75^\circ$

10. Find the value of x for which $DE \parallel AB$ is



(1) 4 (2) 1

(3) 3 (4) 2 **[Ans : (4)]**

Sol :

Since $DE \parallel AB$

$$\frac{AD}{DB} = \frac{AE}{EC} \text{ [By Thales Theorem]}$$

$$\frac{x+3}{3x+19} = \frac{x}{3x+4}$$

$$(x + 3)(3x + 4) = x(3x + 19)$$

$$3x^2 + 13x + 12 = 3x^2 + 19x$$

$$6x = 12$$

$$x = \frac{12}{6}$$

$$x = 2$$

Pythogoras Theorem

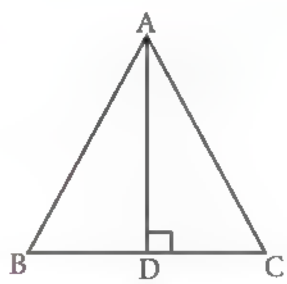
11. In an equilateral triangle ΔABC, if $AD \perp BC$ then

(1) $2AB^2 = 3AD^2$ (2) $4AB^2 = 3AD^2$

(3) $3AB^2 = 4AD^2$ (4) $3AB^2 = 2AD^2$

[Ans : (3)]

Sol :



AB = BC = AC and BD = DC = $\frac{1}{2}$ BC

By Pythogoras theorem

$$AB^2 = BD^2 + AD^2$$

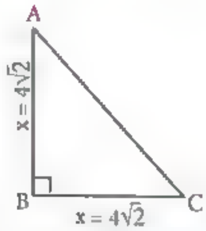
$$AB^2 = \left(\frac{1}{2}BC\right)^2 + AD^2$$

$$AB^2 = \frac{1}{4}BC^2 + AD^2$$
$$4AB^2 = BC^2 + 4AD^2$$
$$4AB^2 - BC^2 = 4AD^2$$
$$4AB^2 - AB^2 = 4AD^2 \quad [\because AB = BC]$$
$$3AB^2 = 4AD^2$$

12. The length of the hypotenuse of an isosceles right triangle whose one side is $4\sqrt{2}$ cm is

(1) 12 cm (2) 8 cm
(3) $8\sqrt{2}$ cm (4) $12\sqrt{2}$ cm **[Ans : (2)]**

Sol :



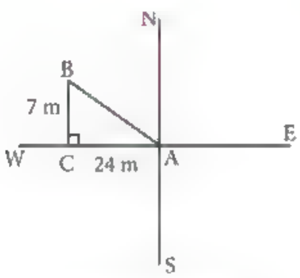
$$AC^2 = AB^2 + BC^2$$
$$= x^2 + x^2$$
$$= 2x^2$$
$$= 2(4\sqrt{2})^2$$
$$= 2 \times 16 \times 2$$
$$AC^2 = 64$$
$$AC = 8 \text{ cm}$$

Another Method:
$$AC^2 = AB^2 + BC^2$$
$$= (4\sqrt{2})^2 + (4\sqrt{2})^2$$
$$= (16 \times 2) + (16 \times 2)$$
$$= 32 + 32$$
$$AC^2 = 64$$
$$AC = 8 \text{ cm}$$

13. A man goes 24 m due west and then 7 m due north. How are is he from the starting point?

(1) 31 m (2) 17 m
(3) 25 m (4) 26 m **[Ans : (3)]**

Sol :

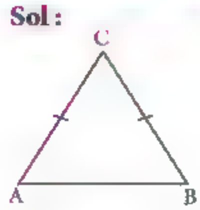


$$AB^2 = AC^2 + BC^2$$
$$= 24^2 + 7^2$$
$$= 576 + 49$$
$$AB^2 = 625$$
$$AB = 25 \text{ m}$$

14. In an isosceles triangle ΔABC if AC = BC and $AB^2 = 2AC^2$, then $\angle C$ = ____

(1) 30° (2) 45°
(3) 90° (4) 60° **[Ans : (3)]**

Sol :



$$AB^2 = 2AC^2$$
$$\Rightarrow \quad AB^2 = AC^2 + BC^2 \quad [\because AC = BC]$$

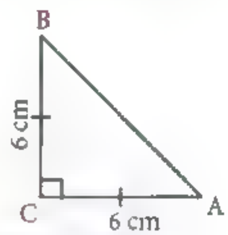
By the converse of Pythagoras theorem

$$\angle C = 90°$$

15. ΔABC is an isosceles triangle in with $\angle C = 90°$, If AC = 6cm, then AB = ____

(1) $6\sqrt{2}$ cm (2) 6 cm
(3) $2\sqrt{6}$ cm (4) $4\sqrt{2}$ **[Ans : (1)]**

Sol :



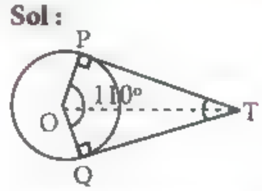
$$AB^2 = AC^2 + BC^2$$
$$= 6^2 + 6^2$$
$$= 36 + 36$$
$$= 72$$
$$AB = \sqrt{72} = 6\sqrt{2} \text{ cm}$$

Circles and tangents and Alternate segment theorem

16. If TP and TQ are two tangents to a circle with centre 'O' so that $\angle POQ = 110°$, then $\angle PTQ$ is

(1) 60° (2) 70°
(3) 80° (4) 90° **[Ans : (2)]**

Sol :



In the quadrilateral POQT

$$\angle O = 110°$$
$$\angle P = 90°$$
$$\angle Q = 90°$$
$$\therefore \angle T = 360° - (110° + 90° + 90°)$$

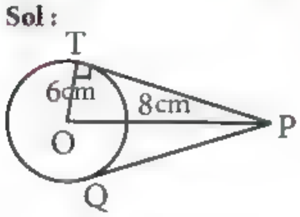
[Sum of 4 angles of a quadrilateral is 360°]

$$= 360° - 290°$$
$$\angle T = 70°$$
$$\angle PTQ = 70°$$

17. **The length of the tangent drawn from a point 8 cm away from the centre of a circle of radius 6 cm is _____**

(1) $\sqrt{7}$ cm (2) $2\sqrt{7}$ cm
(3) 10 cm (4) 5 cm **[Ans : (2)]**

Sol :

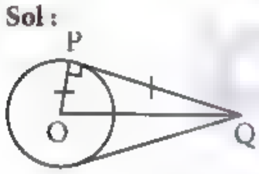


$$OP^2 = PT^2 + TO^2$$
$$8^2 = PT^2 + 6^2$$
$$64 - 36 = PT^2$$
$$PT^2 = 28$$
$$PT = 2\sqrt{7}\text{ cm}$$

18. **PQ is a tangent to a circle with center 'O' at the point P, if ΔOPQ is an isosceles triangle, then $\angle OQP$ is =**

(1) 30° (2) 45°
(3) 60° (4) 90° **[Ans : (2)]**

Sol :

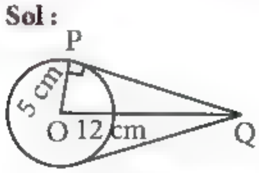


$$\angle P = 90° \; [\because \text{ radius} \perp \text{tangent}]$$
$$OP = PQ \; [\Delta OPQ \text{ is an isosceles}]$$
$$\Rightarrow \angle POQ = \angle PQO$$
$$\angle POQ + \angle PQO = 180° - \angle P$$
$$= 180° - 90° = 90°$$
$$\therefore \angle POQ = \angle PQO = \frac{1}{2} \times 90°$$
$$\therefore \angle OQP = 45°$$

19. **A tangent PQ at a point P of circle of radius 5 cm meats a line through the center 'O' at a point Q such that OQ = 12 cm, Length PQ is.**

(1) 12 cm (2) 13 cm
(3) 8.5 cm (4) $\sqrt{119}$ cm **[Ans : (4)]**

Sol :

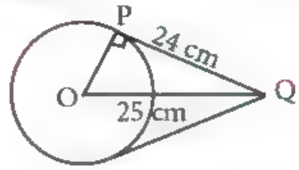


$$OQ^2 = OP^2 + PQ^2 \quad \text{[By Pythagoras Theorem]}$$
$$12^2 = 5^2 + PQ^2$$
$$144 - 25 = PQ^2$$
$$PQ^2 = 119$$
$$PQ = \sqrt{119}\text{ cm}$$

20. **From a point Q, the length of the tangent to a circle is 24 cm an d the distance of a Q from the center is 25 cm. The radius of the circle is _____**

(1) 7 cm (2) 12 cm
(3) 15 cm (4) 24.5 cm **[Ans : (1)]**

Sol :



$$(\text{radius})^2 + 24^2 = 25^2 \quad \text{[By Pythagoras Theorem]}$$
$$(\text{radius})^2 + 576 = 625$$
$$(\text{radius})^2 = 625 - 576$$
$$= 49$$
$$(\text{radius})^2 = \sqrt{49}$$
$$\text{radius} = 7\text{ cm}$$

III. Very Short Answer Questions

1. **The areas of two similar triangles ΔABC and ΔDEF are 81 cm² and 100 cm² respectively.If EF = 5 cm, then find BC.**

Sol :

Given $\Delta ABC \sim \Delta DEF$

$$\frac{area(\Delta ABC)}{area(\Delta DEF)} = \frac{BC^2}{EF^2}$$
$$\frac{81}{100} = \frac{BC^2}{(5)^2}$$
$$\frac{BC}{5} = \frac{9}{10}$$
$$BC = \frac{9}{10} \times 5 = \frac{9}{2}\text{ cm}$$
$$= 4.5\text{ cm}$$
$$\therefore BC = 4.5\text{ cm}$$

2. **The area of ΔPQR = 64 m². Find the area of ΔLMN if $\frac{PQ}{LM} = \frac{4}{5}$ and $\Delta PQR \sim \Delta LMN$.**

Sol :

Given $\Delta PQR \sim \Delta LMN$

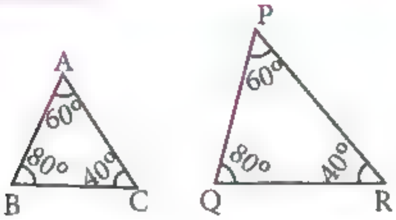
$$\frac{area(\Delta PQR)}{area\ of\ (\Delta LMN)} = \frac{PQ^2}{LM^2}$$
$$\frac{64}{area(\Delta LMN)} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

$\therefore$ Area of $\Delta LMN = \dfrac{64 \times 25}{16}$

$= 100 \text{ m}^2$

Area of $\Delta LMN = 100 \text{ m}^2$

3. Check whether the given pair of triangles are similar or not.



Sol :

In ΔABC and ΔPQR

we have $\angle A = \angle P = 60^\circ$

$\angle B = \angle Q = 80^\circ$

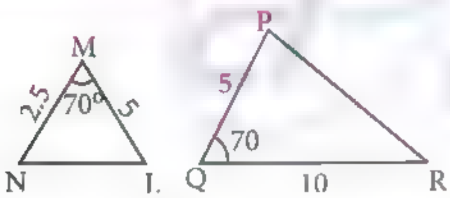
$\angle C = \angle R = 40^\circ$

The corresponding angles are equal

Using AAA similarity rule

$\Delta ABC \sim \Delta PQR$

4. Check the similarity of the given triangles.



Sol :

In ΔMNL and ΔQPR

$$\frac{MN}{QP} = \frac{2.5}{5} = \frac{1}{2}$$

$$\frac{ML}{QR} = \frac{5}{10} = \frac{1}{2}$$

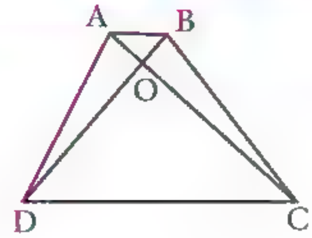
$$\frac{ML}{QR} = \frac{MN}{QP}$$

$$\angle NML = \angle PQR$$

Using SAS criteria of similarity, we have,

$\Delta MNL \sim \Delta QPR$

5. Diagonals AC and BD of a trapezium ABCD with AB || DC intersect each other at the point O. Using a similarity criterion for two triangles show that $\dfrac{OA}{OC} = \dfrac{OB}{OD}$.



Sol :

We have a trapezium ABCD in which AB || DC.

The diagonals AC and BD intersect at O

In ΔOAB and ΔOCD

AB || DC

and BD intersects them

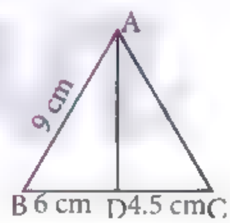
$\therefore \angle OBA = \angle ODC$ (Alternate interior angles)

Also $\angle OAB = \angle OCD$ (Alternate interior angles)

Using AA similarity criteria

$\Delta OAB \sim \Delta OCD$

6. In the figure AD is the bisector of $\angle A$. If BD = ■ cm, DC = 4.5 cm and AB = 9 cm, Find AC.



Sol :

In ΔABC, AD is the bisector of $\angle A$

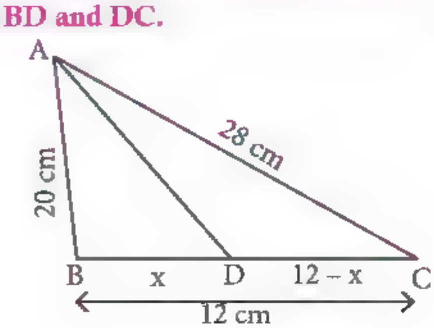
$$\therefore \frac{BD}{DC} = \frac{AB}{AC}$$

$$\frac{6}{4.5} = \frac{9}{AC}$$

$$AC = \frac{9}{6} \times 4.5$$

$$AC = 6.75 \text{ cm}$$

7. In the figure AD is the bisector of $\angle BAC$. If AB = 20 cm, AC = 28 cm and BC = 12 cm, find BD and DC.



Sol :

Let BD = x cm,

Then DC = (12 – x) cm

Since AD is the bisector of $\angle A$.

$$\frac{AB}{AC} = \frac{BD}{DC} \Rightarrow \frac{20}{28} = \frac{x}{12-x}$$

$$\frac{5}{7} = \frac{x}{12-x} \Rightarrow 5(12-x) = 7x$$

$$60 - 5x = 7x$$

$$12x = 60$$

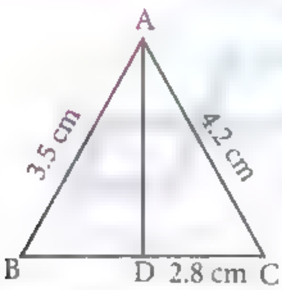
$$x = \frac{60}{12} = 5$$

BD = 5 cm

DC = 12 – 5 = 7 cm

8. In ΔABC, AD is the bisector of $\angle A$, meeting BC at D If AB = 3.5 cm, AC = 4.2 cm and DC = 2.8 cm, find BD.

Sol :



Since AD is the bisector of $\angle A$

$$\frac{AB}{AC} = \frac{BD}{DC} \Rightarrow \frac{3.5}{4.2} = \frac{BD}{2.8}$$

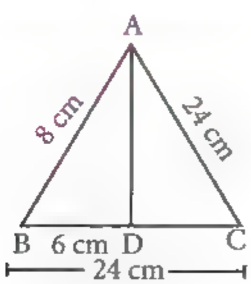
$$BD = \frac{3.5 \times 2.8}{4.2}$$

$$= \frac{0.70}{0.3} = \frac{7}{3} = 2.33$$

BD = 2.33 cm

9. In a ΔABC, if AB = [redacted] cm, AC = 24 cm, BD = 6 cm and BC = 24 cm. Check whether AD is the bisector of $\angle A$ of ΔABC.

Sol :



Given BC = 24 cm and BD = 6 cm

$\therefore$ DC = BC – BD

= 24 – 6 = 18 cm

$$\text{Now, } \frac{AB}{AC} = \frac{8}{24} = \frac{1}{3}$$

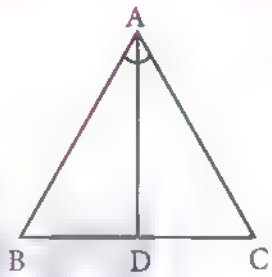
$$\frac{BD}{DC} = \frac{6}{18} = \frac{1}{3}$$

$$\frac{AB}{AC} = \frac{BD}{DC}$$

$\therefore$ By the converse of angular bisector theorem. AD is the bisector of $\angle A$ in ΔABC.

10. If the bisector of an angle of a triangle bisects the opposite side, prove that the triangle is isosceles.

Sol :



Given: In ΔABC, the bisector AD of $\angle A$ bisects the side BC.

To prove: AB = AC

Proof: In ΔABC, AD is the bisector of $\angle A$

$$\frac{AB}{AC} = \frac{BD}{DC}$$

$$\frac{AB}{AC} = 1$$

[$\because$ D is the mid point of BC, BD = DC]

AB = AC.

$\therefore$ The triangle ABC is isosceles.

11. Check the given sides are the sides of a right angled triangle.

(i) a = 6 cm, b = 8 cm and c = 10 cm

(ii) a = 5 cm b = 8 cm and c = 11 cm

Sol :

(i) We have a = 6 cm,

b = 8 cm and c = 10 cm

Here the larger side is c = 10 cm

$$\therefore a^2 + b^2 = 6^2 + 8^2$$

$$= 36 + 64 = 100$$

$$= (10)^2 = c^2$$

$\therefore$ The triangle with the given sides is a right triangle.

(ii) We have a = 5 cm,

b = 8 cm and c = 11 cm

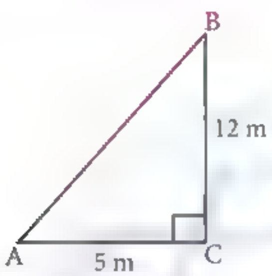
Here the larger side is c = 11 cm

$$a^2 + b^2 = 5^2 + 8^2 = 25 + 64$$
$$a^2 + b^2 = 89$$
$$\text{But } c^2 = 11^2 = 121$$

$\therefore$ $a^2 + b^2 \neq c^2$. The triangle with the given sides are not a right angled triangle.

12. A ladder is placed in such a way that its foot is at a distance of 5 m from a wall and its tip reaches a window 12 m above the ground. Determine the length of the ladder.

Sol :



Let AB be the ladder and B be the window, then

BC = 12 m and AC = 5 m

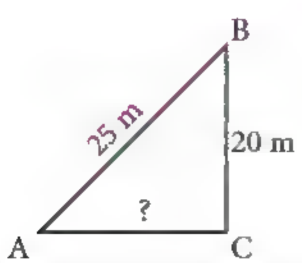
Since ΔABC is right triangle; right angled at C

$$AB^2 = AC^2 + BC^2$$
$$AB^2 = 5^2 + 12^2 = 25 + 144$$
$$AB^2 = 169$$
$$AB = 13 \text{ m}$$

Hence the length of the ladder is 13 m

13. A ladder 25 m long reaches a window of a building 20 m above the ground. Determine the distance of the foot of the ladder from the building.

Sol :



Suppose that AB is the ladder, B is the window and CB is the building.

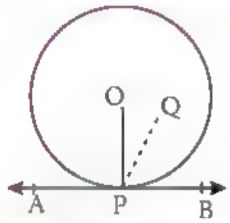
ΔABC is a right triangle and $\angle C = 90°$

$$AB^2 = AC^2 + BC^2$$
$$(25)^2 = AC^2 + 20^2$$
$$625 = AC^2 + 400$$
$$AC^2 = 625 - 400$$
$$AC^2 = 225$$
$$AC = 15$$

Hence the foot of the ladder is at a distance of 15 m from the building.

14. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.

Sol :



In the figure, the center of the circle O and tangent AB touches the circle at P. If possible, let PQ be perpendicular to AB such that it is not passing through 'O'. Join OP.

Since tangent at a point to a circle is perpendicular to the radius thought that point

$$AB \perp OP$$
$$\text{i.e., } \angle OPB = 90° \quad \dots (1)$$

But by construction

$$AB \perp PQ \Rightarrow \angle QPB = 90° \quad \dots (2)$$

From (1) and (2)

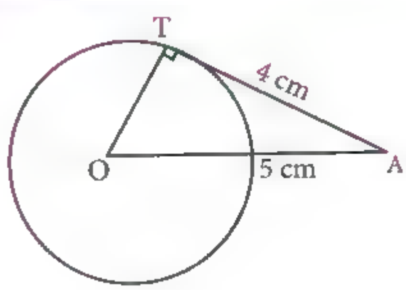
$$\angle QPB = \angle OPB$$

Which is possible only when O and Q coincide

$\therefore$ The perpendicular at the point of contact to the tangent passes through the center.

15. The length of a tangent from a point at a distance of 5 cm from the center of the circle is 4 cm. Find the radius of the circle.

Sol :



The tangent to a circle is perpendicular to the radius through the point of contact

$$\angle OTA = 90^\circ$$

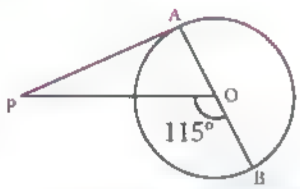
Now in right ΔOTA

$$\begin{aligned} OA^2 &= OT^2 + TA^2 \\ 5^2 &= OT^2 + 4^2 \\ OT^2 &= 5^2 - 4^2 \\ &= 25 - 16 = 9 \\ OT &= 3 \end{aligned}$$

Thus the radius of the circle is 3 cm.

16. In the figure PA is a tangent from an external point P to a circle with centre O. If $\angle POB = 115^\circ$ then find $\angle APO$.

Sol :



Here PA is a tangent and OA is radius. Also a radius through the point of contact is perpendicular to the tangent.

$$OA \perp PA$$

$$\angle PAO = 90^\circ$$

In ΔOAP, $\angle POB$ in an external angle.

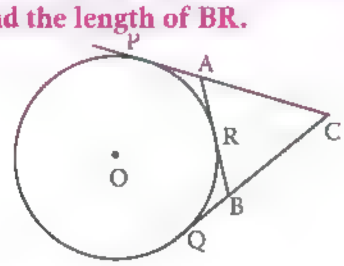
$$\therefore \angle APO + \angle PAO = \angle POB$$

$$\angle APO + 90^\circ = 115^\circ$$

$$\angle APO = 115^\circ - 90^\circ = 25^\circ$$

$$\angle APO = 25^\circ$$

17. In the figure CP and CQ are tangents to a circle with center O. ARB is another tangent touching the circle at R. If QC = 11 cm, BC = 7 cm. Then find the length of BR.



Sol :

Tangents drawn from an external point are equal.

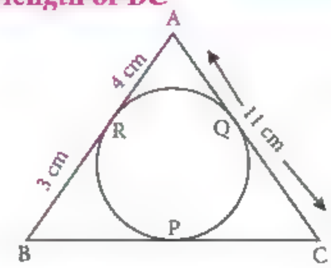
$$BQ = BR \text{ and } CQ = CP$$

Since $BC + BQ = QC$

$$7 + BR = 11 \qquad [\because BQ = BR]$$

$$BR = 11 - 7 = 4 \text{ cm}$$

18. In the figure, ΔABC is circumscribing a circle. Find the length of BC



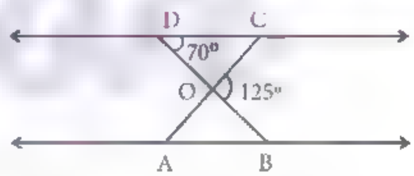
Sol :

Since tangents drawn from an external point to the circle are equal.

$$\begin{aligned} AR &= AQ = 4 \text{ cm} \\ BR &= BP = 3 \text{ cm} \\ PC &= QC \\ QC &= AC - AQ = 11 - 4 = 7 \text{ cm} \\ BC &= BP + PC = 3 + QC \\ &= (3 + 7) \text{ cm} = 10 \text{ cm} \end{aligned}$$

III. Short Answer Questions:

1. In the figure $\Delta ODC \sim \Delta OBA$, $\angle BOC = 125^\circ$ and $\angle CDO = 70^\circ$, find $\angle DOC$, $\angle DCO$ and $\angle OAB$.



Sol :

We have $\angle BOC = 125^\circ$ and

$$\angle CDO = 70^\circ$$

Since $\angle DOC + \angle BOC = 180^\circ$ [linear pair]

$$\angle DOC = 180^\circ - 125^\circ = 55^\circ \qquad \text{... (1)}$$

In ΔDOC Using angle sum property, we get

$$\angle DOC + \angle ODC + \angle DCO = 180^\circ$$

$$55^\circ + 70^\circ + \angle DCO = 180^\circ$$

$$\angle DCO = 180^\circ - 55^\circ - 70^\circ$$

$$\angle DCO = 55^\circ \qquad \text{... (2)}$$

Again $\Delta ODC \sim \Delta OBA$ (given)

$\therefore$ Their corresponding angles are equal.

$$\angle OCD = \angle OAB$$

$$= 55^\circ \qquad \text{... (3)}$$

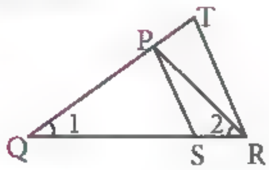
$\therefore$ From (1), (2) and (3)

$$\angle DOC = 55^\circ,$$

$$\angle DCO = 55^\circ \text{ and}$$

$$\angle OAB = 55^\circ$$

2. In the figure $\frac{QR}{QS} = \frac{QT}{PR}$ and $\angle 1 = \angle 2$ show that $\Delta PQS \sim \Delta TQR$



Sol :

In ΔPQR $\angle 1 = \angle 2$ (given)

$PR = QP$ ... (1)

[$\because$ In a triangle, sides opposite to equal angles are equal]

Given $\frac{QR}{QS} = \frac{QT}{PR}$ ... (2)

From (1) and (2)

$$\frac{QR}{QS} = \frac{QT}{QP}$$

$$\frac{QS}{QR} = \frac{QP}{QT} \quad \text{... (3)}$$

Now in ΔPQS and ΔTQR

$$\frac{QS}{QR} = \frac{QP}{QT}$$

$$\angle SQP = \angle RQT = \angle 1 \text{ [From (3)]}$$

$\therefore$ Using SAS similarity criteria

$$\Delta PQS \sim \Delta TQR.$$

3. S and T are points on sides PR and QR of ΔPQR such that $\angle P = \angle RTS$.

Show that $\Delta RPQ \sim \Delta RTS$.

Sol :

T is a point on QR and S is a point or PR such that

$\angle RTS = \angle P$

Now in ΔRPQ and ΔRTS

$\angle RPQ = \angle RTS$ (given)

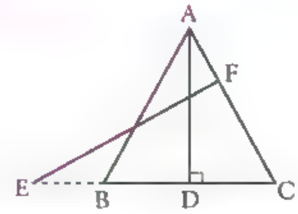
$\angle PRQ = \angle TRS$ [common]

$\therefore$ Using AA similarity, we have

$$\Delta RPQ \sim \Delta RTS$$

4. In the figure E is a point on side CB produced of an isosceles triangle ABC with AB = AC. If AD $\perp$ BC and EF $\perp$ AC.

Prove that $\Delta ABD \sim \Delta ECF$.



Sol :

We have an isosceles triangle ΔABC in which

AB = AC.

In ΔABD and ΔECF

AB = AC (given)

$\Rightarrow$ Angles opposite to them are equal.

$\therefore$ $\angle ACB = \angle ABC$

$\angle ECF = \angle ABD$ ... (1)

AD $\perp$ BC and EF $\perp$ AC

$\angle ADB = \angle EFC = 90°$ ... (2)

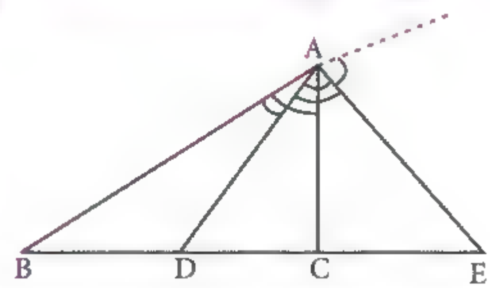
From (1) and (2) we have

By AA criteria of similarity

$$\Delta ABD \sim \Delta ECF$$

5. The bisector of interior $\angle A$ of ΔABC meets BC in D and the bisector of exterior $\angle A$ meets BC produced in E. Prove that $\frac{BD}{BE} = \frac{CD}{CE}$.

Sol :



Given: In ΔABC AD and AE respectively the bisectors of the interior and exterior angles at A.

To prove: $\frac{BD}{BE} = \frac{CD}{CE}$

Proof: Since AD is the internal bisector of $\angle A$ meeting BC at D.

$$\frac{AB}{AC} = \frac{BD}{DC} \quad \text{... (1)}$$

Since AE is the external bisector of $\angle A$ meeting BC produced in E.

$$\frac{AB}{AC} = \frac{BE}{CE} \quad \text{... (2)}$$

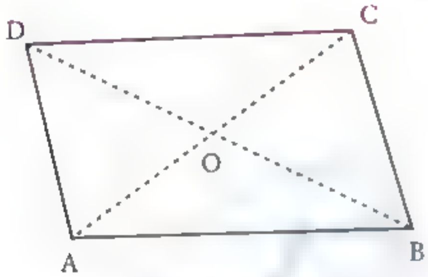
From (1) and (2) we get

$$\frac{BD}{DC} = \frac{BE}{CE}$$

$$\frac{BD}{BE} = \frac{CD}{CE}$$

6. If the diagonal BD of a quadrilateral ABCD bisects both $\angle B$ and $\angle D$, Show that $\frac{AB}{BC} = \frac{AD}{CD}$.

Sol :



Given: A quadrilateral ABCD in which the diagonal BD bisects $\angle B$ and $\angle D$.

To prove: $\frac{AB}{BC} = \frac{AD}{CD}$.

Construction: Join AC intersecting BD in O.

Proof

In ΔABC, BO is the bisector of $\angle B$.

$$\frac{AO}{OC} = \frac{BA}{BC}$$

$$\frac{OA}{OC} = \frac{AB}{BC} \quad \text{.....(1)}$$

In ΔADC, DO is the bisector of $\angle D$

$$\frac{AO}{OC} = \frac{DA}{DC}$$

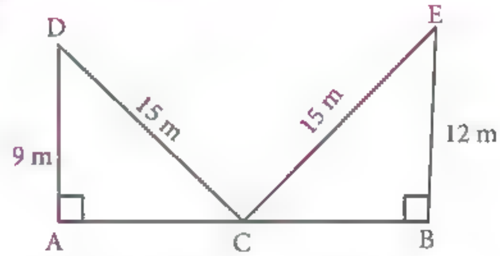
$$\frac{OA}{OC} = \frac{AD}{CD} \quad \text{.....(2)}$$

From (1) and (2) we get

$$\frac{AB}{BC} = \frac{AD}{CD}$$

7. A ladder 15 m long reaches a window which is 9 m above the ground on one side of a street. Keeping its foot at the same point, the ladder is turned to other side of the street to reach a window 12 m high. Find the width of the street.

Sol :



Let AB be the width of the street; C be the foot of the ladder. Let D and E be the windows at heights of 9 m and 12 m respectively from the ground.

Then CD and CE are the two positions of the ladder.

Clearly AD = 9 m, BE = 12 m, CD = CE = 15 m

From the right triangle ΔACD we have

$$CD^2 = AC^2 + AD^2$$

$$15^2 = AC^2 + 9^2$$

$$AC^2 = 225 - 81 = 144$$

$$AC = 12 \text{ m}$$

In ΔBCE, we have

$$CE^2 = BC^2 + BE^2$$

$$15^2 = BC^2 + 12^2$$

$$BC^2 = 225 - 144 = 81$$

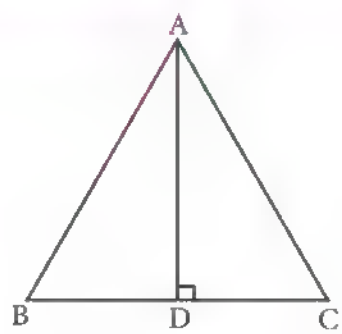
$$BC = 9 \text{ m}$$

Hence width of the street AB = AC + CB

= 12 + 9 = 21 m

8. Prove that three times the square of any side of an equilateral triangle in equal to four times the square of the altitude.

Sol :



Let ABC be an equilateral triangle and AD $\perp$ BC.

In ΔADB and ΔADC, we have

$$AB = AC$$
$$\angle B = \angle C$$
$$\text{and } \angle ADB = \angle ADC$$

$\therefore$ By RHS Criteria

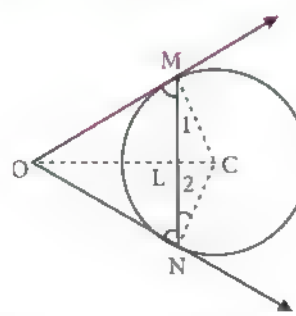
$$\Delta ADB \cong \Delta ADC$$
$$BD = DC \quad [\because \text{By CPCTC}]$$
$$\Rightarrow BD = DC = \frac{1}{2} BC$$

Since ΔADB is right triangle right angled at D

$$AB^2 = AD^2 + BD^2$$
$$\Rightarrow AB^2 = AD^2 + \left(\frac{1}{2} BC\right)^2$$
$$\Rightarrow AB^2 = AD^2 + \frac{BC^2}{4}$$
$$AB^2 = AD^2 + \frac{AB^2}{4} \quad [\because BC = AB]$$
$$AD^2 = AB^2 - \frac{AB^2}{4}$$
$$= \frac{4AB^2 - AB^2}{4}$$
$$AD^2 = \frac{3AB^2}{4}$$
$$4AD^2 = 3AB^2$$

9. Prove that the tangents drawn at the ends of a chord of a circle make equal angles with the chord.

Sol :



Let NM be the chord of a circle with centre C. Let the tangents at M and N meet at O

$\therefore$ OM is a tangent at M

$$\therefore \angle OMC = 90^\circ$$
$$\text{Similarly } \angle ONC = 90^\circ$$

$$\text{Since } CM = CN$$
$$\text{In } \Delta CMN \ \angle 1 = \angle 2$$

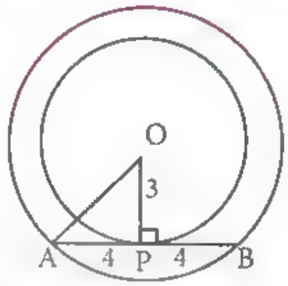
[$\because$ Angles opposite to equal sides are equal in a triangle]

$$\angle OMC - \angle 1 = \angle ONC - \angle 2$$
$$\angle OML = \angle ONL$$

Thus tangents make equal angles with the chord.

10. Two concentric circles have a common center 'O' the chord AB to the bigger circles touches the smaller circle at P. If OP = 3 cm and AB = 8 cm then find the radiu of the bigger circle.

Sol :



$\therefore$ AB touches the smaller circle at P

$\therefore$ OP $\perp$ AB $\Rightarrow \angle OPA = 90^\circ$

Now AB is the chord of the bigger circle. Since the perpendicular from the centre to a chord, bisects the chord.

$\therefore$ P is the mid point of AB

$$AP = \frac{8}{2} = 4 \text{ cm}$$

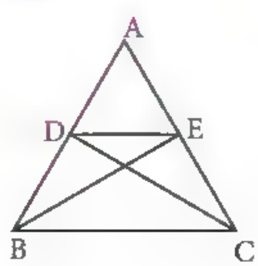
In right ΔAPO, we have

$$AO^2 = OP^2 + AP^2$$
$$AO^2 = 3^2 + 4^2 = 9 + 16$$
$$AO^2 = 25$$
$$AO = 5 \text{ cm}$$

$\therefore$ The radius of the bigger circle = 5 cm.

IV. Long Answer Questions

1. In the figure, if $\Delta ABE \cong \Delta$ ACD, show that $\Delta ADE \sim \Delta ABC$.



Sol :

We have $\Delta ABE \cong \Delta ACD$

Their corresponding parts are equal.

$$AB = AC$$
$$AE = AD$$
$$\therefore \frac{AB}{AC} = \frac{AE}{AD} \Rightarrow \frac{AB}{AE} = \frac{AC}{AD}$$
$$\Rightarrow \frac{AB}{AD} = \frac{AC}{AE} \quad [\because AE = AD] \quad \ldots(1)$$

Now in ΔADE and ΔABC

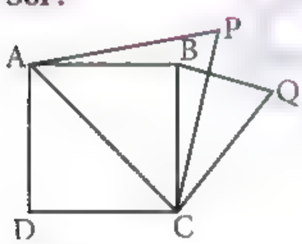
$$\frac{AB}{AD} = \frac{AC}{AE} \quad \text{[From (1)]}$$
$$\angle DAE = \angle BAC \quad \text{[common]}$$

Using SAS similarity criteria we have

$$\Delta ADE \sim \Delta ABC$$

2. Prove that the area of an equilateral triangle described on one side of a square in equal to half the area of the equilateral triangle described on one of its diagonals

Sol :



We have a square ABCD, whose diagonal AC. Equilateral triangle ΔBQC is described on the side BC and another equilateral ΔAPC is described on the diagonal AC.

$\therefore$ All equilateral triangles are similar.

$$\Delta APC \sim \Delta BQC$$

$\therefore$ The ratio of their areas is equal to the square of the ratio of their corresponding sides.

$$\frac{area\,(\Delta\,APC)}{area\,(\Delta\,BQC)} = \left(\frac{AC}{BC}\right)^2 \quad \ldots(1)$$

Since the length of a diagonal of a square $= \sqrt{2} \times side$

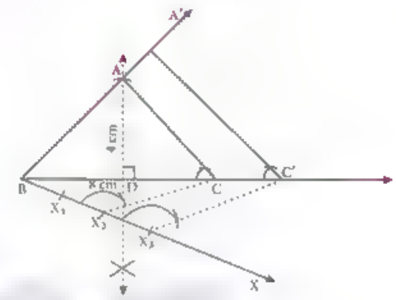
$$AC = \sqrt{2} \times BC \quad \ldots(2)$$

From (1) and (2) we have

$$\frac{area(\Delta\,APC)}{area(\Delta\,BQC)} = \left(\frac{\sqrt{2}\,BC}{BC}\right)^2$$
$$= \left(\sqrt{2}\right)^2 = 2$$
$$\therefore \text{area } (\Delta BQC) = \frac{1}{2} \text{ area } (\Delta APC)$$

3. Construct an isosceles triangle whose base is 8 cm and altitude 4 cm and then another triangle whose sides are $1\frac{1}{2}$ times the corresponding sides of the isosceles triangle.

Sol :

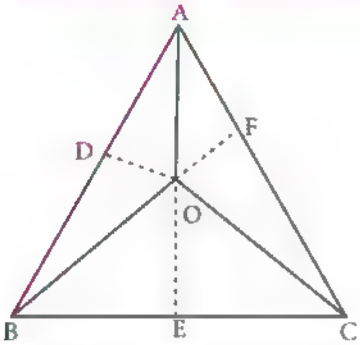


Steps of construction:

1. Drawn BC = 8 cm.
2. Drawn the perpendicular bisector of BC which intersects BC at D.
3. Marked a point A on the above perpendicular such that DA = 4 cm.
4. Joined AB and AC. Thus ΔABC is the required isosceles triangle.
5. Then drawn a ray BX such that $\angle CBX$ is an acute angle.
6. On BX, marked three points (since $1\frac{1}{2} = \frac{3}{2}$) X_1, X_2 and X_3 such that $BX_1 = X_1X_2 = X_2X_3$.
7. Joined X_2 to C.
8. Drawn a line through X_3 parallel to X_2C and intersecting BC extended to C'.
9. Drawn a line through C' parallel to CA intersecting BA extended at A'. Thus $\Delta A'BC'$ is the required triangle.

4. O is any point inside a triangle ΔABC. The bisector of $\angle AOB, \angle BOC, \angle COA$ meet the sides AB, BC and CA in point D, E and F respectively. Show that AD × BE × CF = DB × EC × FA.

Sol :



In ΔAOB, OD is the bisector of $\angle AOB$

$$\frac{OA}{OB} = \frac{AD}{DB} \quad \text{... (1)}$$

In ΔBOC, OE in the bisector of $\angle BOC$

$$\frac{OB}{OC} = \frac{BE}{EC} \quad \text{... (2)}$$

In ΔCOA, OF is the bisector of $\angle COA$

$$\frac{OC}{OA} = \frac{CF}{FA} \quad \text{... (3)}$$

Multiplying the corresponding sides of (1), (2) and (3) we get

$$\frac{OA}{OB} \times \frac{OB}{OC} \times \frac{OC}{OA} = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

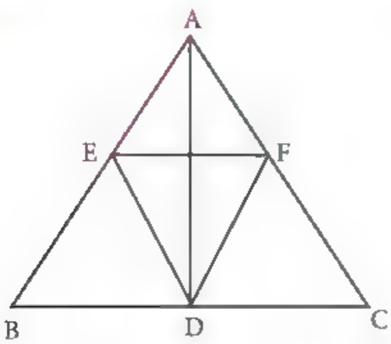
$$1 = \frac{AD}{DB} \times \frac{BE}{EC} \times \frac{CF}{FA}$$

$$DB \times EC \times FA = AD \times BE \times CF$$

$$AD \times BE \times CF = DB \times EC \times FA$$

5. In AD is the median of ΔABC. The bisector of $\angle ADB$ and $\angle ADC$ meet AB and AC in E and F respectively. Prove that EF || BC.

Sol :



Given: In ΔABC, , AD is the median and DE and DF are the bisectors of $\angle ADB$ and $\angle ADC$ respectively meeting AB and AC in E and F respectively.

To prove: $EF \parallel BC$

Proof

In ΔADB, DE is the bisector of $\angle ADB$

$$\therefore \frac{AD}{DB} = \frac{AE}{EB} \quad \text{... (1)}$$

In ΔADC, DF is the bisector of $\angle ADC$

$$\frac{AD}{DC} = \frac{AF}{FC}$$

$$\Rightarrow \frac{AD}{DB} = \frac{AF}{FC}$$

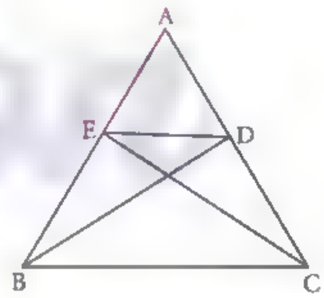
[$\because$ AD is the median BD = DC] ... (2)

From (1) and (2), we get

$$\frac{AE}{EB} = \frac{AF}{FC}$$

6. The bisectors of the angle B and C of a triangle ABC, meet the opposite sides in D and E respectively. If DE $DE \parallel BC$ BC, prove that the triangle is isosceles.

Sol :



Given: ΔABC is a triangle in which the bisectors of $\angle B$ and $\angle C$ meet the sides AC and AB at D and E respectively.

To prove: AB = AC

Construction: Join DE

Proof: In ΔABC, BD is the bisector of $\angle B$

$$\therefore \frac{AB}{BC} = \frac{AD}{DC} \quad \text{... (1)}$$

In ΔABC, CE is the bisector of $\angle C$

$$\therefore \frac{AC}{BC} = \frac{AE}{BE} \quad \text{... (2)}$$

Now DE || BC

$$\Rightarrow \frac{AE}{BE} = \frac{AD}{DC} \quad \text{... (3)}$$

[By thales theorem]

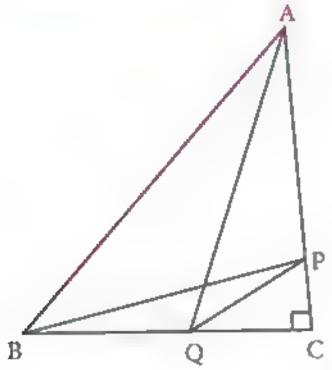
From (1), (2) and (3), we have

$$\frac{AB}{BC} = \frac{AC}{BC} \Rightarrow AB = AC$$

Hence ΔABC is isosceles.

7. **P and Q are points on the sides CA and CB respectively of ΔABC right angled at C. Prove that $AC^2 + BP^2 = AB^2 + PQ^2$**

Sol :



In right angled triangles ACQ and PCB, we have

$$AQ^2 = AC^2 + CQ^2 \text{ and}$$
$$PB^2 = PC^2 + CB^2$$
$$\Rightarrow AQ^2 + BP^2 = (AC^2 + CQ^2) + (PC^2 + CB^2)$$
$$\Rightarrow AQ^2 + BP^2 = (AC^2 + BC^2) + (PC^2 + QC^2)$$

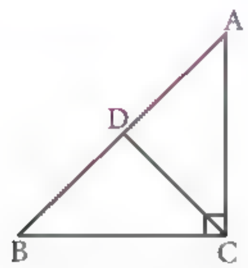
[By pythagoras theorem we have

$AC^2 + BC^2 = AB^2$ and $PC^2 + QC^2 = PQ^2$]

$$\therefore AQ^2 + BP^2 = AB^2 + PQ^2$$

8. **ABC is a right triangle right angled at C and $AC = \sqrt{3}$ BC. Prove that $\angle ABC = 60°$.**

Sol :



Let D be the midpoint of AB. Join CD.

Since ABC is a right angled triangle, $\angle ACB = 90°$

$$AB^2 = AC^2 + BC^2$$
$$AB^2 = \left(\sqrt{3}\,BC\right)^2 + BC^2$$

[$\because AC = \sqrt{3}\,BC$ given]

$$AB^2 = 3BC^2 + BC^2$$
$$\Rightarrow AB^2 = 4\,BC^2$$
$$AB = 2\,BC \qquad ...(1)$$

But $BD = \frac{1}{2}\,AB$

$$AB = 2\,BD \qquad ...(2)$$

(1) and (2) $\Rightarrow$ BD = BC

We know that the midpoint of the hypotenuse of a right triangle is equidistance from the vertices

$$\therefore CD = AD = BD$$
$$\Rightarrow CD = BC$$

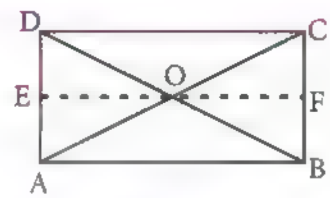
Thus in ΔABC we have BD = CD = BC

$\therefore \Delta BCD$ is equilateral

$$\Rightarrow \therefore \angle ABC = 60°$$

9. **A point O in the interior of a rectangle ABCD is joined with each of the vertices A, B, C and D. Prove that $OB^2 + OD^2 = OC^2 + OA^2$**

Sol :



Let ABCD be the given rectangle. Let 'O' be the point within it. Join OA, OB, OC and OD.

Through O draw EOF || AB. Then ABFE is a rectangle.

In right triangles ΔOEA and ΔOFC, we have

$$OA^2 = OE^2 + AE^2 \text{ and } OC^2 = OF^2 + CF^2$$
$$OA^2 + OC^2 = (OE^2 + AE^2) + (OF^2 + CF^2)$$
$$OA^2 + OC^2 = OE^2 + OF^2 + AE^2 + CF^2 \qquad ... (1)$$

Now in right triangles ΔOFB and ΔDOE we have

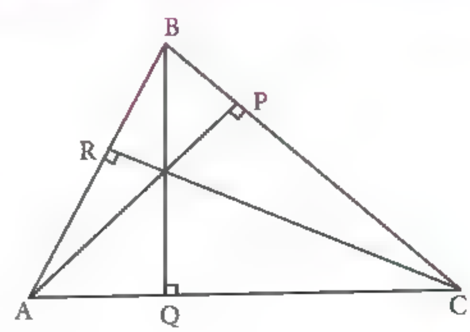
$$OB^2 = OF^2 + FB^2 \text{ and } OD^2 = OE^2 + DE^2$$
$$OB^2 + OD^2 = (OF^2 + FB^2) + (OE^2 + DE^2)$$
$$OB^2 + OD^2 = OE^2 + OF^2 + DE^2 + BF^2$$
$$OB^2 + OD^2 = OE^2 + OF^2 + CF^2 + AE^2 \qquad ... (2)$$

[$\because$ DE = CF and AE = BF]

From (1) and (2) we get $OA^2 + OC^2 = OB^2 + OD^2$

10. **Show that the altitudes of a triangle are concurrent.**

Sol :



Let in ΔABC , P, Q and R are the foot of the perpendiculars drawn from the vertices A, B and C respectively.

$\Delta BRC \sim \Delta BPA$

$\because \angle BRC = \angle BPA = 90°$,

$\angle$B is common [$\therefore$ By AA criteria]

$$\frac{BR}{BP} = \frac{BC}{BA} \quad \text{... (1)}$$

Similarly $\Delta AQB \sim \Delta ARC$

$$\frac{AQ}{AR} = \frac{AP}{AC} \quad \text{... (2)}$$

and $\Delta CPA \sim \Delta CQB$

$$\frac{CP}{CQ} = \frac{AC}{BC} \quad \text{... (3)}$$

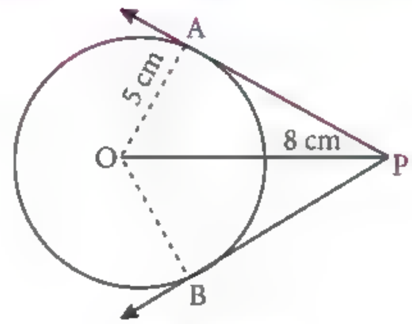
Multiplying (1), (2) and (3), we have

$$\frac{BR}{BP} \times \frac{AQ}{AR} \times \frac{CP}{CQ} = \frac{BC}{AB} \times \frac{AB}{AC} \times \frac{AC}{BC} = 1$$

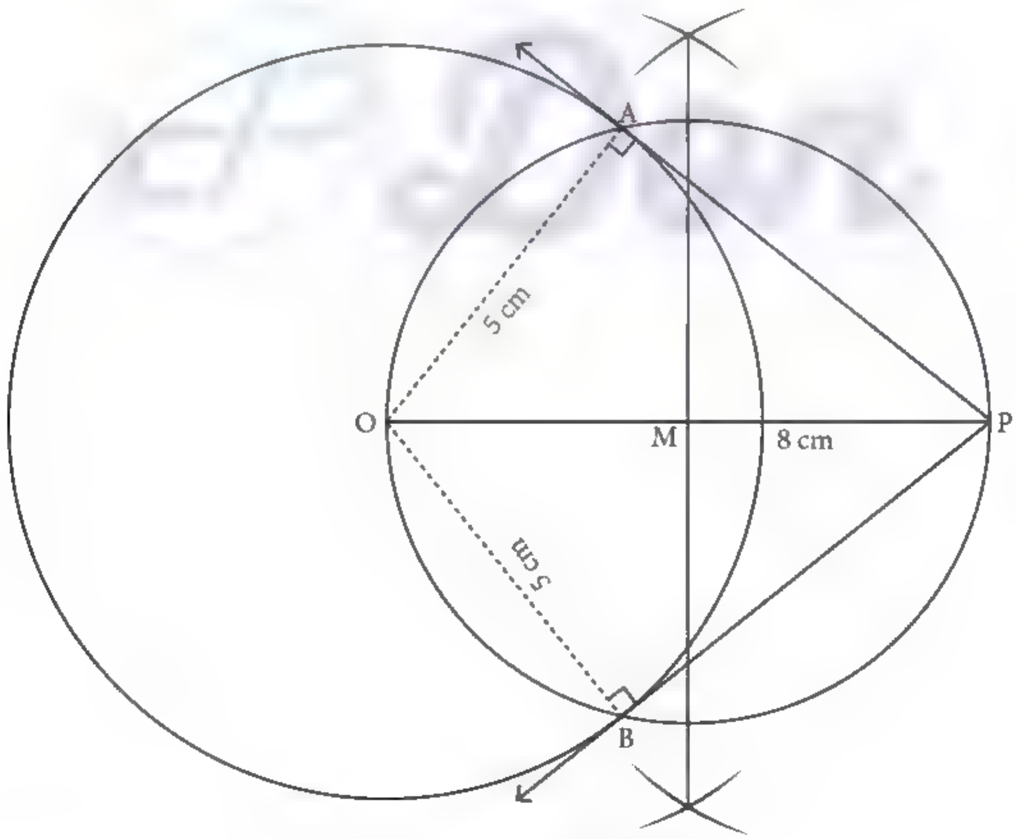
$\therefore$ By Ceva's theorem altitudes are concurrent.

11. Draw a circle of radius 5 cm. Consider a point P at a distance of 8 cm from the circle's centre and draw two tangents from P.

Sol :



Rough Diagram



AP = BP = 6.2 cm

Construction:

Step 1: With center 'O' drawn a circle of radius 5 cm

Step 2: Drawn a line OP = 8 cm

Step 3: Draw a perpendicular bisector of OP which cuts OP at M

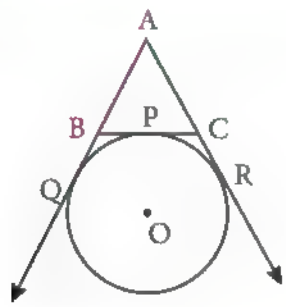
Step 4: With M as center and MO as radius drawn a circle which cuts previous circle at A and B.

Step 5: Joined AP and BP. AP and BP are the required tangents.
$AP = BP = 6.2$ cm.

12. A circle is touching the side BC of a ΔABC at P and touching AB and AC produced at Q and R.

Prove that $AQ = \frac{1}{2}$ (Perimeter of ΔABC)

Sol :



Since the two tangents drawn to a circle from an external point are equal.

$$AQ = AR \quad \text{... (1)}$$

$$\text{Similarly, } BQ = BP \quad \text{... (2)}$$

$$\text{and } CR = CP \quad \text{... (3)}$$

Now perimeter of ΔABC

$$= AB + BC + AC$$

$$= AB + (BP + PC) + AC$$

$$= AB + (BQ + CR) + AC \quad \text{[From (2) and (3)]}$$

$$= (AB + BQ) + (CR + AC)$$

$$= AQ + AR$$

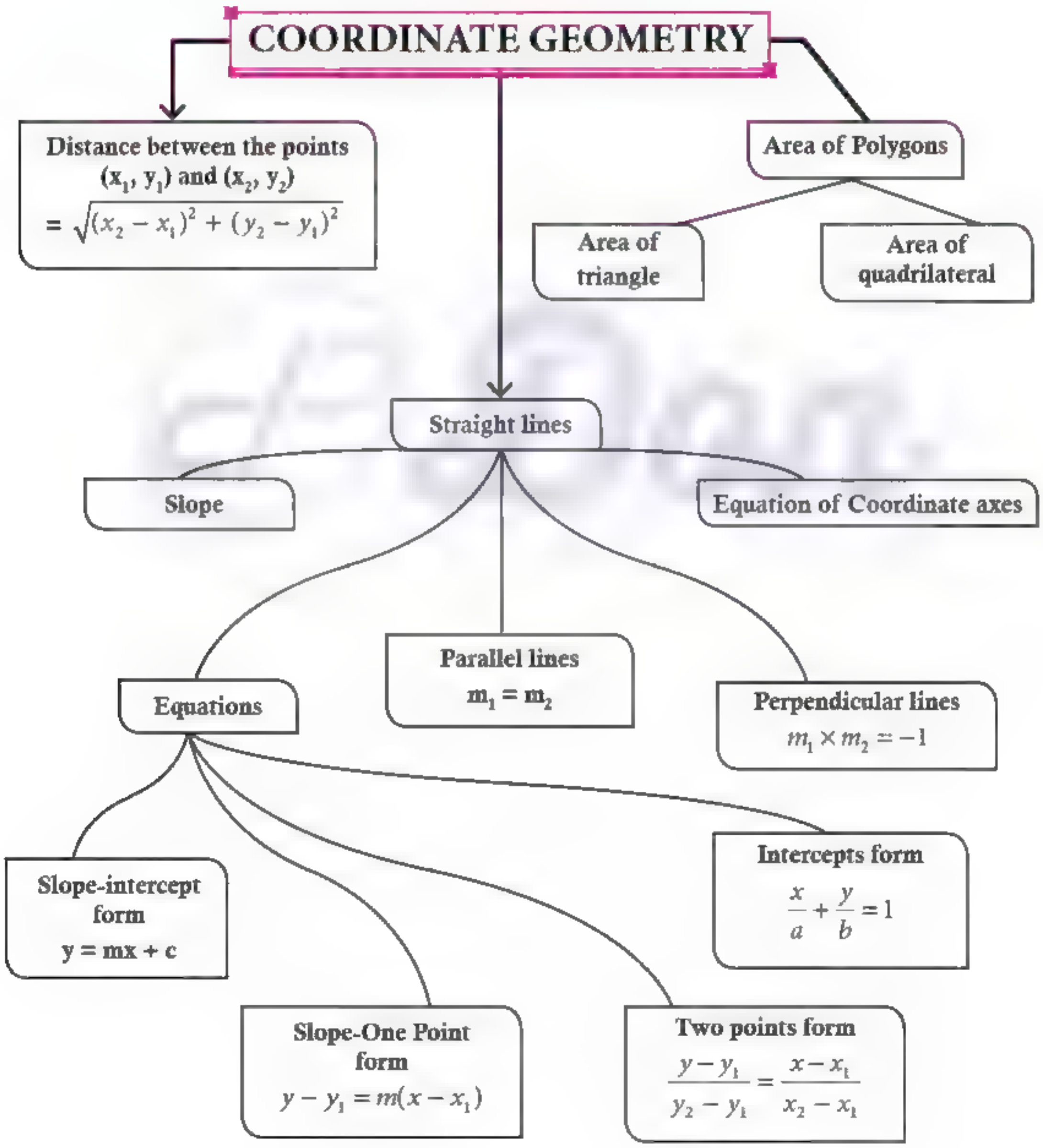
$$= AQ + AQ \quad \text{[From (1)]}$$

$$= 2\,AQ$$

$$AQ = \frac{1}{2} \text{ (Perimeter of } \Delta ABC\text{)}$$

UNIT 5 COORDINATE GEOMETRY

MIND MAP



Area of a Triangle

Key Points

Area of triangle with the given vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

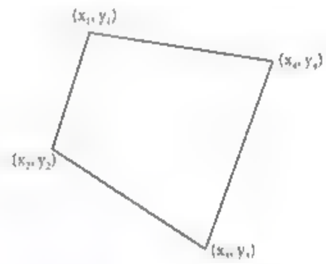
$$= \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \text{ sq. units}$$

(or) $$\frac{1}{2}\left[(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)\right] \text{ sq. units.}$$

Condition for collinearity

- If the three distinct points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then area of the triangle formed by the points is zero.

  i.e., $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$



Area of Quadrilateral formed by the points

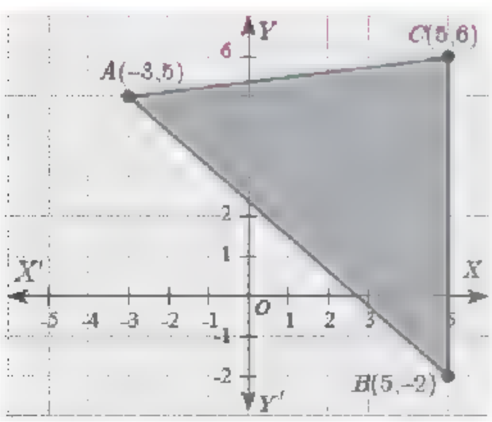
(x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) is

$$= \frac{1}{2}\left[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\right] \text{ sq. units.}$$

Worked Examples

5.1 Find the area of the triangle whose vertices are at (–3, 5), (5, 6) and (5, –2).

Sol :



Plot the points in a rough diagram and take them in counter-clockwise order.

Let the vertices be A(–3, 5), B(5, – 2), C(5, 6)

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ (x_1, y_1) & (x_2, y_2) & (x_3, y_3) \end{array}$$

The area of ΔABC is

$$= \frac{1}{2}\left\{(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)\right\}$$

$$= \frac{1}{2}\left\{(6 + 30 + 25) - (25 - 10 - 18)\right\}$$

$$= \frac{1}{2}\left\{61 + 3\right\}$$

$$= \frac{1}{2}(64) = 32 \text{ sq. units}$$

5.2 Show that the points P(–1.5, 3), Q(6, – 2), R(–3, 4) are collinear.

Sol :

The points are P(–1.5, 3), Q(6, – 2), R(–3, 4)

Area of ΔPQR

$$= \frac{1}{2}\left\{(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)\right\}$$

$$= \frac{1}{2}\left\{(3 + 24 - 9) - (18 + 6 - 6)\right\}$$

$$= \frac{1}{2}\left\{18 - 18\right\} = 0 \text{ sq. units}$$

Therefore, the given points are collinear.

5.3 If the area of the triangle formed by the vertices A(–1, 2), B(k, – 2) and C(7, 4) (taken in order) is 22 sq. units, find the value of k.

Sol :

The vertices are A(–1, 2), B(k, – 2) and C(7, 4)

Area of triangle ABC is 22 sq. units

$$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 22$$

$$\frac{1}{2}\ \{(2 + 4k + 14) - (2k - 14 - 4)\} = 22$$

$$2k + 34 = 44 \Rightarrow 2k = 10 \Rightarrow k = 5$$

5.4 If the points P(-1, -4), Q(b, c) and R(5, – 1) are collinear and 2b + c = 4, then find the values of b and c.

Sol :

Since the three points P(–1, –4), Q(b, c) and R(5, –1) are collinear,

Area of triangle PQR = 0

$$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 0$$

$$\frac{1}{2}\ \{(-c - b - 20) - (-4b + 5c + 1)\} = 0$$

$$-c - b - 20 + 4b - 5c - 1 = 0$$

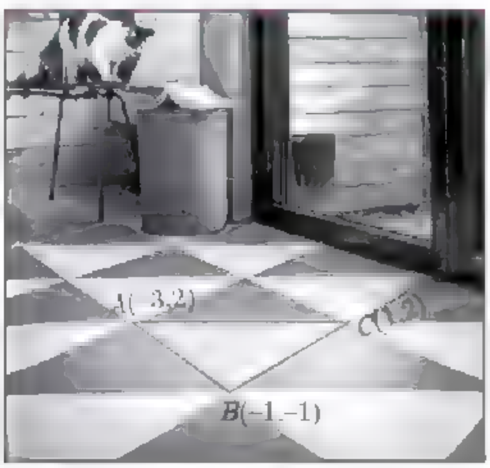
$$b - 2c = 7 \quad \ldots (1)$$

Also, 2b + c =4 (from given information) ... (2)

Solving (1) and (2) we get b = 3, c = – 2

5.5 The floor of a hall is covered with identical tiles which are in the shapes of triangles. One such triangle has the vertices at (– 3, 2), (– 1, – 1) and (1, 2). If the floor of the hall is completely covered by 110 tiles, find the area of the floor.

Sol :



Vertices of one triangular tile are at (– 3, 2), (– 1, – 1) and (1, 2)

$\therefore$ Area of this tile

$$= \frac{1}{2}\ \{(3 - 2 + 2) - (-2 - 1 - 6)\} \text{ sq. units}$$

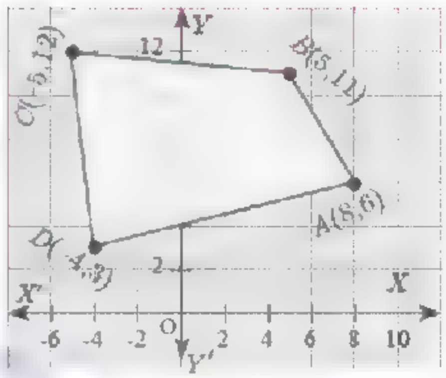
$$= \frac{1}{2}\ (12) = 6 \text{ sq. units}$$

Since the floor is covered by 110 triangle shaped identical tiles,

Area of floor = 110 × 6 = 660 sq. units

5.6 Find the area of the quadrilateral formed by the points (8, 6), (5, 11), (– 5, 12) and (– 4, 3).

Sol :



Before determining the area of quadrilateral, plot the vertices in a graph.

Let the vertices be A(8, 6), B(5, 11), C(– 5, 12) and D(– 4, 3)

Therefore, area of the quadrilateral ABCD

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\}$$

$$= \frac{1}{2}\ \{(88 + 60 - 15 - 24) - (30 - 55 - 48 + 24)\}$$

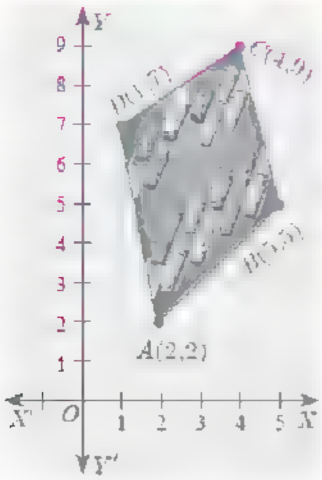
$$= \frac{1}{2}\ \{109 + 49\}$$

$$= \frac{1}{2}\ \{158\} = 79 \text{ sq. units}$$

5.7 The given diagram shows a plan for constructing a new parking lot at a campus. It is estimated that such construction would cost ₹ 1300 per square feet. What will be the total cost for making the parking lot?

Sol :

The new parking lot is a quadrilateral whose vertices are at A(2, 2), B(5, 5), C(4, 9) and D(1, 7).



Therefore, Area of parking lot

$= \frac{1}{2}\ \{(10 + 45 + 28 + 2) - (10 + 20 + 9 + 14)\}$

$= \frac{1}{2}\ \{85 - 53\}$

$= \frac{1}{2}(32) = 16$ sq. units.

Therefore, area of parking lot = 16 sq. ft

Construction rate per square feet = ₹ 1300

Therefore, total cost for constructing the parking lot = 16 × 1300 = ₹ 20800.

Progress Check

1. Complete the following table.

Sl. No.	Points	Distance	Mid Point	Internal		External	
				Point	Ratio	Point	Ratio
(i)	(3, 4), (5, 5)	...	...	...	2 : 3	...	2 : 3
(ii)	(– 7, 13), (– 3, 1)	...	...	$\left(-\frac{13}{3}, 5\right)$	...	(– 13, 15)	...

Ans : (i) (3, 4), (5, 5)

$$\text{Distance} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$
$$= \sqrt{(3-5)^2+(4-5)^2}$$
$$= \sqrt{4+1} = \sqrt{5} \text{ units}$$

$$\text{Mid point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{3+5}{2}, \frac{4+5}{2}\right)$$
$$= \left(4, \frac{9}{2}\right)$$

$$\text{Internal division} = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$
$$= \left(\frac{2(5)+3(3)}{2+3}, \frac{2(5)+3(4)}{2+3}\right) = \left(\frac{19}{5}, \frac{22}{5}\right)$$

$$\text{External division} = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$
$$= \left(\frac{2(5)-3(3)}{2-3}, \frac{2(5)-3(4)}{2-3}\right)$$
$$= \left(\frac{1}{-1}, \frac{-2}{-1}\right) = (-1, 2).$$

(ii) (– 7, 13), (– 3, 1)

$$\text{Distance} = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$
$$= \sqrt{(-7+3)^2+(13-1)^2}$$
$$= \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \text{ units.}$$

$$\text{Mid point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$
$$= \left(\frac{-7-3}{2}, \frac{13+1}{2}\right) = (-5, 7)$$

Internal Division:

Let the Ratio be k : 1, given point $\left(-\frac{13}{3}, 5\right)$

$$\text{Now } \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right) = \left(-\frac{13}{3}, 5\right)$$
$$\left(\frac{k(-3)+1(-7)}{k+1}, \frac{k(1)+1(13)}{k+1}\right) = \left(-\frac{13}{5}, 5\right)$$
$$\left(\frac{-3k-7}{k+1}, \frac{k+13}{k+1}\right) = \left(-\frac{13}{3}, 5\right)$$
$$\Rightarrow \quad \frac{k+13}{k+1} = 5$$
$$k + 13 = 5k + 5$$
$$8 = 4k \Rightarrow 2 = k$$

∴ Ratio is 2 : 1

External Division:

Let the ratio be k : 1 and given point (– 13, 15)

$$\text{Now } \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right) = (-13, 15)$$

$$\left[\frac{k(-3)-1(-7)}{k-1}, \frac{k(1)-1(13)}{k-1}\right] = (-13, 15)$$

$$\Rightarrow \quad \frac{k-13}{k-1} = 15$$

$$k - 13 = 15k - 15$$

$$15 - 13 = 15k - k$$

$$2 = 14k$$

$$k = \frac{2}{14} = \frac{1}{7}$$

$\therefore$ Ratio is 1 : 7

2. A (0, 5), B (5, 0) and C (– 4, – 7) are vertices of a triangle then its centroid will be at _____.

Ans: A (0, 5), B (5, 0) and C (– 4, – 7)

Centroid of a triangle $= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$$= \left(\frac{0+5-4}{3}, \frac{5+0-7}{3}\right)$$

$$= \left(\frac{1}{3}, \frac{-2}{3}\right)$$

3. The vertices of ΔPQR are P (0, – 4), Q (3, 1) and R (– 8, 1)

(i) Draw ΔPQR on a graph paper.

(ii) Check if ΔPQR is equilateral.

(iii) Find the area of ΔPQR.

(iv) Find the co-ordinates of M, the mid-point of QP.

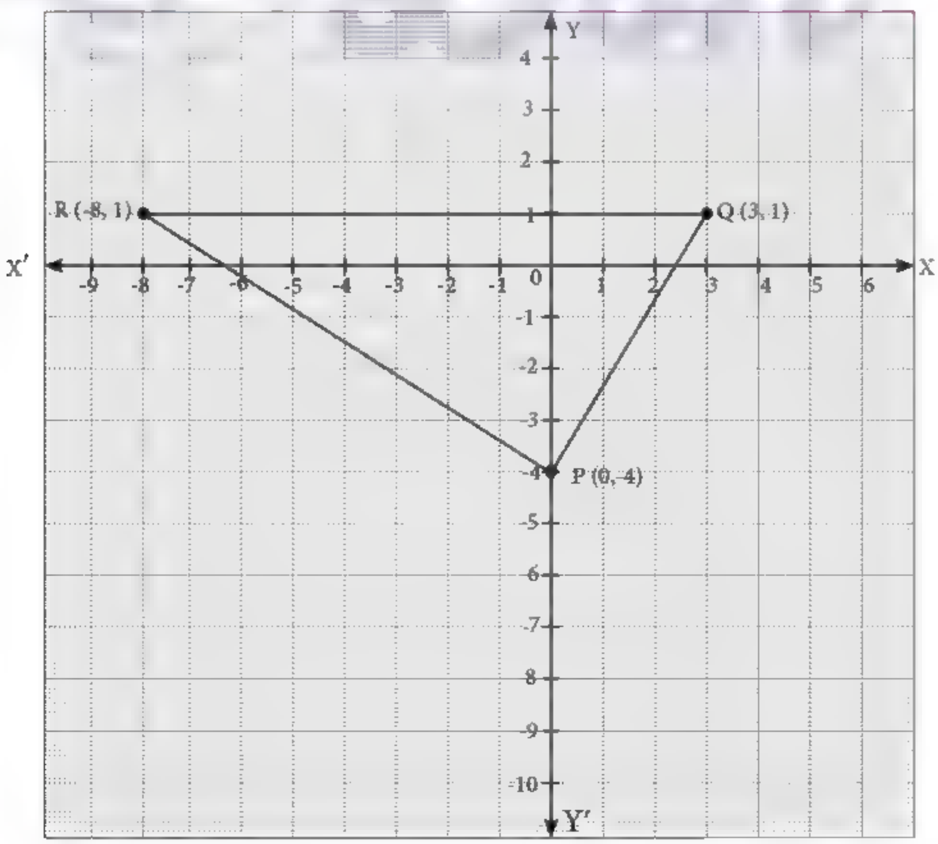
(v) Find the co-ordinates of N, the mid-point of QR.

(vi) Find the area of ΔMPN.

(vii) What is the ratio between the areas of ΔMPN and ΔPQR?

Ans: Given vertices are P (0, – 4), Q (3, 1) and R (– 8, 1)

(i)



(ii) $\Delta\, PQR$ is not equilateral.

(iii) Area of $\Delta\, PQR$

$$= \frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\,[0\,(1-1) + 3\,(1+4) - 8\,(-4-1)]$$

$$= \frac{1}{2}\,[15 + 40] = \frac{55}{2} \text{ sq. units.}$$

(iv) Mid point of

$$QP = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= M\left(\frac{0+3}{2}, \frac{-4+1}{2}\right) = M\left(\frac{3}{2}, -\frac{3}{2}\right)$$

(v) Mid point of

$$QR = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$= N\left(\frac{3-8}{2}, \frac{1+1}{2}\right) = N\left(-\frac{5}{2}, 1\right)$$

(vi) Area of $\Delta\, MPN$

$$M\left(\frac{3}{2}, -\frac{3}{2}\right), P\,(0, -4), N\left(-\frac{5}{2}, 1\right)$$

$$\text{Area } = \frac{1}{2}\left[\frac{3}{2}(-4-1) + 0\left(1+\frac{3}{2}\right) - \frac{5}{2}\left(-\frac{3}{2}+4\right)\right]$$

$$= \frac{1}{2}\left[-\frac{15}{2} - \frac{25}{4}\right]$$

$$= -\frac{1}{2}\left[\frac{30+25}{4}\right] = \frac{55}{8} \text{ sq. units}$$

[$\because$ area cannot be negative.]

(vii) Area of $\Delta\, PQR$ > area of $\Delta\, MPN$.

$$\text{Area of PQR : Area of MPN} = \frac{55}{2} : \frac{55}{4} = 4 : 2$$

4. In a quadrilateral ABCD with vertices A (– 3, – 8), B (6, – 6), C (4, 2), D (– 8, 2)

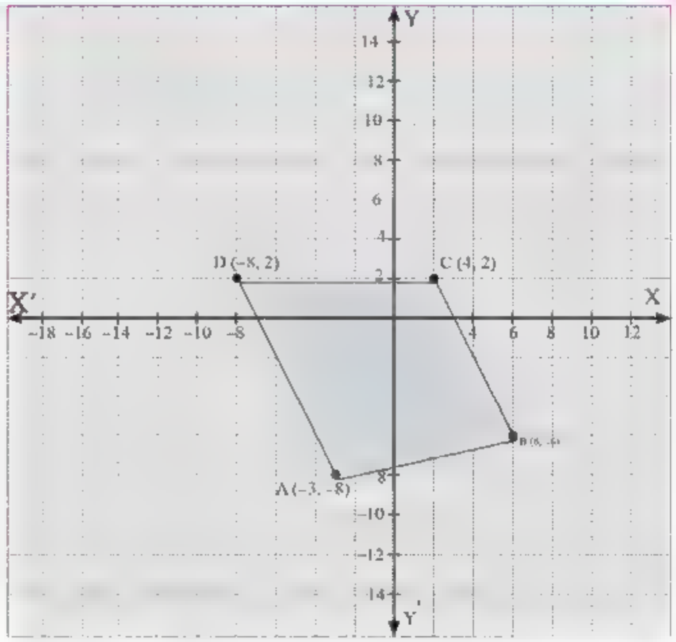
(i) Find the area of $\Delta\, ABC$

(ii) Find the area of $\Delta\, ACD$

(iii) Calculate area of $\Delta\, ABC$ + area of $\Delta\, ACD$

(iv) Find the area of quadrilateral ABCD

(v) Compare the answer (iii) and (iv)



Ans : Quadrilateral ABCD with vertices A (– 3, – 8), B (6, – 6), C (4, 2) and D (– 8, 2)

(i) Area of $\Delta\, ABC = \frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq. units.

$$= \frac{1}{2}\,[-3\,(-6-2) + 6\,(2+8) + 4\,(-8+6)]$$

$$= \frac{1}{2}\,(24 + 60 - 8) = \frac{1}{2}\,(76) = 38 \text{ sq. units.}$$

(ii) Area of $\Delta\, ACD$

$$= \frac{1}{2}\,[-3\,(2-2) + 4\,(2+8) - 8\,(-8-2)]$$

$$= \frac{1}{2}\,[40 + 80] = 60 \text{ sq. units.}$$

(iii) Area of $\Delta\, ABC$ + Area of $\Delta\, ACD$

$= 38 + 60 = 98$ sq. units.

(iv) Area of Quadrilateral ABCD.

$$= \frac{1}{2}\,[(x_1-x_3)(y_2-y_4) - (x_2-x_4)(y_1-y_3)] \text{ sq. units}$$

$$= \frac{1}{2}\,[(-3-4)(-6-2) - (6+8)(-8-2)]$$

$$= \frac{1}{2}\,[56 + 140] = \frac{196}{2} = 98 \text{ sq. units.}$$

(v) From **(iii)** and **(iv)**, we know that

Area of Quadrilateral ABCD = Area of $\Delta\, ABC$ + Area of $\Delta\, ACD$.

Thinking Corner

1. How many triangles exist, whose area is zero?

Ans : No such triangles exist.

2. If the area of a quadrilateral formed by the points (a, a), (– a, a), (a, – a) and (– a, – a), where $a \neq 0$ is 64 square units, then

(i) identify the type of the quadrilateral

(ii) find all possible values of a.

Ans : Given points (a, a), (– a, a), (a, – a) and (– a, –a)

Area of the quadrilateral = 64 sq. units

$$\frac{1}{2}[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)] = 64$$

$$(a + a)(a + a) - (-a - a)(a + a) = 128$$

$$4a^2 + 4a^2 = 128$$

$$8a^2 = 128$$

$$a^2 = 16$$

$$a = \pm 4$$

Since area = 64 [Perfect square and points are equidistant], the quadrilateral is a square.

Exercise 5.1

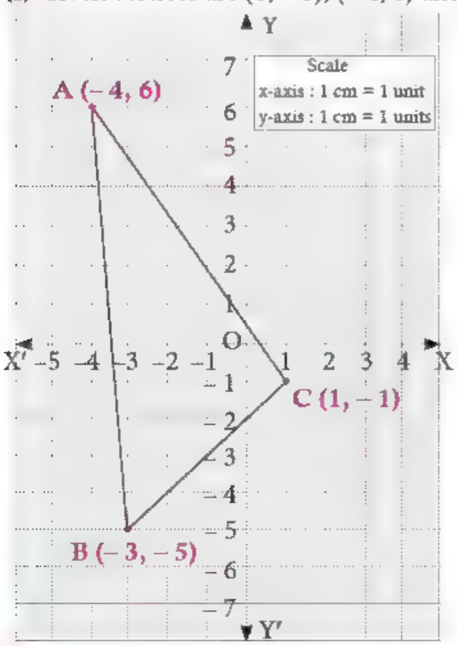
1. Find the area of the triangle formed by the points

(i) (1, – 1), (– 4, 6) and (– 3, – 5)

(ii) (– 10, – 4), (– 8, – 1) and (– 3, – 5)

Sol :

(i) Given vertices are (1, – 1), (– 4, 6) and (– 3, – 5)



A(–4, 6), B(–3, –5), C(1, –1)

Area of triangle ABC $= \frac{1}{2}[(x_1 y_2 + x_2 y_3 + x_3 y_1) - (x_2 y_1 + x_3 y_2 + x_1 y_3)]$

$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq. units

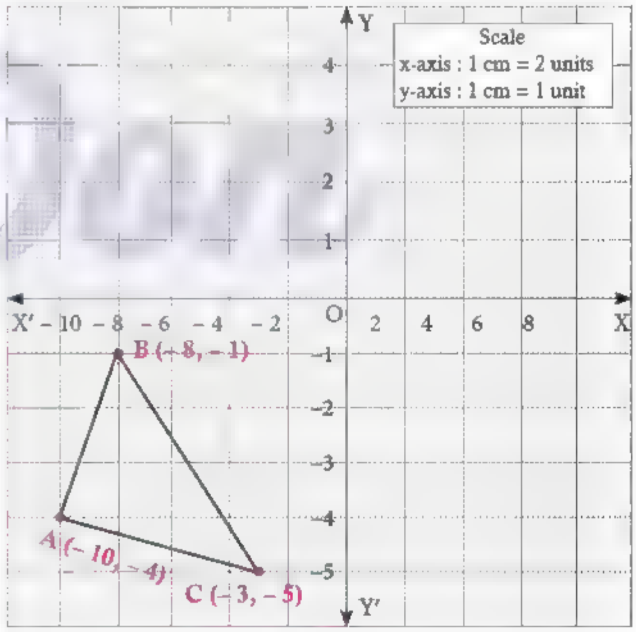
$= \frac{1}{2}[-4(-5+1) - 3(-1-6) + 1(6+5)]$

$= \frac{1}{2}[-4 \times (-4) - 3 \times (-7) + 1 \times (11)]$

$= \frac{1}{2}[16 + 21 + 11]$

$= \frac{1}{2}(48) = 24$ sq. units.

(ii) Given vertices are A(– 10, – 4), B(– 8, – 1) and B(– 3, – 5)



Area of triangle $= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq. units

$= \frac{1}{2}[-10(-1+5) - 8(-5+4) - 3(-4+1)]$

$= \frac{1}{2}[-10(4) - 8(-1) - 3(-3)]$

$= \frac{1}{2}[-40 + 8 + 9]$

$= \frac{1}{2}[-40 + 17] = -\frac{23}{2} = -11.5$

[$\because$ Area cannot be negative]

$\therefore$ Area of triangle = 11.5 sq. units.

2. Determine whether the set of points are collinear.

(i) $\left(-\frac{1}{2}, 3\right)$, (– 5, 6) and (– 8, 8)

(ii) (a, b + c), (b, c + a) and (c, ■ + b)

Sol :

(i) Given points are $\left(-\frac{1}{2}, 3\right)$, (– 5, 6) and (– 8, 8).

Let us use area of triangle formula

Area of triangle= $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$= \frac{1}{2}\left[-\frac{1}{2}(6-8) - 5(8-3) - 8(3-6)\right]$

$= \frac{1}{2}\left[-\frac{1}{2}(-2) - 5(5) - 8(-3)\right]$

$= \frac{1}{2}[1 - 25 + 24] = \frac{1}{2}(0) = 0$

Since, the area of triangle is zero, the given points are collinear.

(ii) Given points are (a, b + c), (b, c + a) and (c, a + b)

Area of triangle= $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$= \frac{1}{2}[a(c + a - a - b) + b(a + b - b - c) + c(b + c - c - a)]$

$= \frac{1}{2}[a(c - b) + b(a - c) + c(b - a)]$

$= \frac{1}{2}[ac - ab + ab - bc + bc - ac]$

$= \frac{1}{2}(0) = 0$

Since, the area of triangle is zero, the given points are collinear.

3. Vertices of given triangles are taken in order and their areas are provided below. In each of the following find the value of 'p'.

Sl. No.	Vertices	Area (sq. units)
(i)	(0, 0), (p, 8), (6, 2)	20
(ii)	(p, p), (5, 6), (5, – 2)	32

Sol :

(i) Given vertices are (0, 0), (p, 8) and (6, 2)

Area of triangle= 20 sq. units.

Area of triangle= $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$\frac{1}{2}[0(8-2) + p(2-0) + 6(0-8)] = 20$

$2p - 48 = 40$

$2p = 40 + 48$

$2p = 88$

$p = \frac{88}{2} = 44$

(ii) Given vertices are (p, p), (5, 6) and (5, – 2)

Area of triangle= –32 sq. units

Area of triangle= $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$

$\frac{1}{2}[p(6+2) + 5(-2-p) + 5(p-6)] = 32$

$8p - 10 - 5p + 5p - 30 = 64$

$8p - 40 = 64$

$8p = 64 + 40 = 104$

$p = \frac{104}{8} = 13$

4. In each of the following, find the value of 'a' for which the given points are collinear.

(i) (2, 3), (4, a) and (6, – 3)

(ii) (a, 2 – 2a), (– a + 1, 2a) and (– 4 – a, 6 – 2 a)

Sol :

(i) Given points are (2, 3), (4, a) and (6, – 3)

Since the points are colinear, Area of triangle is zero.

i.e., $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

$\Rightarrow 2(a + 3) + 4(-3 - 3) + 6(3 - a) = 0$

$2a + 6 - 24 + 18 - 6a = 0$

$-4a + 0 = 0$

$-4a = 0$

$a = 0$

(ii) Given points are (a, 2 – 2a), (– a + 1, 2a) and (– 4 – a, 6 – 2a)

Since the points are collinear, Area of triangle is zero.

i.e., $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

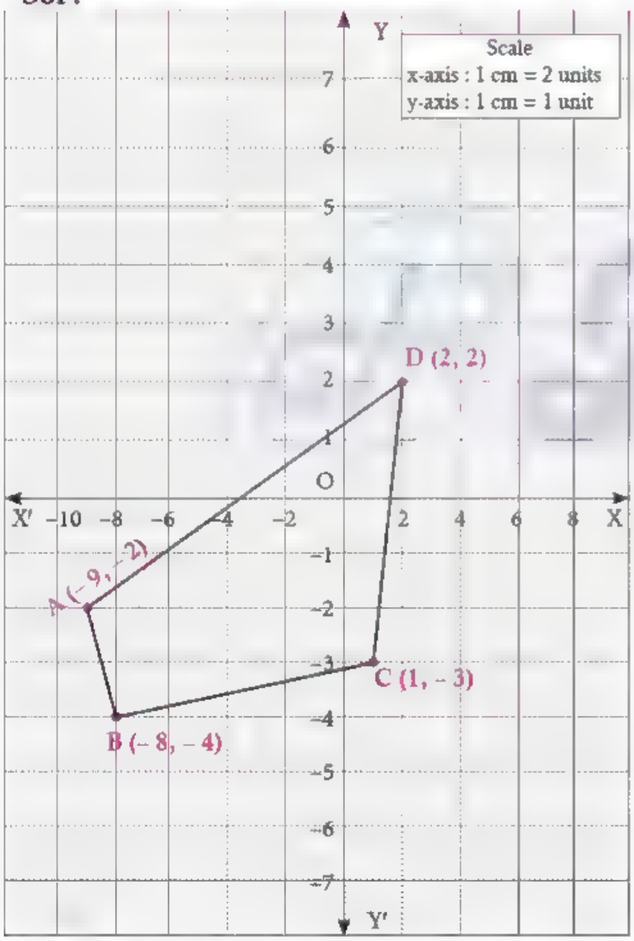
$a(2a - 6 + 2a) + (-a + 1)(6 - 2a - 2 + 2a) + (-4 - a)(2 - 2a - 2a) = 0$

$a(4a - 6) + (-a + 1)(4) + (-4 - a)(2 - 4a) = 0$

$$4a^2 - 6a - 4a + 4 - 8 + 16a - 2a + 4a^2 = 0$$
$$8a^2 + 4a - 4 = 0$$
(Dividing by 4)
$$2a^2 + a - 1 = 0$$
$$(2a - 1)(a + 1) = 0$$
$$2a - 1 = 0, a + 1 = 0 \Rightarrow a = \frac{1}{2}, -1.$$

5. Find the area of the quadrilateral whose vertices are at
(i) (– 9, – 2), (– 8, – 4), (2, 2) and (1, – 3)
(ii) (– 9, 0), (– 8, 6), (– 1, – 2) and (– 6, – 3)

Sol :



(i) Given vertices are A(– 9, – 2), B(– 8, – 4), C(1, –3) and D(2, 2)

Area of the Quadrilateral ABCD

$$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)] \text{ sq. units}$$

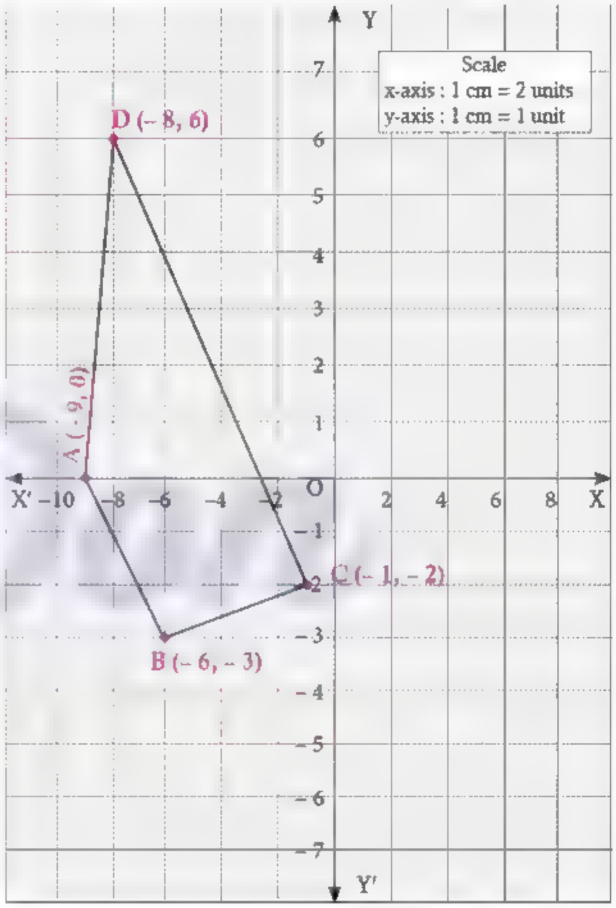
$$= \frac{1}{2}[(-9)(-4) + (-8)(-3) + (1)(2) + (2)(-2)] - [(-8)(-2) + (1)(-4) + (2)(-3) + (-9)(2)]$$

$$= \frac{1}{2}[(36 + 24 + 2 - 4) - (16 - 4 - 6 - 18)]$$

$$= \frac{1}{2}[58 - (-12)] = \frac{1}{2}[70] = 35 \text{ sq. units}$$

$\therefore$ Area of quadrilateral = 35 sq. units.

(ii) Given vertices are (– 9, 0), (– 8, 6), (– 1, – 2) and (– 6, – 3)



Area of quadrilateral ABCD

$$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)]$$

(or) $\frac{1}{2} = [(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)]$ sq. units

$$= \frac{1}{2}[(-9 + 1)(-3 - 6) - (-6 + 8)(0 + 2)]$$

$$= \frac{1}{2}[(-8)(-9) - (2)(2)]$$

$$= \frac{1}{2}[72 - 4] = \frac{68}{2} = 34$$

$\therefore$ Area of quadrilateral = 34 sq. units.

6. Find the value of k, if the area of a quadrilateral is 28 sq. units, whose vertices are (– 4, – 2), (– 3, k), (3, – 2) and (2, 3)

Sol : Given vertices are (– 4, – 2), (– 3, k), (3, – 2) and (2, 3) and area of quadrilateral is 28 sq. units.

$$\text{Area of quadrilateral} = \frac{1}{2}\,[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)]$$

$$\frac{1}{2}\,[(-4-3)(k-3) - (-3-2)(-2+2)] = 28$$

$$(-7)(k-3) - (-5)(0) = 56$$

$$-7k + 21 = 56$$

$$-7k = 56 - 21 = 35$$

$$k = \frac{35}{-7} = -5$$

$$k = -5$$

7. If the points A (– 3, 9), B (a, b) and C (4, – 5) are collinear and if a + b = 1, then find a and b.

Sol :

Given points A (– 3, 9), B(a, b) and C (4, – 5)

Since the points are collinear, Area of triangle ABC = 0

$$\therefore \frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$$

$$\Rightarrow \quad -3(b+5) + a(-5-9) + 4(9-b) = 0$$

$$-3b - 15 - 14a + 36 - 4b = 0$$

$$(\div -7) \quad -14a - 7b + 21 = 0$$

$$2a + b = 3 \quad \dots (1)$$

$$\text{Given that } a + b = 1 \quad \dots (2)$$

Solving (1) and (2)

$$2a + b = 3 \quad \dots (1)$$

$$a + b = 1 \quad \dots (2)$$

$$(1) - (2) \Rightarrow \quad a = 2$$

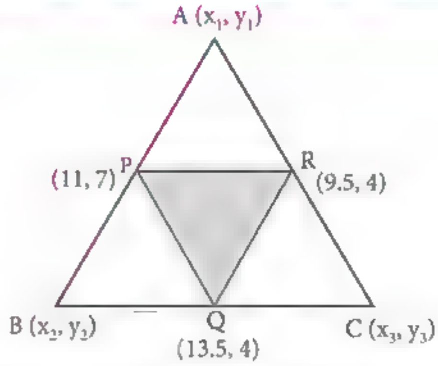
Substituting in (2)

$$2 + b = 1 \Rightarrow b = 1 - 2 = -1$$

$\therefore$ a = 2, b = – 1.

8. Let P (11, 7), Q (13.5, 4) and R (9.5, 4) be the mid-points of the sides AB, BC and AC respectively of ΔABC. Find the coordinates of the vertices A, B and C. Hence find the area of ΔABC and compare this with area of ΔPQR.

Sol :



Given P, Q, R are the mid points of the sides of a triangle.

Let the vertices of triangle ABC be A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3)

P is the mid point of AB

$$\Rightarrow \left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right] = (11, 7)$$

Comparing the co-ordinates, we get

$$x_1 + x_2 = 22 \text{ and} \quad \dots (1)$$

$$y_1 + y_2 = 14 \quad \dots (2)$$

Q is the mid point of BC

$$\Rightarrow \left[\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right] = (13.5, 4)$$

$$x_2 + x_3 = 27 \quad \dots (3)$$

$$y_2 + y_3 = 8 \quad \dots (4)$$

R is the mid point of AC

$$\Rightarrow \left[\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}\right] = (9.5, 4)$$

$$x_1 + x_3 = 19 \quad \dots (5)$$

$$y_1 + y_3 = 8 \quad \dots (6)$$

Solving (1), (3) and (5), we get

$x_1 = 7, x_2 = 15$ and $x_3 = 12$.

Solving (2), (4) and (6)

we get $y_1 = 7, y_2 = 7, y_3 = 1$

$\therefore$ The vertices are A (7, 7), B (15, 7) and C (12, 1)

$$\text{Area of } \Delta ABC = \frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\,[7(7-1) + 15(1-7) + 12(7-7)]$$

$$= \frac{1}{2}\,[42 - 90] = \frac{1}{2}(-48) = -24$$

$$= 24 \text{ sq. units.} \quad [\because \text{ Area cannot be negative}]$$

Area of ΔPQR

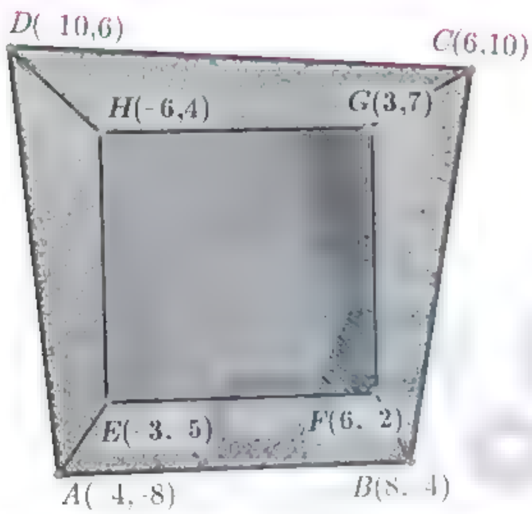
$$= \frac{1}{2}[11(4-4) + 13.5(4-7) + 9.5(7-4)]$$

$$= \frac{1}{2}[0 - 40.5 + 28.5] = \frac{-12}{2} = -6$$

Area $= 6$ sq. units [$\because$ area cannot be negative]

$\therefore$ Area of ΔABC = 4 (Area of ΔPQR).

9. In the figure, the quadrilateral swimming pool shown is surrounded by concrete patio. Find the area of the patio.



Sol : Area of patio = Area of Quadrilateral ABCD – Area of Quadrilateral EFGH

Area of Quadrilateral ABCD

A (– 4, – 8), B (8, – 4), C (6, 10) and D (– 10, 6)

$$\text{Area} = \frac{1}{2}[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)]$$

$$= \frac{1}{2}[(-4-6)(-4-6) - (8+10)(-8-10)]$$

$$= \frac{1}{2}[100 + 324] = \frac{424}{2} = 212 \text{ sq. units}$$

Area of Quadrilateral EFGH

E (– 3, – 5), F (6, – 2), G (3, 7) and H (– 6, 4)

$$\text{Area} = \frac{1}{2}[(-3,-3)(-2-4) - (6+6)(-5-7)]$$

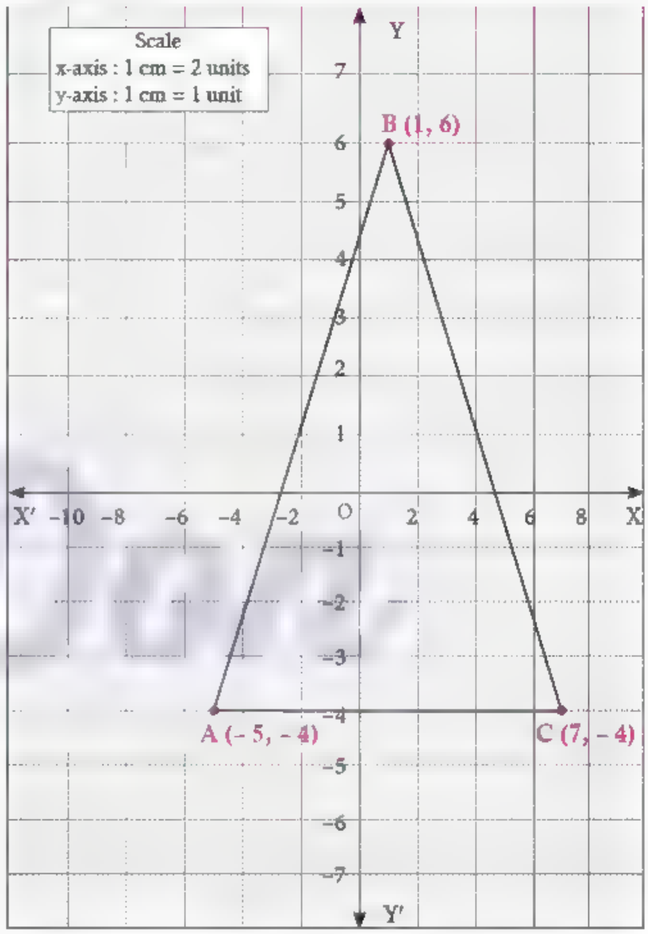
$$= \frac{1}{2}[36 + 144] = \frac{180}{2} = 90 \text{ sq. units}$$

$\therefore$ Area of patio = Area of Quadrilateral ABCD – Area of Quadrilateral EFGH

$= 212 - 90 = 122$ sq. units

10. A triangular shaped glass with vertices at A(– 5, – 4), B(1, 6) and C(7, – 4) has to be painted. If one bucket of paint covers 6 square feet, how many buckets of paint will be required to paint the whole glass, if only one coat of paint is applied.

Sol :



Given vertices are A (– 5, – 4), B (1, 6) and C (7, – 4)

$$\text{Area of triangle} = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \text{ sq. units}$$

$$\text{Area of triangle ABC} = \frac{1}{2}[-5(6+4) + 1(-4+4) + 7(-4-6)]$$

$$= \frac{1}{2}[-50 + 0 - 70] = \frac{-120}{2} = -60$$

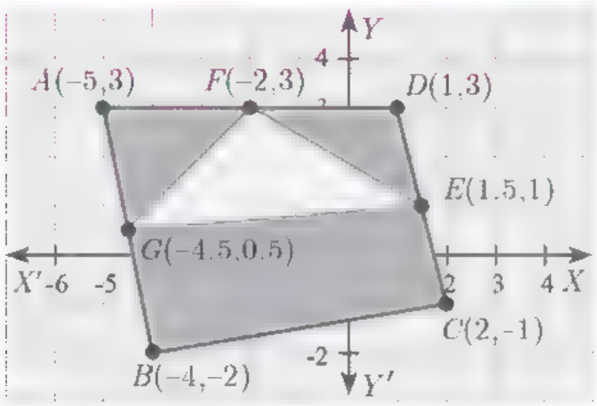
[$\because$ Area cannot be negative].

$\therefore$ Area $= 60$ sq. units.

Given that one bucket of paint can be applied for 6 sq. ft

$\therefore$ No . of buckets $= \frac{60}{6} = 10.$

11. In the figure, find the area of (i) triangle AGF (ii) triangle FED (iii) quadrilateral BCEG.



Sol :

(i) Area of triangle AGF

Vertices A (– 5, 3), G (– 4.5, 0.5) and F (– 2, 3).

Area of triangle $= \frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq. units

$= \frac{1}{2}\,[-5\,(0.5 - 3) - 4.5\,(3 - 3) - 2\,(3 - 0.5)]$

$= \frac{1}{2}\,[12.5 - 5] = \frac{7.5}{2} = 3.75$ sq. units

(ii) Area of triangle FED

Vertices are F (– 2, 3), E (1.5, 1) and D (1, 3)

Area of triangle $= \frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq. units

$= \frac{1}{2}\,[-2\,(1 - 3) + 1.5\,(3 - 3) + 1\,(3 - 1)]$

$= \frac{1}{2}\,[4 + 2] = \frac{6}{2} = 3$ sq. units

(iii) Area of quadrilateral BCEG

Vertices are B (– 4, – 2), C (2, – 1), E (1.5, 1) and G (– 4.5, 0.5)

Area of quadrilateral $= \frac{1}{2}\,[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)]$ sq. units

$= \frac{1}{2}\,[(-4 - 1.5)(-1 - 0.5) - (2 + 4.5)(-2 - 1)]$

$= \frac{1}{2}\,[(-5.5)(-1.5) - (6.5)(-3)]$

$= \frac{1}{2}\,[8.25 + 19.5] = \frac{1}{2}\,[27.75]$

$= 13.875 \approx 13.88$ sq. units

Inclination of a Line

Key Points

The angle of inclination of a line is the angle which a straight line makes with the positive direction of X-axis and it is usually denoted by θ.

- The angle of inclination of any line parallel to X-axis is 0°.
- The angle of inclination of any line perpendicular to X-axis (or) parallel to Y-axis is 90°.
- Slope: If 'θ' is the angle of inclination, then '$\tan\theta$' is called slope of a straight line and it is usually denoted by 'm'.
- Slope $m = \tan\theta$, $0^\circ \le \theta \le 180^\circ$, $\theta \ne 90^\circ$
- Slope $= \dfrac{\textit{change in ordinates}}{\textit{change in abscissae}} = \dfrac{y_2 - y_1}{x_2 - x_1}$
- The slope of a vertical line is undefined as tan 90° = undefined.
- The slope of the line through the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$ (or) $\dfrac{y_1 - y_2}{x_1 - x_2}$
- If two lines are parallel, then their slopes are equal. i.e., $m_1 = m_2$
- If two lines are perpendicular, then the product of their slopes is – 1 i.e., $m_1 \times m_2 = -1$.
- Slope is also called Gradient of the line.

Worked Examples

5.8 (i) **What is the slope of a line whose inclination is 30°?**

(ii) **What is the inclination of a line whose slope is $\sqrt{3}$?**

Sol :

(i) Here $\theta = 30^\circ$

Slope $m = \tan\theta$

Therefore, slope $= m = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$

(ii) $m = \sqrt{3}$, let θ be the inclination of the line

$\tan\theta = \sqrt{3} \quad \Rightarrow \theta = 60^\circ$

5.9 **Find the slope of a line joining the given points**

(i) (– 6, 1) and (– 3, 2)

(ii) $\left(-\dfrac{1}{3}, \dfrac{1}{2}\right)$ *and* $\left(\dfrac{2}{7}, \dfrac{3}{7}\right)$

(iii) (14, 10) and (14, – 6)

Sol :

(i) (– 6, 1) and (– 3, 2)

The slope $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2-1}{-3+6} = \dfrac{1}{3}$

(ii) $\left(-\dfrac{1}{3}, \dfrac{1}{2}\right)$ *and* $\left(\dfrac{2}{7}, \dfrac{3}{7}\right)$

The slope $= \dfrac{\frac{3}{7} - \frac{1}{2}}{\frac{2}{7} + \frac{1}{3}} = \dfrac{\frac{6-7}{14}}{\frac{6+7}{21}}$

$= -\dfrac{1}{14} \times \dfrac{21}{13} = -\dfrac{3}{26}$.

(iii) (14, 10) and (14, – 6)

The slope $= \dfrac{-6-10}{14-14} = \dfrac{-16}{0}$

which is undefined.

The slope is undefined.

5.10 The line r passes through the points (– 2, 2) and (5, 8) and the line s passes through the points (– 8, 7) and (– 2, 0). Is the line r perpendicular to s?

Sol :

The slope of line r is $m_1 = \dfrac{8-2}{5+2} = \dfrac{6}{7}$

The slope of line s is $m_2 = \dfrac{0-7}{-2+8} = \dfrac{-7}{6}$

The product of slopes $= \dfrac{6}{7} \times \dfrac{-7}{6} = -1$

That is, $m_1 m_2 = -1$

Therefore, the line r is perpendicular to line s.

5.11 The line P passes through the points (3, – 2), (12, 4) and the line Q passes through the points (6, – 2) and (12, 2). Is P parallel to Q?

Sol :

The slope of line P is $m_1 = \dfrac{4+2}{12-3} = \dfrac{6}{9} = \dfrac{2}{3}$

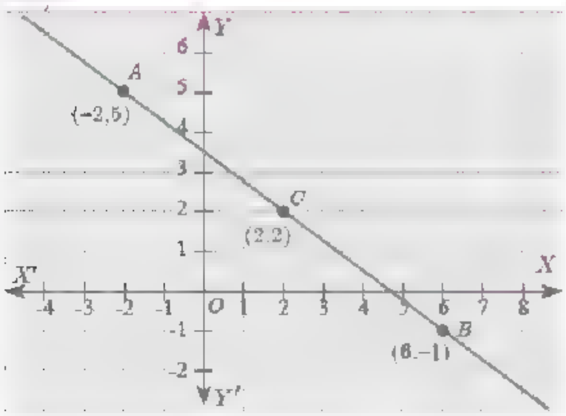
The slope of line Q is $m_2 = \dfrac{2+2}{12-6} = \dfrac{4}{6} = \dfrac{2}{3}$

Thus slope of line P = slope of line Q

Therefore, the line P is parallel to the line Q.

5.12 Show that the points (– 2, 5), (6, – 1) and (2, 2) are collinear.

Sol :



The vertices are A(– 2, 5), B(6, – 1) and C(2, 2).

Slope of AB $= \dfrac{-1-5}{6+2}$

$= \dfrac{-6}{8} = \dfrac{-3}{4}$

Slope of BC $= \dfrac{2+1}{2-6}$

$= \dfrac{3}{-4} = \dfrac{-3}{4}$

$\Rightarrow$ Slope of AB = Slope of BC

Therefore, the points A, B, C all lie in a same straight line.

Hence the points A, B, and C are collinear.

5.13 Let A(1, – 2), B(6, – 2), C(5, 1) and D(2, 1) be four points

(i) Find the slope of the line segment (a) AB (b) CD

(ii) Find the slope of the line segment (a) BC (b) AD

(iii) What can you deduce from your answer.

Sol :

(i) (a) Slope of AB $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{-2+2}{6-1} = 0$

(b) Slope of CD $= \dfrac{1-1}{2-5}$

$= \dfrac{0}{-3} = 0$

(ii) (a) Slope of BC $= \dfrac{1+2}{5-6}$

$= \dfrac{3}{-1} = -3$

(b) Slope of AD $= \dfrac{1+2}{2-1}$

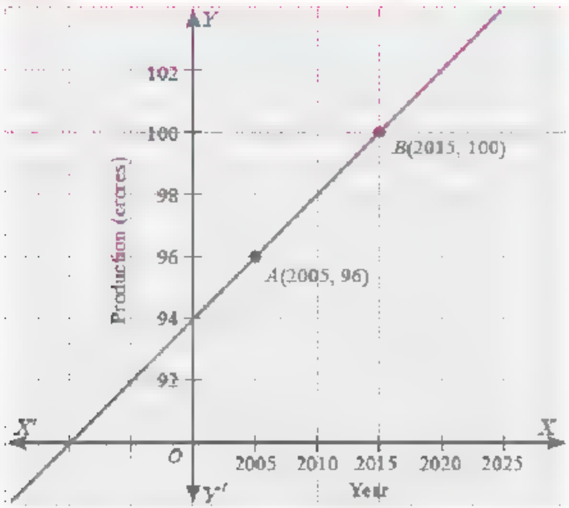
$= \dfrac{3}{1} = 3$

(iii) The slope of AB and CD are equal so they are parallel.

Similarly the lines AD and BC are not parallel, since their slopes are not equal. So we can deduce that the quadrilateral ABCD is a trapezium.

5.14 Consider the given graph representing (in crores) growth of population graph. Find the slope of the line AB and hence estimate the population in the year 2030.

Sol :



The points A(2005, 96) and B(2015, 100) are on the line AB.

$$\text{Slope of AB} = \frac{100-96}{2015-2005} = \frac{4}{10} = \frac{2}{5}$$

Let the growth of population in 2030 be k crores. Assuming that the point C(2030, k) is on AB, we have

$$\text{slope of AC} = \text{slope of AB}$$

$$\Rightarrow \quad \frac{k-96}{2030-2005} = \frac{2}{5}$$

$$\Rightarrow \quad \frac{k-96}{25} = \frac{2}{5}$$

$$\Rightarrow \quad k-96 = 10$$

$$\Rightarrow \quad k = 106$$

Hence the estimated population in 2030 = 106 Crores.

5.15 Without using Pythagoras theorem, show that the vertices (1, – 4), (2, – 3) and (4, – 7) form a right angled triangle.

Sol :

The vertices are A(1, – 4), B(2, – 3) and C(4, – 7).

$$\text{The slope of AB} = \frac{-3+4}{2-1} = \frac{1}{1} = 1$$

$$\text{The slope of BC} = \frac{-7+3}{4-2} = \frac{-4}{2} = -2$$

$$\text{The slope of AC} = \frac{-7+4}{4-1} = \frac{-3}{3} = -1$$

Slope of AB × Slope of AC = (1) (–1) = – 1

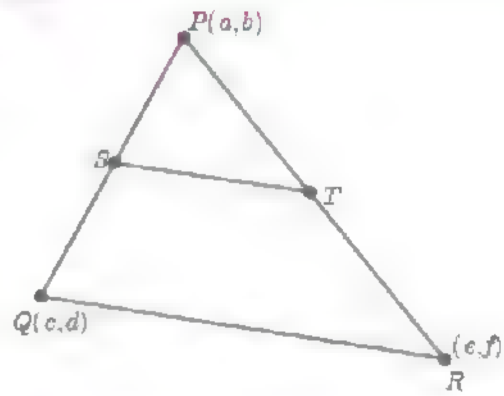
That is, $m_1 \times m_2 = -1$

Therefore, AB is perpendicular to AC. $\angle A = 90^\circ$

Therefore, ΔABC is a right angled triangle.

5.16 Prove analytically that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is equal to half of its length.

Sol :



Let P(a, b), Q(c, d) and R(e, f) be the vertices of a triangle.

Let S be the mid-point of PQ and T be the mid-point of PR

Therefore,

$$S = \left(\frac{a+c}{2}, \frac{b+d}{2}\right) \text{ and } T = \left(\frac{a+e}{2}, \frac{b+f}{2}\right)$$

$$\text{Now, Slope of ST} = \frac{\frac{b+f}{2} - \frac{b+d}{2}}{\frac{a+e}{2} - \frac{a+c}{2}}$$

$$= \frac{f-d}{e-c}$$

$$\text{And slope of QR} = \frac{f-d}{e-c}$$

Therefore, ST is parallel to QR. (since, their slopes are equal)

Also,

$$ST = \sqrt{\left(\frac{a+e}{2} - \frac{a+c}{2}\right)^2 + \left(\frac{b+f}{2} - \frac{b+d}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(e-c)^2 + (f-d)^2}$$

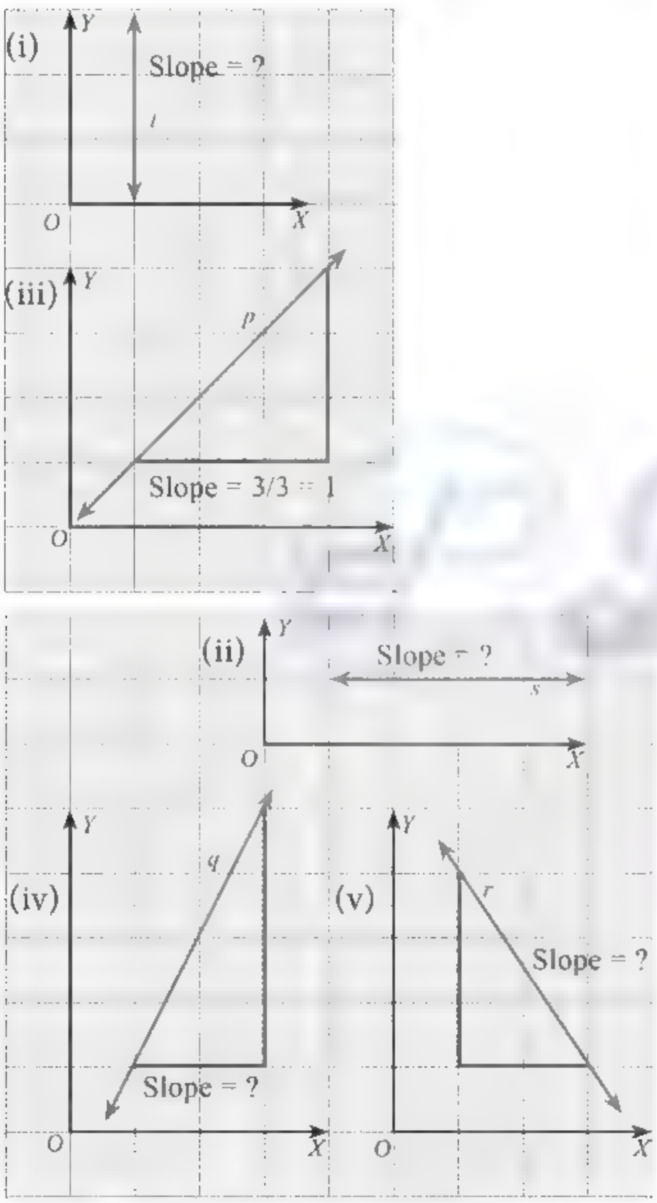
$$ST = \frac{1}{2} QR$$

Thus ST is parallel to QR and half of it.

Progress Check

1. Write down the slope of each of the lines shown on the grid below.

Write down the slope of the lines shown on the grid below.



Sol : (iii) Slope of the line

$$P = \frac{\text{change in } y \text{ coordinate}}{\text{change in } x \text{ coordinate}} = \frac{3}{3} = 1.$$

Ans : Slope of T is undefined.

Slope of S is 0.

2. Fill the missing boxes.

Sl. No.	Points	Slope
1	A (– a, b), B (3a, – b)	...
2	A (2, 3), B (–, –)	2
3	...	0
4	...	undefined

Ans :

1. Points A (–a, b), B (3a, – b)

$$\text{Slope} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{b + b}{-a - 3a} = \frac{2b}{-4a} = -\frac{b}{2a}$$

2. A (2, 3), B (___, ___), slope = 2

$$\text{Slope} = \frac{y_1 - y_2}{x_1 - x_2} = 2$$

$$\frac{3 - y_2}{2 - x_2} = 2$$

Any values satisfying this, can be the second point. For example, $x_2 = -2$, $y_2 = -5$

3. Any line parallel to X-axis, is having slope '0'.
4. Any line perpendicular to X-axis, is having slope 'undefined'.

Thinking Corner

1. The straight lines x-axis and y-axis are perpendicular to each other. Is the condition $m_1 m_2 = -1$ true?

Ans : $m_1\, m_2 = -1$ is true only when well-defined slopes are given.

X-axis with slope '0' and y-axis with slope 'undefined'.

2. Provide three examples of using the concept of slope in real-life situations.

Ans : Real life situation of concept of slope.

(i) While building the roads, need to consider the slope.

(ii) Wheel-chair ramp in Hospitals.

(iii) While constructing Bridges.

Exercise 5. 2

1. What is the slope of a line whose inclination with positive direction of x-axis is

(i) 90° **(ii) 0°**

Sol :

(i) Given angle of inclination $\theta = 90°$

$\therefore$ Slope of a line 'm' $= \tan\theta$

$= \tan 90° = \infty$ (undefined)

(ii) Angle of inclination $\theta = 0°$

Slope of a line 'm' $= \tan\theta = \tan 0° = 0$

2. What is the inclination of a line whose slope is

(i) 0 **(ii) 1**

Sol :

(i) Given slope 'm' = 0

$\Rightarrow \tan\theta = 0 = \tan 0°$

$\therefore \theta = 0°$

(ii) Slope 'm' = 1

$\tan\theta = 1 = \tan 45°$

$\therefore \theta = 45°$

3. Find the slope of a line joining the points

(i) $(5, \sqrt{5})$ with the origin

(ii) $(\sin\theta, -\cos\theta)$ and $(-\sin\theta, \cos\theta)$

Sol :

(i) Given points $(5, \sqrt{5})$ and $(0, 0)$

$$\text{Slope of a line} = \frac{y_1 - y_2}{x_1 - x_2} \text{ (or) } \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\sqrt{5} - 0}{5 - 0} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

(ii) Given points $(\sin\theta, -\cos\theta)$ and $(-\sin\theta, \cos\theta)$

$$\text{Slope of a line} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{-\cos\theta - \cos\theta}{\sin\theta + \sin\theta} = -\frac{2\cos\theta}{2\sin\theta} = -\cot\theta$$

4. What is the slope of a line perpendicular to the line joining A (5, 1) and P where P is the mid-point of the segment joining (4, 2) and (– 6, 4).

Sol : Mid point of line segment joining (4, 2) and (– 6, 4)

$$\text{Mid point} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= P\left(\frac{4-6}{2}, \frac{2+4}{2}\right)$$

$$= P\left(-\frac{2}{2}, \frac{6}{2}\right) = P(-1, 3)$$

Now, slope of a line joining A (5, 1) and P (– 1, 3)

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{1-3}{5+1} = \frac{-2}{6} = -\frac{1}{3}$$

$\therefore$ Slope of a perpendicular to the line joining A and P

$$= -\frac{1}{m} = -\frac{1}{\left(-\frac{1}{3}\right)} = 3$$

5. Show that the given points are collinear: (– 3, – 4), (7, 2) and (12, 5)

Sol :

Given points (– 3, – 4), (7, 2) and (12, 5)

Let the points be A (– 3, – 4), B (7, 2) and C (12, 5)

$$\text{Slope of a line} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\text{Slope of AB} = \frac{-4-2}{-3-7} = \frac{-6}{-10} = \frac{3}{5}$$

$$\text{Slope of BC} = \frac{2-5}{7-12} = \frac{-3}{-5} = \frac{3}{5}$$

Slope of AB = Slope of BC

$\therefore$ The points A, B and C are collinear.

6. If the three points (3, – 1), (a, 3) and (1, – 3) are collinear, find the value of a.

Sol :

Given points (3, – 1), (a, 3) and (1, – 3)

Let the points be A (3, – 1), B (a, 3) and C (1, – 3)

$$\text{Slope of AB} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-1-3}{3-a} = \frac{-4}{3-a}$$

$$\text{Slope of BC} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{3+3}{a-1} = \frac{6}{a-1}$$

Since, the points A, B and C are collinear.

Slope of AB = Slope of BC

$$\frac{-4}{3-a} = \frac{6}{a-1}$$

$$-2(a-1) = 3(3-a)$$
$$-2a+2 = 9-3a$$
$$3a-2a = 9-2$$
$$a = 7.$$

7. The line through the points (– 2, a) and (9, 3) has slope $-\frac{1}{2}$. Find the value of a.

Sol : Given points (– 2, a) and (9, 3) and

Slope $= -\frac{1}{2}$

$$\text{Slope of the line} = -\frac{1}{2}$$
$$\frac{y_1-y_2}{x_1-x_2} = -\frac{1}{2}$$
$$\frac{a-3}{-2-9} = -\frac{1}{2}$$
$$\frac{a-3}{-11} = -\frac{1}{2}$$
$$2(a-3) = 11$$
$$2a = 11+6$$
$$a = \frac{17}{2}$$

8. The line through the points (– 2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

Sol :

$$\text{Slope of the line} = \frac{y_1-y_2}{x_1-x_2}$$

Slope of the line joining the points (– 2, 6) and (4, 8)

$$m_1 = \frac{6-8}{-2-4} = \frac{-2}{-6} = \frac{1}{3}$$

Slope of the line joining the points (8, 12) and (x, 24)

$$m_2 = \frac{12-24}{8-x} = -\frac{12}{8-x}$$

Given that the lines are perpendicular

$\therefore$ $m_1 \times m_2 = -1.$

$$\frac{1}{3} \times \frac{-12}{8-x} = -1$$
$$4 = 8-x$$
$$x = 8-4$$
$$x = 4.$$

9. Show that the given vertices form a right angled triangle and check whether its satisfies Pythagoras theorem

(i) A (1, – 4), B (2, – 3) and C (4, – 7)

(ii) L (0, 5), M (9, 12) and N (3, 14)

Sol :

(i) Given vertices A (1, – 4), B (2, – 3) and C (4, – 7)

$$\text{Slope of the line} = \frac{y_1-y_2}{x_1-x_2}$$

$$\text{Slope of AB} = \frac{-4+3}{1-2} = \frac{-1}{-1} = 1$$
$$\text{Slope of BC} = \frac{-3+7}{2-4} = \frac{4}{-2} = -2$$
$$\text{Slope of AC} = \frac{-4+7}{1-4} = \frac{3}{-3} = -1$$

(Slope of AB) × (Slope of AC) = – 1

$\therefore$ AB is perpendicular to AC.

Hence, the given vertices form a right angled triangle.

Distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ units.

$$AB = \sqrt{(1-2)^2+(-4+3)^2} = \sqrt{1+1} = \sqrt{2}$$
$$AB^2 = (\sqrt{2})^2 = 2$$
$$BC = \sqrt{(2-4)^2+(-3+7)^2} = \sqrt{4+16} = \sqrt{20}$$
$$BC^2 = 20$$
$$AC = \sqrt{(1-4)^2+(-4+7)^2} = \sqrt{9+9} = \sqrt{18}$$
$$AC^2 = 18$$

Now, $AB^2 + AC^2 = BC^2$

Hence, the Pythagoras theorem is satisfied.

(ii) L (0, 5), M (9, 12) and N (3, 14)

$$\text{Slope of a line} = \frac{y_1-y_2}{x_1-x_2}$$

$$\text{Slope of LM} = \frac{5-12}{0-9} = \frac{-7}{-9} = \frac{7}{9}$$
$$\text{Slope of MN} = \frac{12-14}{9-3} = \frac{-2}{6} = \frac{-1}{3}$$
$$\text{Slope of LN} = \frac{5-14}{0-3} = \frac{-9}{-3} = 3$$

$$\text{(Slope of MN)} \times \text{(Slope of LN)} = \left(-\frac{1}{3}\right)\times 3 = -1.$$

$\therefore$ MN is perpendicular to LN.

Hence, the given vertices form a right angled triangle.

Distance formula $= \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$$LM = \sqrt{(0-9)^2+(5-12)^2} = \sqrt{81+49} = \sqrt{130}$$

$$LM^2 = 130$$

$$MN = \sqrt{(9-3)^2+(12-14)^2} = \sqrt{36+4} = \sqrt{40}$$

$$MN^2 = 40$$

$$LN = \sqrt{(0-3)^2+(5-14)^2} = \sqrt{9+81} = \sqrt{90}$$

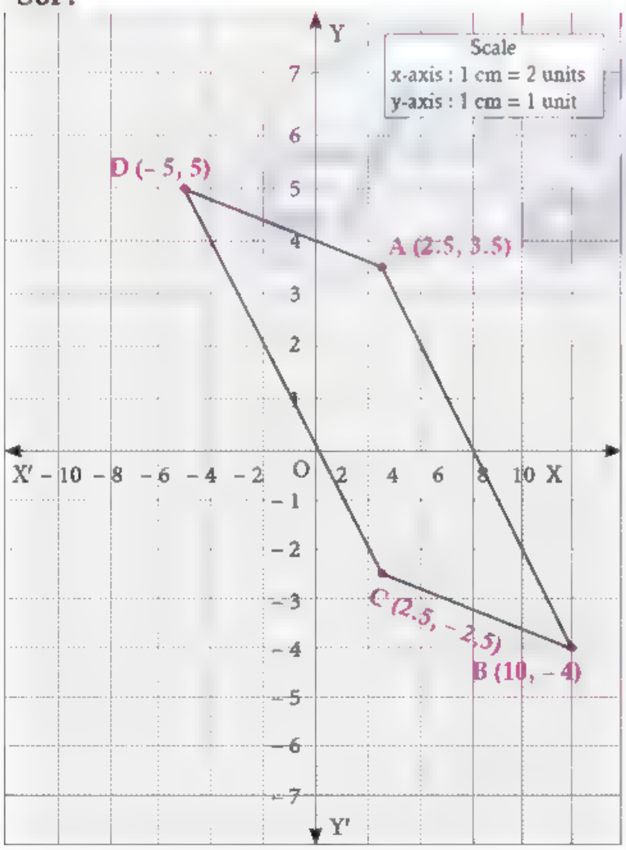
$$LN^2 = 90$$

$\therefore$ $LN^2 + MN^2 = LM^2$

Hence, the Pythagoras theorem is satisfied.

10. Show that the given points form a parallelogram: A (2.5, 3.5), B (10, – 4), C (2.5, – 2.5) and D (– 5, 5)

Sol :



Given points A (2.5, 3.5), B (10, – 4), C (2.5, – 2.5) and D (– 5, 5)

$$\text{Slope of a line} = \frac{y_1-y_2}{x_1-x_2}$$

$$\text{Slope of AB} = \frac{3.5+4}{2.5-10} = \frac{7.5}{-7.5} = -1$$

$$\text{Slope of BC} = \frac{-4+2.5}{10-2.5} = \frac{-1.5}{7.5} = -\frac{1}{5}$$

$$\text{Slope of CD} = \frac{-2.5-5}{2.5+5} = -\frac{7.5}{7.5} = -1$$

$$\text{Slope of AD} = \frac{3.5-5}{2.5+5} = -\frac{1.5}{7.5} = -\frac{1}{5}$$

Slope of AB = Slope of CD = – 1

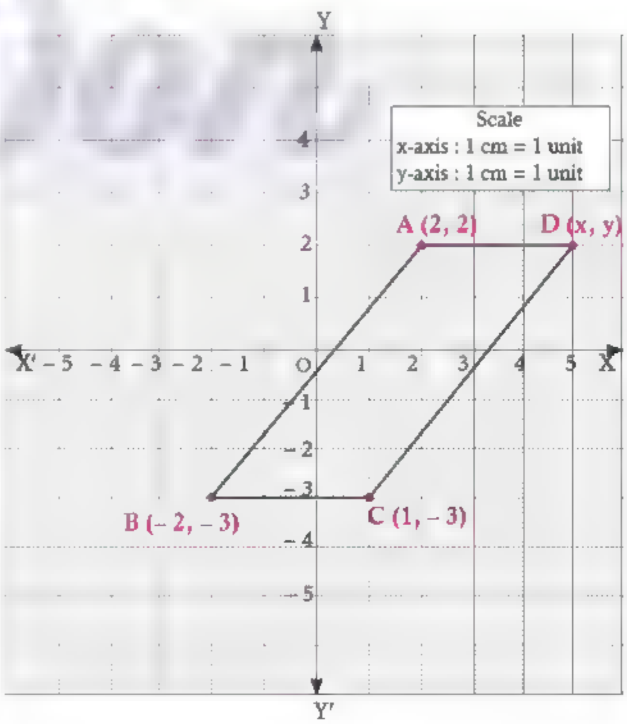
Slope of BC = Slope of AD = $-\frac{1}{5}$

$\therefore$ AB is parallel to CD and BC is parallel to AD.

Hence, the given points form a parallelogram.

11. If the points A (2, 2), B (– 2, – 3), C (1, – 3) and D (x, y) form a parallelogram then find the value of x and y.

Sol : Given points A (2, 2), B (– 2, – 3), C (1, – 3) and D (x, y)



$$\text{Slope of a line} = \frac{y_1-y_2}{x_1-x_2}$$

$$\text{Slope of AB} = \frac{2+3}{2+2} = \frac{5}{4}$$

$$\text{Slope of BC} = \frac{-3+3}{-2-1} = 0$$

$$\text{Slope of CD} = -\frac{-3-y}{1-x}$$

$$\text{Slope of AD} = \frac{2-y}{2-x}$$

Since, the points form a parallelogram
AB is parallel to CD and BC is parallel to AD

$\therefore$ Slope of AB = Slope of CD

$$\frac{5}{4} = \frac{-3-y}{1-x}$$

$$5(1-x) = 4(-3-y)$$

$$5-5x = -12-4y$$

$$\Rightarrow 5x-4y = 17 \quad \ldots(1)$$

Slope of BC = Slope of AD

$$0 = \frac{2-y}{2-x}$$

$$\Rightarrow 2-y = 0$$

$$y = 2$$

Substituting in (1)

$$5x-4(2) = 17$$

$$5x = 17+8 = 25$$

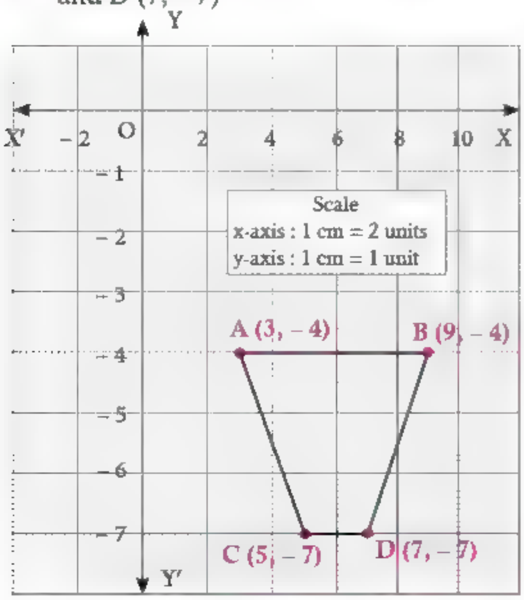
$$x = \frac{25}{5} = 5$$

$\therefore$ $x = 5, y = 2.$

12. Let A (3, – 4), B (9, – 4), C (5, – 7) and D (7, – 7). Show that ABCD is a trapezium.

Sol :

(i) Given points A (3, – 4), B (9, – 4), C (5, – 7) and D (7, – 7)



$$\text{Slope of a line} = \frac{y_1-y_2}{x_1-x_2}$$

$$\text{Slope of AB} = \frac{-4+4}{3-9} = 0$$

$$\text{Slope of BD} = \frac{-4+7}{9-7} = \frac{3}{2}$$

$$\text{Slope of CD} = \frac{-7+7}{5-7} = 0$$

$$\text{Slope of AC} = \frac{-4+7}{3-5} = \frac{3}{-2} = \frac{-3}{2}$$

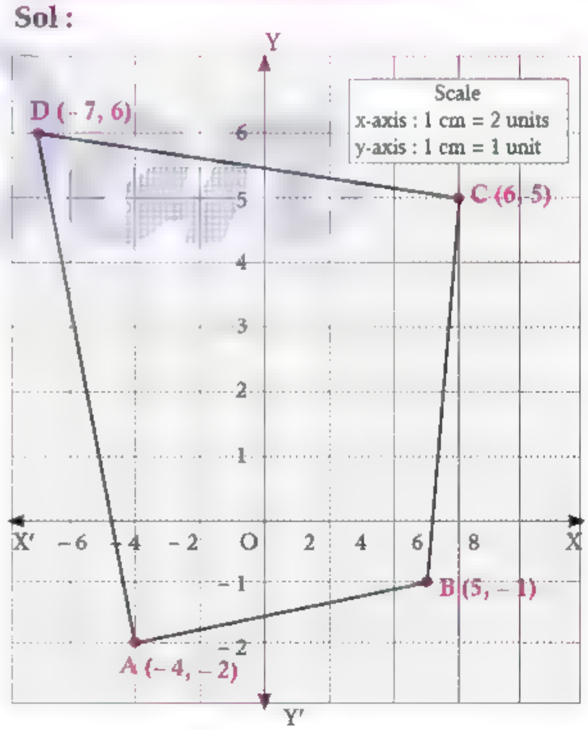
Slope of AB = Slope of CD

$\therefore$ AB is parallel to CD

Hence, the given points form a Trapezium.

13. A quadrilateral has vertices A(– 4, – 2), B(5, – 1), C(6, 5) and D(– 7, 6). Show that the mid-points of its sides form a parallelogram.

Sol :



Given, vertices of a quadrilateral are A(– 4, – 2), B(5, – 1), C(6, 5) and D(– 7, 6).

Let P, Q, R and S be the mid points of the sides AB, BC, CD and AD respectively

Mid point of

$$AB = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = P\left(\frac{-4+5}{2}, \frac{-2-1}{2}\right)$$

$$= P\left(\frac{1}{2}, -\frac{3}{2}\right)$$

Mid point of BC

$$= Q\left(\frac{5+6}{2}, \frac{-1+5}{2}\right) = Q\left(\frac{11}{2}, 2\right)$$

Mid point of CD

$$= R\left(\frac{6-7}{2}, \frac{5+6}{2}\right) = R\left(-\frac{1}{2}, \frac{11}{2}\right)$$

Mid point of AD

$$= S\left(\frac{-4-7}{2}, \frac{-2+6}{2}\right) = S\left(-\frac{11}{2}, 2\right)$$

$$\text{Slope of PQ} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-\frac{3}{2}-2}{\frac{1}{2}-\frac{11}{2}} = \frac{-\frac{7}{2}}{-\frac{10}{2}} = \frac{7}{10}$$

$$\text{Slope of QR} = \frac{2-\frac{11}{2}}{\frac{11}{2}+\frac{1}{2}} = \frac{-\frac{7}{2}}{\frac{12}{2}} = -\frac{7}{12}$$

$$\text{Slope of RS} = \frac{\frac{11}{2}-2}{-\frac{1}{2}+\frac{11}{2}} = \frac{\frac{7}{2}}{\frac{10}{2}} = \frac{7}{10}$$

$$\text{Slope of PS} = \frac{-\frac{3}{2}-2}{\frac{1}{2}+\frac{11}{2}} = \frac{-\frac{7}{2}}{\frac{12}{2}} = -\frac{7}{12}$$

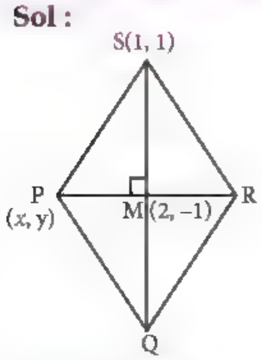
Slope of PQ = Slope of RS ⇒ PQ || RS

Slope of QR = Slope of PS ⇒ QR || PS

Hence, the mid points form a parallelogram.

14. PQRS is a rhombus. Its diagonals PR and QS intersect at the point M and satisfy QS = 2PR. If the co-ordinates of S and M are (1, 1) and (2, – 1) respectively, find the co-ordinates of P.

Sol :



PQRS is a rhombus

Slope of QS=Slope of SM

$$\text{Slope SM} = \frac{-1-1}{2-1} = -\frac{2}{1} = -2$$

Diagonals PR and QS intersect at right angle.

$\therefore$ Slope of PR = Slope of PM = $\frac{1}{2}$

Let P be (x, y)

$$\text{Then the slope of PM} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y+1}{x-2} = \frac{1}{2}$$

$$y + 1 = \frac{x-2}{2}$$

$$y = \frac{x-2}{2} - 1 = \frac{x-4}{2}$$

$$\therefore \text{ P is } \left(x, \frac{x-4}{2}\right)$$

Given QS = 2 PR

$$\Rightarrow \frac{QS}{2} = PR$$

$$\Rightarrow SM = PR$$

$$\Rightarrow \frac{SM}{2} = PM \text{ (M is the midpoint of PR)}$$

$$\therefore \frac{SM}{2} = PM = \frac{\sqrt{(2-1)^2 + (-1-1)^2}}{2}$$

$$\frac{SM}{2} = PM = \frac{\sqrt{5}}{2}$$

$$PM^2 = \frac{5}{4}$$

Now using distance formula,

$$PM^2 = (x-2)^2 + \left(\frac{x-4}{2} + 1\right)^2 = \frac{5}{4}$$

$$x^2 - 4x + 4 + \frac{x^2 - 4x + 4}{4} = \frac{5}{4}$$

$$x^2 - 4x + 4 = 1$$

$$x^2 - 4x + 3 = 0$$

$$(x-1)(x-3) = 0$$

$$x = 1, 3$$

When x = 1, $y = -\frac{3}{2}$

When x = 3, $y = -\frac{1}{2}$

$\therefore$ The co-ordinates of P can be $\left(1, -\frac{3}{2}\right), \left(3, -\frac{1}{2}\right)$

Straight Line

Key Points

- The linear equation is of the form $ax + by + c = 0$ is the equation of straight line where a, b and c are real numbers.
- Equation of X-axis is $y = 0$
- Equation of Y-axis is $x = 0$
- Equation of any line parallel to X-axis is $y = b$
- Equation of any line parallel to Y-axis is $x = a$
- Equation of straight line in slope-intercept form is $y = mx + c$ where 'm' is slope and 'c' is y intercept.
- Equation of straight line in slope-point form is $y - y_1 = m(x - x_1)$ where 'm' is the slope and (x_1, y_1) is the point that the line passes through.
- Equation of straight line in two points form is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are the points.
- Equation of straight line in intercepts form is $\frac{x}{a} + \frac{y}{b} = 1$ where 'a' is x-intercept and 'b' is y-intercept.
- $y = mx$ is the equation of straight line passing through origin.
- In the point (x, y), 'x' is called 'Abscissa' and 'y' is called 'Ordinate'.

Worked Examples

5.17 **Find the equation of the straight line passing through (5,7) and is (i) parallel to X-axis (ii) parallel to Y-axis.**

Sol :

(i) The equation of any straight line parallel to X-axis is $y = b$.
Since it passes through (5, 7), $b = 7$.
Therefore, the required equation of the line is $y = 7$.

(ii) The equation of any straight line parallel toY-axis is $x = a$
Since it passes through (5, 7), $a = 5$
Therefore, the required equation of the line is $x = 5$.

5.18 **Find the equation of a straight line whose**
(i) Slope is 5 and Y-intercept is – 9
(ii) Inclination is 45° and Y-intercept is 11

Sol :

(i) Given, Slope = 5, Y- intercept $c = -9$
Therefore, equation of a straight line is

$$y = mx + c$$

$$y = 5x - 9 \Rightarrow 5x - y - 9 = 0$$

(ii) Given, $\theta = 45°$, Y-intercept $c = 11$
Slope $m = \tan\theta = \tan 45° = 1$
Therefore, equation of a straight line is $y = mx + c$

$$\Rightarrow \quad y = x + 11 \Rightarrow x - y + 11 = 0$$

5.19 **Calculate the slope and Y-intercept of the straight line $8x - 7y + 6 = 0$**

Sol :

Equation of the given straight line is

$$8x - 7y + 6 = 0$$

$$\Rightarrow \quad 7y = 8x + 6$$

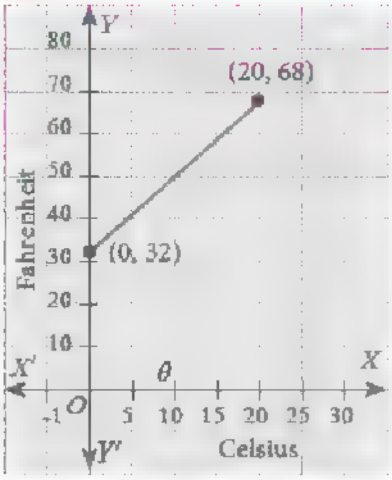
$$y = \frac{8}{7}x + \frac{6}{7} \quad \dots (1)$$

Comparing (1) with $y = mx + c$

Slope $m = \frac{8}{7}$ and Y-intercept $c = \frac{6}{7}$

5.20 **The graph relates temperatures y (in Fahrenheit degree) to temperatures x (in Celsius degree) (a) Find the slope and Y intercept (b) Write an equation of the line (c) What is the mean temperature of the earth in Fahrenheit degree if its mean temperature is 25° Celsius?**

Sol :



(a) Slope = $\dfrac{\textit{Change in y co-ordinate}}{\textit{Change in x co-ordinate}}$

$$= \frac{68-32}{20-0} = \frac{36}{20} = \frac{9}{5} = 1.8$$

The line crosses the Y-axis at (0, 32)

$\Rightarrow$ So the slope is $\frac{9}{5}$ and Y-intercept is 32.

(b) Use the slope and Y-intercept to write an equation

$\Rightarrow$ The equation is $y = \frac{9}{5}x + 32$

(c) In Celsius, the mean temperature of the earth is 25°. To find the mean temperature in Fahrenheit, we find the value of y when x = 25

$$y = \frac{9}{5}x + 32$$

$$y = \frac{9}{5}(25) + 32$$

$$y = 77$$

Therefore, the mean temperature of the earth is 77°F

5.21 Find the equation of a line passing through the point (3, – 4) and having slope $\frac{-5}{7}$

Sol : $(x_1, y_1) = (3, -4)$ and $m = \frac{-5}{7}$

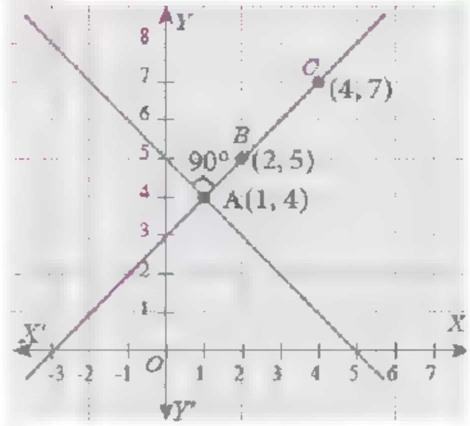
The equation of the point-slope form of the straight line is $y - y_1 = m(x - x_1)$

$$y + 4 = -\frac{5}{7}(x-3)$$

$$5x + 7y + 13 = 0$$

5.22 Find the equation of a line passing through the point A(1, 4) and perpendicular to the line joining points (2, 5) and (4, 7).

Sol :



Let the given points be A(1, 4), B(2, 5) and C(4, 7).

Slope of line BC $= \frac{7-5}{4-2} = \frac{2}{2} = 1$

Let m be the slope of the required line

Since the required line is perpendicular to BC,

$$\Rightarrow m \times 1 = -1 \quad \Rightarrow m = -1$$

The required line also passes through the point A(1, 4).

The equation of the required straight line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -1(x-1)$$

$$y - 4 = -x + 1$$

$$x + y - 5 = 0$$

5.23 Find the equation of a straight line passing through (5, – 3) and (7, – 4)

Sol : The equation of a straight line passing through the two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

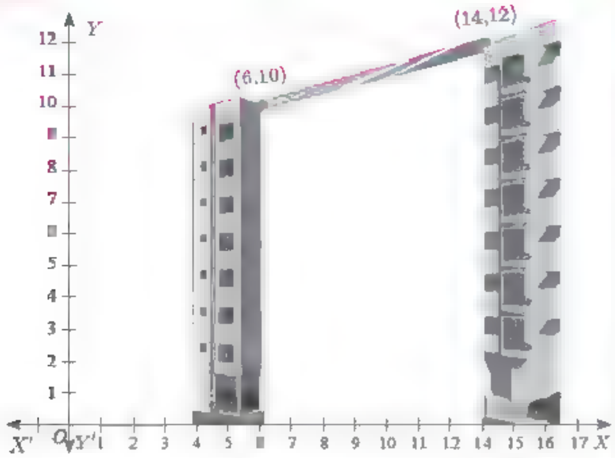
$$\Rightarrow \frac{y+3}{-4+3} = \frac{x-5}{7-5}$$

$$\Rightarrow 2y + 6 = -x + 5$$

Therefore, $x + 2y + 1 = 0$

5.24 Two buildings of different heights are located at opposite sides of each other. If a heavy rod is attached joining the terrace of the buildings from (6, 10) to (14, 12), find the equation of the rod joining the buildings.

Sol :



Let A(6, 10), B(14, 12) be the points denoting the terrace of the buildings.

The equation of the rod is the equation of the straight line passing through A(6, 10) and B(14, 12)

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-10}{12-10} = \frac{x-6}{14-6}$$

$$\frac{y-10}{2} = \frac{x-6}{8}$$

Therefore, $x - 4y + 34 = 0$

Hence, equation of the rod is $x - 4y + 34 = 0$

5.25 Find the equation of a line which passes through (5, 7) and makes intercepts on the axes equal in magnitude but opposite in sign.

Sol :

Let the X-intercept be 'a' and Y-intercept be '– a'.
The equation of the line in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{-a} = 1 \quad \text{(Here } b = -a\text{)}$$

Therefore, $x - y = a$ ... (1)

Since (1) passes through (5, 7)

Therefore, $5 - 7 = a \Rightarrow a = -2$

Thus the required equation of the straight line is $x - y = -2$; or $x - y + 2 = 0$

5.26 Find the intercepts made by the line $4x - 9y + 36 = 0$ on the co-ordinate axes.

Sol :

Equation of the given line is
$4x - 9y + 36 = 0 \Rightarrow 4x - 9y = -36$

Dividing by – 36 we get, $\frac{x}{-9} + \frac{y}{4} = 1$ ... (1)

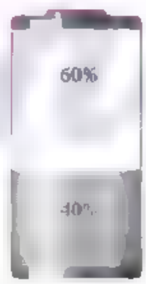
Comparing (1) with intercept form, we get
X-intercept $a = -9$; Y-intercept $b = 4$.

5.27 A mobile phone is put to use when the battery power is 100%. The percent of battery power 'y' (in decimal) remaining after using the mobile phone for x hours is assumed as $y = -0.25x + 1$

(i) Draw a graph of the equation.

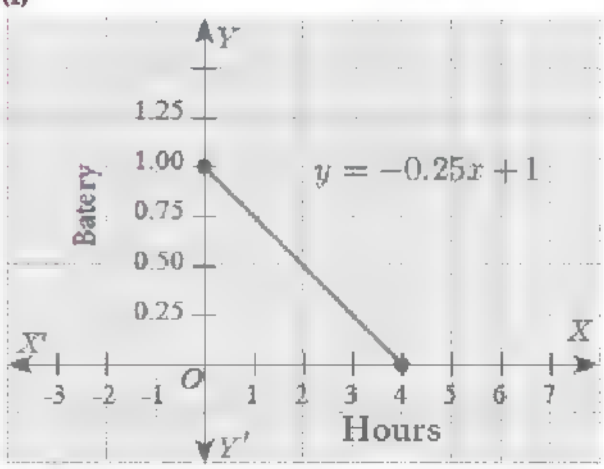
(ii) Find the number of hours elapsed if the battery power is 40%.

(iii) How much time does it take so that the battery has no power?



Sol :

(i)



(ii) To find the time when the battery power is 40%, we have to take $y = 0.40$
$0.40 = -0.25x + 1 \quad \Rightarrow \quad 0.25x = 0.60$

$$\Rightarrow x = \frac{0.60}{0.25} = 2.4 \text{ hours.}$$

(iii) If the battery power is 0 then y = 0

Therefore, $0 = -0.25x + 1 \Rightarrow -0.25x = 1$

$\Rightarrow x = 4$ hours.

Thus, after 4 hours, the battery of the mobile phone will have no power.

5.28 A line makes the positive intercepts on co-ordinate axes whose sum is 7 and it passes through (– 3, 8). Find its equation.

Sol :

If a and b are the intercepts then

$a + b = 7$ or $b = 7 - a$

By intercept form $\frac{x}{a} + \frac{y}{b} = 1$ ... (1)

We have $\frac{x}{a} + \frac{y}{7-a} = 1$

As this line passes through the point (– 3, 8), we have

$$\frac{-3}{a} + \frac{8}{7-a} = 1 \Rightarrow -3(7-a) + 8a = a(7-a)$$

$$\Rightarrow -21 + 3a + 8a = 7a - a^2$$

$$\Rightarrow a^2 + 4a - 21 = 0$$

Solving this equation $(a - 3)(a + 7) = 0$

$a = 3$ or $a = -7$

Since a is positive,

we have $a = 3$ and $b = 7 - a = 7 - 3 = 4$.

Hence $\frac{x}{3} + \frac{y}{4} = 1$

Therefore, $4x + 3y - 12 = 0$ is the required equation.

5.29 A circular garden is bounded by East Avenue and Cross Road. Cross Road intersects North Street at D and East Avenue at E. AD is tangential to the circular garden at A(3, 10) using the figure.

(a) Find the equation of

(i) East Avenue

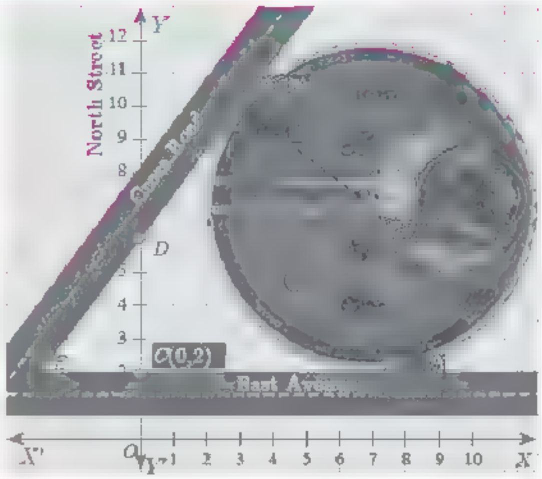
(ii) North Street

(iii) Cross Road

(b) Where does the Cross Road intersect the

(i) East Avenue?

(ii) North Street



Sol :

(a) (i) East Avenue is the straight line joining C(0, 2) and B(7, 2). Thus the equation of East Avenue is obtained by using two-point form which is

$$\frac{y-2}{2-2} = \frac{x-0}{7-0}$$

$$\frac{y-2}{0} = \frac{x}{7} \Rightarrow y = 2$$

(ii) Since the point D lie vertically above C(0, 2). The x-coordinate of D is 0. Since any point on North Street has x-coordinate value 0.

Therefore, the equation of North Street is X = 0

(iii) To find equation of Cross Road.

Center of circular garden M is (7, 7), A is (3, 10)

We first find slope of MA, which we call m_1

Thus $m_1 = \frac{10-7}{3-7} = \frac{-3}{4}$.

Since the Cross Road is perpendicular to MA, if m_2 is the slope of the Cross Road then, $m_1 m_2 = 1$

$$\Rightarrow \frac{-3}{4} m_2 = -1 \Rightarrow m_2 = \frac{4}{3}.$$

Now, the corss road has slope $\frac{4}{3}$ and it passes through the point A(3, 10).

The equation of the Cross Road is

$$y - 10 = \frac{4}{3}(x - 3)$$

$$\Rightarrow 3y - 30 = 4x - 12$$

$$\Rightarrow 4x - 3y + 18 = 0$$

(b) **(i)** If D is (0, k) then D is a point on the Cross Road.

Therefore, substituting x = 0, y = k in the equation of Cross Road, we get

$0 - 3k + 18 = 0$

$\Rightarrow \quad k = 6$

D is (0, 6)

(ii) To find E, Let E be (q, 2)

put y = 2 in the equation of the Cross Road, we get

$4q - 6 + 18 = 0$

$4q = -12 \Rightarrow q = -3$

Therefore, The point E is (– 3, 2)

Thus the Cross Road meets the North Street at D(0, 6) and East Avenue at E (– 3, 2).

Progress Check

1. Fill the details in Respective boxes.

Form	When to use?	Name
$y = mx + c$	Slope = m, Intercept = c are given	...
$\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$	...	...
...	The intercepts are given	Intercept form

Ans :

$y = mx + c$	Slope m, y Intercept c	Slope-intercept form
$\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$	When two points (x_1, y_1) and (x_2, y_2) are given.	Two points form
$\dfrac{x}{a} + \dfrac{y}{b} = 1$	Intercepts are given	Intercepts form

2.

Sl. No.	Equation	Slope	x intercept	y intercept
1	$3x - 4y + 2 = 0$	...	...	...
2	$y = 14x$	...	...	0
3	...	...	2	– 3

Ans :

1. $3x - 4y + 2 = 0$

$\Rightarrow \quad \text{Slope} = \dfrac{-3}{-4} = \dfrac{3}{4}$

$3x - 4y = -2$

$\dfrac{x}{\left(-\dfrac{2}{3}\right)} + \dfrac{y}{\left(\dfrac{1}{2}\right)} = 1 \Rightarrow$ x intercept $-\dfrac{2}{3}$

y intercept $\dfrac{1}{2}$

2. $y = 14x \Rightarrow$ Slope m = 14

[$\because$ y = mx + c form]

x intercept is '0'

y intercept is '0'

3. Given 'x' intercept is 2, 'y' intercept is – 3

Equation of the line is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

$\Rightarrow \quad \dfrac{x}{2} + \dfrac{y}{-3} = 1$

$3x - 2y - 6 = 0$

Thinking Corner

1. Is it possible to express, the equation of a straight line in slope-Intercept form, when it is parallel to Y-axis?

Ans :

Any line parallel to Y-axis, is of the form x = a.

$\therefore$ It cannot be expressed in slope-intercept form.

Exercise 5. 3

1. Find the equation of a straight line passing through the mid-point of a line segment joining the points (1, – 5), (4, 2) and parallel to: (i) X-axis (ii) Y-axis.

Sol : Given points (1, – 5) and (4, 2)

Mid-point of the line joining the points (1, – 5), (4, 2)

$= \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right) = \left(\dfrac{1+4}{2}, \dfrac{-5+2}{2}\right)$

$= \left(\dfrac{5}{2}, -\dfrac{3}{2}\right)$

(i) Equation of a straight line passing through $\left(\frac{5}{2}, -\frac{3}{2}\right)$ and parallel to X-axis is $y = b$.

i.e., $y = -\frac{3}{2} \Rightarrow 2y + 3 = 0.$

(ii) Equation of a straight line passing through $\left(\frac{5}{2}, -\frac{3}{2}\right)$ and parallel to Y-axis is $x = a$.

i.e., $x = \frac{5}{2} \Rightarrow 2x - 5 = 0$

2. The equation of a straight line is 2 (x – y) + 5 = 0. Find its slope, inclination and intercept on the Y-axis.

Sol :

Given line is $2(x - y) + 5 = 0$

$\Rightarrow 2x - 2y + 5 = 0$ is in the form $ax + by + c = 0$

$$\text{Slope} = -\frac{a}{b} = \frac{-2}{-2} = 1$$

$$\text{Slope} = 1$$

Inclination:

i.e., $\tan\theta = 1 = \tan 45°$

$$\theta = 45°$$

$$2x - 2y + 5 = 0$$

$$2x - 2y = -5 \text{ (Dividing by } -5)$$

$$\frac{x}{\left(-\frac{5}{2}\right)} + \frac{y}{\left(\frac{5}{2}\right)} = 1$$

y intercept is $\frac{5}{2}$

3. Find the equation of a whose inclination is 30° and making an intercept – 3 on the Y-axis.

Sol :

Given inclination $\theta = 30°$

$\therefore$ Slope $m = \tan 30° = \frac{1}{\sqrt{3}}$

y intercept $c = -3$

Equation of straight line in slope and 'y' intercept form is

$$y = mx + c$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{3}}x - 3$$

$$\sqrt{3}y = x - 3\sqrt{3}$$

$$x - \sqrt{3}y - 3\sqrt{3} = 0$$

4. Find the slope and Y intercept of $\sqrt{3}x + (1 - \sqrt{3})$ y = 3.

Sol :

Given line is $\sqrt{3}x + (1 - \sqrt{3})y = 3$.

$\sqrt{3}x + (1 - \sqrt{3})y - 3 = 0$ is in the form

$$ax + by + c = 0$$

$$\text{Slope} = -\frac{a}{b}$$

$$= -\frac{\sqrt{3}}{1 - \sqrt{3}}$$

$$= -\frac{\sqrt{3}}{1 - \sqrt{3}} \times \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{\sqrt{3} + 3}{2}$$

$\sqrt{3}x + (1 - \sqrt{3})y = 3$ [Dividing throughout by '3'

$$\frac{\sqrt{3}x}{3} + \frac{(1 - \sqrt{3})}{3}y = 1$$

$$\frac{x}{(\sqrt{3})} + \frac{y}{\left(\frac{3}{1 - \sqrt{3}}\right)} = 1$$

$$\text{y intercept} = \frac{3}{1 - \sqrt{3}} \times \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{3 + 3\sqrt{3}}{1 - 3} = \frac{3 + 3\sqrt{3}}{-2}$$

5. Find the value of 'a', the line through (– 2, 3) and (8, 5) is perpendicular to y = ax + 2.

Sol :

Slope of a line passing through (– 2, 3) and (8, 5) is

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{3 - 5}{-2 - 8} = \frac{-2}{-10} = \frac{1}{5} = m_1$$

Slope of the line $y = ax + 2$ is $'a' = m_2$

Given that the lines are perpendicular

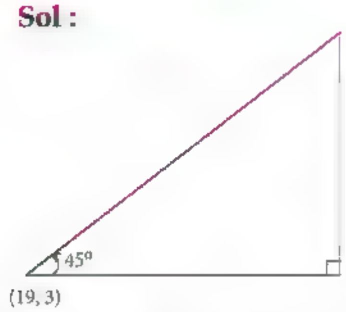
$$\therefore \quad m_1 \times m_2 = -1$$

$$\frac{1}{5} \times a = -1$$

$$a = -5.$$

6. The hill in the form of a triangle has its foot at (19, 3). The inclination of the hill to the ground is 45°. Find the equation of the hill joining the foot and top.

Sol :



Foot of the hill is at (19, 3)

Angle of inclination $\theta = 45°$

Slope $m = \tan 45° = 1$

Equation of line passing through (x_1, y_1) and having

Slope 'm' is $y - y_1 = m(x - x_1)$

$\Rightarrow y - 3 = 1(x - 19)$

$y - 3 = x - 19 \Rightarrow x - y - 16 = 0.$

7. Find the equation of a line through the given pair of points

(i) $\left(2, \frac{2}{3}\right)$ and $\left(\frac{-1}{2}, -2\right)$

(ii) (2, 3) and (– 7, – 1)

Sol :

(i) Given points $\left(2, \frac{2}{3}\right)$ and $\left(-\frac{1}{2}, -2\right)$

Equation of the line passing through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - \frac{2}{3}}{-2 - \frac{2}{3}} = \frac{x - 2}{-\frac{1}{2} - 2}$$

$$\Rightarrow \frac{3y - 2}{-6 - 2} = \frac{2x - 4}{-1 - 4}$$

$\Rightarrow -5(3y - 2) = -8(2x - 4)$

$\Rightarrow -15y + 10 = -16x + 32$

$\Rightarrow 16x - 15y - 22 = 0$

(ii) Given points (2, 3) and (– 7, – 1)

Equation of the line passing through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 3}{-1 - 3} = \frac{x - 2}{-7 - 2}$$

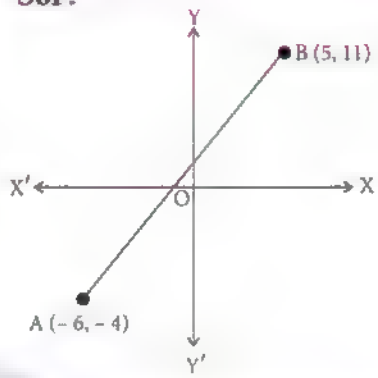
$9(y - 3) = 4(x - 2)$

$9y - 27 = 4x - 8$

$4x - 9y + 19 = 0$

8. A cat is located at the point (– 6, – 4) in XY-plane. A bottle of milk is kept at (5, 11). The cat wishes to consume the milk travelling through shortest possible distance. Find the equation of the path it needs to take its milk.

Sol :



Let A be the location of the cat and B be the point where bottle of milk is kept.

Given A (–6, – 4), B (5, 11)

Equation of line passing through (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y + 4}{11 + 4} = \frac{x + 6}{5 + 6}$$

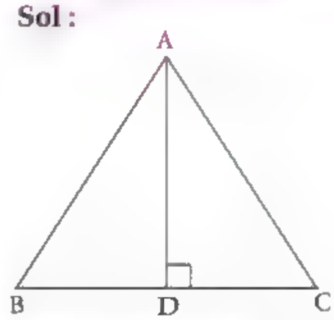
$11(y + 4) = 15(x + 6)$

$11y + 44 = 15x + 90$

$15x - 11y + 46 = 0$

9. Find the equation of the median and altitude of ΔABC through A where the vertices are A (6, 2), B (– 5, – 1) and C (1, 9).

Sol :



Given vertices are A (6, 2), B (– 5, – 1) and C (1, 9)

Median through A:

Let D be the mid point of BC

Mid point of BC $= D\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= D\left(\frac{-5+1}{2}, \frac{-1+9}{2}\right)$

$= D(-2, 4)$

Now AD is the median.

Equation of AD $\Rightarrow \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

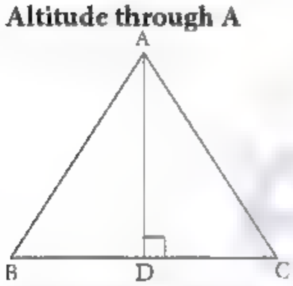
$$\frac{y-2}{4-2} = \frac{x-6}{-2-6}$$

$$\frac{y-2}{2} = \frac{x-6}{-8}$$

$$-4y+8 = x-6$$

$$x+4y-14 = 0$$

Altitude through A



Altitude is passing through 'A' and perpendicular to BC.

Now,

Slope of BC $= \frac{y_1-y_2}{x_1-x_2} = \frac{-1-9}{-5-1} = \frac{-10}{-6} = \frac{5}{3}$

$\therefore$ Slope of Altitude $= -\frac{3}{5}$

Equation of the altitude which is passing through A (6, 2) and having slope $-\frac{3}{5}$ is

$$y-y_1 = m(x-x_1)$$

$$y-2 = -\frac{3}{5}(x-6)$$

$$5y-10 = -3x+18$$

$$3x+5y-28 = 0$$

10. Find the equation of a straight line which has slope $\frac{5}{-4}$ and passing through the point (– 1, 2).

Sol : Given point (– 1, 2), Slope $m = -\frac{5}{4}$

Equation of the line passing through (x_1, y_1) and having slope 'm' is

$$y-y_1 = m(x-x_1)$$

$$y-2 = -\frac{5}{4}(x+1)$$

$$4y-8 = -5x-5$$

$$5x+4y-3 = 0$$

11. You are downloading a song. The percent y (in decimal form) of mega bytes remaining to get downloaded in *x* seconds is given by $y = -0.1x + 1$. Find

(i) graph the equation.

(ii) the total MB of the song.

(iii) after how many seconds will 75% of the song gets downloaded.

(iv) after how many seconds the song will be downloaded completely.

Sol :

Given equation is $y = -0.1x + 1$ where 'x' is time (in seconds) and 'y' is percentage of megabytes remaining.

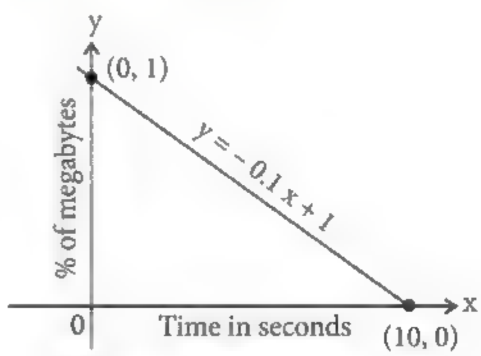
(i) Graph of $y = -0.1x + 1$

$\Rightarrow \quad y = -\frac{x}{10}+1$

$\Rightarrow \quad 10y = -x+10$

Points to be plotted

x	0	10
y	1	0



(ii) $y = -0.1x + 1$

Initially, time $x = 0 \Rightarrow y = 1$

$\therefore$ Total MB of the song is 1.

(iii) After how many seconds will 75% of the song gets downloaded.

$\therefore$ 25% is remaining $\Rightarrow y = 0.25$

$$y = -0.1x+1$$

$$0.25 = -0.1x+1$$

$$0.25-1 = -0.1x$$

$$x = \frac{-0.75}{-0.1} = \frac{75}{10} = 7.5 \text{ seconds.}$$

(iv) After how many seconds, the song will be downloaded completely.

$\therefore$ Remaining megabytes is 0

$$y = 0.1\,x + 1$$
$$0 = -0.1\,x + 1$$
$$0.1\,x = 1$$
$$x = \frac{1}{0.1} = 10 \text{ seconds.}$$

12. Find the equation of a line whose intercepts on the axes are given below

(i) 4, – 6 **(ii) $-5, \frac{3}{4}$**

Sol :

(i) Given intercepts are 4, – 6

$$a = 4, b = -6$$

Equation of the line in the intercepts form is

$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{4} + \frac{y}{-6} = 1$$
$$3x - 2y - 12 = 0$$

(ii) Given intercepts are $-5, \frac{3}{4}$

$$\Rightarrow \quad a = -5, b = \frac{3}{4}$$

Equation of the line in the intercepts form is

$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\Rightarrow \quad \frac{x}{-5} + \frac{y}{\left(\frac{3}{4}\right)} = 1$$
$$\Rightarrow \quad \frac{x}{-5} + \frac{4y}{3} = 1$$
$$\Rightarrow \quad 3x - 20y = -15$$
$$\Rightarrow \quad 3x - 20y + 15 = 0$$

13. Find the intercepts made by the following lines on the co-ordinate axes

(i) $3x - 2y - 6 = 0$ **(ii) $4x + 3y + 12 = 0$**

Sol :

(i) Given line is $3x - 2y - 6 = 0$

$$\Rightarrow \quad 3x - 2y = 6$$

Dividing by 6

$$\Rightarrow \quad \frac{3x}{6} - \frac{2y}{6} = 1$$
$$\Rightarrow \quad \frac{x}{2} + \frac{y}{(-3)} = 1 \text{ is in the form}$$
$$\frac{x}{a} + \frac{y}{b} = 1$$

$\therefore$ 'x' intercept $= a = 2$

y intercept $= b = -3$

(ii) $4x + 3y + 12 = 0$

$$\Rightarrow \quad 4x + 3y = -12$$

Dividing by – 12

$$\Rightarrow \quad \frac{4x}{-12} + \frac{3y}{-12} = 1$$
$$\Rightarrow \quad \frac{x}{(-3)} + \frac{y}{(-4)} = 1 \text{ is in the form}$$
$$\frac{x}{a} + \frac{y}{b} = 1$$

$\therefore$ x intercept $= a = -3$

y intercept $= b = -4.$

14. Find the equation of a straight line

(i) Passing through (1, – 4) and has intercepts which are in the ratio 2 : 5

(ii) Passing through (– 8, 4) and making equal intercepts on the co-ordinate axes.

Sol :

(i) Given that intercepts are in the ratio 2 : 5

i.e., $$\frac{a}{b} = \frac{2}{5}$$
$$a = \frac{2b}{5}$$

Equation of the line in Intercepts form is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \quad \frac{x}{\left(\frac{2b}{5}\right)} + \frac{y}{b} = 1$$
$$\Rightarrow \quad \frac{5x}{2b} + \frac{y}{b} = 1$$
$$\Rightarrow \quad 5x + 2y = 2b$$

This passes through (1, – 4)

$$\therefore \quad 5(1) + 2(-4) = 2b$$
$$5 - 8 = 2b \Rightarrow b = -\frac{3}{2}$$

$$a = \frac{2b}{5} = \frac{2\left(-\frac{3}{2}\right)}{5} = -\frac{3}{5}$$

$\therefore$ Equation of a straight line is

$$\frac{x}{a}+\frac{y}{b} = 1 \Rightarrow \frac{x}{\left(-\frac{3}{5}\right)}+\frac{y}{\left(-\frac{3}{2}\right)} = 1$$

$$\Rightarrow \quad \frac{5x}{-3}+\frac{2y}{-3} = 1$$

$$\Rightarrow \quad 5x+2y+3 = 0.$$

(ii) Given that intercepts are equal.

$\therefore \quad a = b$

Equation of the line in intercepts form is

$$\frac{x}{a}+\frac{y}{b} = 1$$

$$\Rightarrow \quad \frac{x}{a}+\frac{y}{a} = 1$$

$$\Rightarrow \quad x+y = a$$

This passes through (– 8, 4)

$\therefore \quad -8+4 = a$

$\Rightarrow \quad a = -4 \; [\because \; a = b]$

then, $\quad b = -4$

$\therefore$ Equation of the straight line is

$$\frac{x}{a}+\frac{y}{b} = 1$$

$$\Rightarrow \quad \frac{x}{-4}+\frac{y}{-4} = 1$$

$$\Rightarrow \quad x+y+4 = 0.$$

General form of a straight line

Key Points

The Linear equation $ax + by + c = 0$ is known as general form of the straight line where a, b and c are real numbers.

- Condition for parallelism:

  If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, then $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.

- Condition for perpendicularity:

  If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are perpendicular, then $a_1a_2 + b_1b_2 = 0$.

- Slope of the line $ax + by + c = 0$

  $$\text{Slope} = -\frac{\textit{Co-efficient of } x}{\textit{Co-efficient of } y} = -\frac{a}{b}$$

Worked Examples

5.30 **Find the slope of the straight line $6x + 8y + 7 = 0$.**

Sol :

Given $6x + 8y + 7 = 0$

$$\text{slope } m = \frac{-\textit{Co-efficient of } x}{\textit{Co-efficient of } y}$$

$$= -\frac{6}{8} = -\frac{3}{4}$$

Therefore, the slope of the straight line is $-\frac{3}{4}$.

5.31 **Find the slope of the line which is**

(i) parallel to $3x - 7y = 11$

(ii) perpendicular to $2x - 3y + 8 = 0$

Sol :

(i) Given straight line is $3x - 7y = 11$

$$\Rightarrow \quad 3x-7y-11 = 0$$

$$\text{Slope } m = \frac{-3}{-7} = \frac{3}{7}$$

Since parallel lines have same slopes, slope of any line parallel to

$$3x - 7y = 11 \text{ is } \frac{3}{7}$$

(ii) Given straight line is $2x - 3y + 8 = 0$

Slope $m = \dfrac{-2}{-3} = \dfrac{2}{3}$

Since product of slopes is -1 for perpendicular lines, slope of any line perpendicular to

$2x - 3y + 8 = 0$ is $\dfrac{-1}{\frac{2}{3}} = \dfrac{-3}{2}$

5.32 Show that the straight lines $2x + 3y - 8 = 0$ and $4x + 6y + 18 = 0$ are parallel.

Sol : Slope of the straight line $2x + 3y - 8 = 0$ is

$$m_1 = \frac{-co\text{-}efficient\ of\ x}{co\text{-}efficient\ of\ y}$$

$$m_1 = \frac{-2}{3}$$

Slope of the straight line $4x + 6y + 18 = 0$ is

$$m_2 = \frac{-4}{6} = \frac{-2}{3}$$

Here, $m_1 = m_2$

That is, slopes are equal. Hence, the two straight lines are parallel.

5.33 Show that the straight lines $x - 2y + 3 = 0$ and $6x + 3y + 8 = 0$ are perpendicular.

Sol :

Slope of the straight line $x - 2y + 3 = 0$ is

$$m_1 = \frac{-1}{-2} = \frac{1}{2}$$

Slope of the straight line $6x + 3y + 8 = 0$ is

$$m_2 = \frac{-6}{3} = -2$$

Now, $m_1 \times m_2 = \dfrac{1}{2} \times (-2) = -1$

Hence, the two straight lines are perpendicular.

5.34 Find the equation of a straight line which is parallel to the line $3x - 7y = 12$ and passing through the point (6, 4).

Sol : Equation of the straight line, parallel to

$3x - 7y - 12 = 0$ is $3x - 7y + k = 0$.

Since it passes through the point (6, 4).

$$3(6) - 7(4) + k = 0$$

$$k = 28 - 18 = 0$$

Therefore, equation of the required straight line is $3x - 7y + 10 = 0$.

5.35 Find the equation of a straight line perpendicular to the line $y = \frac{4}{3}x - 7$ and passing through the point $(7, -1)$.

Sol :

The equation $y = \dfrac{4x}{3} - 7$ can be written as

$$4x - 3y - 21 = 0.$$

Equation of a straight line perpendicular to

$4x - 3y - 21 = 0$ is $3x + 4y + k = 0$

Since it is passes through the point $(7, -1)$,

$$21 - 4 + k = 0 \Rightarrow k = -17$$

Therefore, equation of the required straight line is $3x + 4y - 17 = 0$.

5.36 Find the equation of a straight line parallel to Y-axis and passing through the point of intersection of the lines $4x + 5y = 13$ and $x - 8y + 9 = 0$.

Sol :

Given lines $4x + 5y - 13 = 0$ ... (1)

$x - 8y + 9 = 0$ ... (2)

To find point of intersection, solve equation (1) and (2)

$$\begin{array}{cccccccc} & x & & y & & 1 & & \\ 5 & & -13 & & 4 & & 5 \\ -8 & & 9 & & 1 & & -8 \end{array}$$

$$\frac{x}{45 - 104} = \frac{y}{-13 - 36} = \frac{1}{-32 - 5}$$

$$\frac{x}{-59} = \frac{y}{-49} = \frac{1}{-37}$$

$$x = \frac{59}{37},\ y = \frac{49}{37}$$

Therefore, the point of intersection

$$(x, y) = \left(\frac{59}{37}, \frac{49}{37}\right)$$

The equation of line parallel to Y-axis is $x = c$.

It passes through $(x, y) = \left(\dfrac{59}{37}, \dfrac{49}{37}\right)$.

Therefore, $c = \dfrac{59}{37}$.

The equation of the line is $x = \dfrac{59}{37}$

$\Rightarrow$ $37x - 59 = 0$.

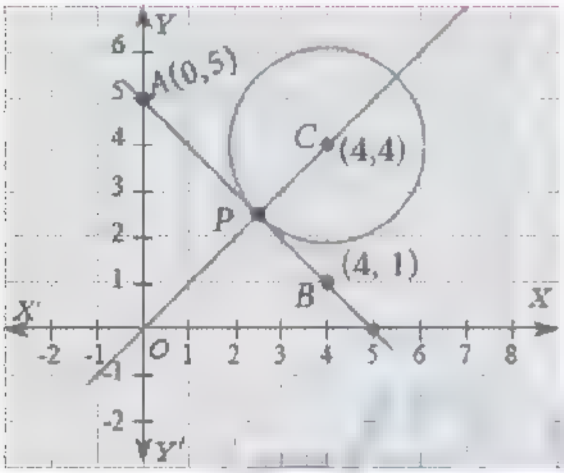
5.37 **The line joining the points A (0, 5) and B (4, 1) is a tangent to a circle whose centre C is at the point (4, 4).**

(i) Find the equation of the line AB.

(ii) Find the equation of the line through C which is perpendicular to the line AB.

(iii) Find the co-ordinates of the point of contact of tangent line AB with the circle.

Sol :



(i) Equation of line AB, A (0, 5) and B (4, 1)

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-5}{1-5} = \frac{x-0}{4-0} \Rightarrow 4(y-5) = -4x$$

$$y-5 = -x$$

$$x+y-5 = 0.$$

(ii) The equation of a line which is perpendicular to the line AB : $x+y-5=0$ is $x-y+k=0$

Since it is passing through the point (4, 4), we have

$$4-4+k = 0 \Rightarrow k=0$$

The equation of a line which is perpendicular to AB and through C is

$$x-y = 0 \quad \dots (2)$$

(iii) The co-ordinate of the point of contact P of the tangent line AB with the circle point of intersection of lines.

$$x+y-5 = 0 \text{ and } x-y = 0$$

Solving, we get $x = \frac{5}{2}$ and $y = \frac{5}{2}$

Therefore, the co-ordinate of the point of contact is $P\left(\frac{5}{2}, \frac{5}{2}\right)$.

Progress Check

1.

Sl. No.	Equations	Parallel or perpendicular
1	$5x+2y+5=0$ $5x+2y-3=0$	
2	$3x-7y-6=0$ $7x+3y+8=0$	
3	$8x-10y+11=0$ $4x-5y+16=0$	
4	$2y-9x-7=0$ $27y+6x-21=0$	

Ans :

1. $5x+2y+5=0 \quad a_1=5, \ b_1=2$

$5x+2y-3=0 \quad a_2=5, \ b_2=2$

$$\frac{a_1}{a_2} = \frac{5}{5} = 1, \ \frac{b_1}{b_2} = \frac{2}{2} = 1$$

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

$\therefore$ Lines are parallel.

2. $3x-7y-6=0 \quad a_1=3, \ b_1=-7$

$7x+3y+8=0 \quad a_2=7, \ b_2=3$

$a_1a_2+b_1b_2 = (3)(7)+(-7)(3) = 0$

$\therefore$ The lines are perpendicular.

3. $8x-10y+11=0 \quad a_1=8, \ b_1=-10$

$4x-5y+16=0 \quad a_2=4, \ b_2=-5$

$$\frac{a_1}{a_2} = \frac{8}{4} = 2, \quad \frac{b_1}{b_2} = \frac{-10}{-5} = 2$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$

$\therefore$ The lines are parallel.

4. $2y-9x-7=0 \quad a_1=2, \ b_1=-9$

$27y+6x-21=0 \quad a_2=27, \ b_2=6$

$a_1a_2+b_1b_2 = (2)(27)+(-9)(6) = 54-54 = 0$

$\therefore$ The lines are perpendicular.

Thinking Corner

1. **How many straight lines do you have with slope 1?**
   Ans: Many

2. **Find the number of point of intersection of two straight lines.**
   Ans: One

3. **Find the number of straight lines perpendicular to the line $2x - 3y + 6 = 0$.**
   Ans: Many

Exercise 5.4

1. **Find the slope of the following straight lines**

   (i) $5y - 3 = 0$ **(ii)** $7x - \frac{3}{17} = 0$

Sol :

(i) $5y - 3 = 0$

$$y = \frac{3}{5}$$

$$y = (0)x + \frac{3}{5}$$

Slope $m = 0$

(or) Comparing with $ax + by + c = 0$

$$\text{Slope} = -\frac{a}{b} = -\frac{0}{5} = 0$$

(ii) $7x - \frac{3}{17} = 0$

Comparing with $ax + by + c = 0$

$$\text{Slope} = -\frac{a}{b} = -\frac{7}{0} = \text{undefined}$$

2. **Find the slope of the line which is**

   (i) parallel to $y = 0.7x - 11$

   (ii) perpendicular to the line $x = -11$

Sol :

(i) Given line is $y = 0.7x - 11$ is in the form

$$y = mx + c$$

Slope $m = 0.7$

$\therefore$ Slope of the line parallel to $y = 0.7x - 11$ is $m = 0.7$.

($\because$ Slopes are equal when the lines are parallel)

(ii) Given line is $x = -11$

(Comparing with $ax + by + c = 0$)

Slope $m = -\frac{a}{b} = -\frac{1}{0} =$ undefined.

$\therefore$ Slope of the line perpendicular to $x = -11$ is 0.

3. **Check whether the given lines are parallel or perpendicular**

   (i) $\frac{x}{3} + \frac{y}{4} + \frac{1}{7} = 0$ **and** $\frac{2x}{3} + \frac{y}{2} + \frac{1}{10} = 0$

   (ii) $5x + 23y + 14 = 0$ **and** $23x - 5y + 9 = 0$.

Sol :

(i) Given lines are $\frac{x}{3} + \frac{y}{4} + \frac{1}{7} = 0$ and

$$\frac{2x}{3} + \frac{y}{2} + \frac{1}{10} = 0$$

$$a_1 = \frac{1}{3}, b_1 = \frac{1}{4} \text{ and } a_2 = \frac{2}{3}, b_2 = \frac{1}{2}$$

$$\text{Now, } \frac{a_1}{a_2} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}, \frac{b_1}{b_2} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, the given lines are parallel.

(ii) Given lines $5x + 23y + 14 = 0$ and $23x - 5y + 9 = 0$

$\therefore$ $a_1 = 5, b_1 = 23$ and $a_2 = 23, b_2 = -5$

Now $a_1a_2 + b_1b_2 = (5)(23) + (23)(-5) = 0$

$\therefore$ The given lines are perpendicular.

4. **If the straight lines $12y = -(p + 3)x + 12$, $12x - 7y = 16$ are perpendicular then find 'p'.**

Sol :

Given lines are $12y = -(p + 3)x + 12$ and

$$12x - 7y = 16$$

$$12y = -(p + 3)x + 12$$

$$\Rightarrow (p + 3)x + 12y - 12 = 0 \quad \text{... (1)}$$

$$12x - 7y - 16 = 0 \quad \text{... (2)}$$

$\therefore$ $a_1 = (p + 3), b_1 = 12$ and $a_2 = 12, b_2 = -7$

Given that the lines are perpendicular,

$$\therefore a_1a_2 + b_1b_2 = 0$$

$$\Rightarrow (p + 3)(12) + (12)(-7) = 0$$

$$\Rightarrow 12[p + 3 - 7] = 0$$

$$\Rightarrow p - 4 = 0$$

$$\Rightarrow p = 4$$

5. Find the equation of a straight line passing through the point P(– 5, 2) and parallel to the line joining the points Q(3, – 2) and R(– 5, 4).

Sol :

The vertices Q(3, – 2) and R(– 5, 4)

$$\text{Slope of the line QR} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-2-4}{3+5} = \frac{-6}{8} = \frac{-3}{4}$$

Slope of the line parallel to QR is $-\frac{3}{4}$

$\therefore$ Equation of the line passing through P(– 5, 2) and having slope $-\frac{3}{4}$ is

$$y - y_1 = m(x - x_1)$$
$$y - 2 = -\frac{3}{4}(x+5)$$
$$4y - 8 = -3x - 15$$
$$3x + 4y + 7 = 0$$

6. Find the equation of a line passing through (6, – 2) and perpendicular to the line joining the points (6, 7) and (2, – 3).

Sol :

Slope of the line joining the points (6, 7) and (2, – 3) is

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{7+3}{6-2} = \frac{10}{4} = \frac{5}{2} = m$$

Slope of the perpendicular line is

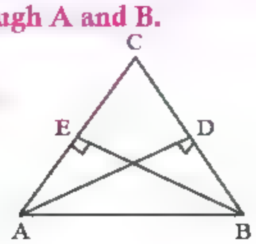
$$-\frac{1}{m} = -\frac{1}{(5/2)} = -\frac{2}{5}$$

Now, equation of the line passing through (6, – 2) and having slope $-\frac{2}{5}$ is

$$y - y_1 = m(x - x_1)$$
$$y + 2 = -\frac{2}{5}(x-6)$$
$$5y + 10 = -2x + 12$$
$$2x + 5y - 2 = 0$$

7. A(– 3, 0) B(10, – 2) and C(12, 3) are the vertices of ΔABC. Find the equation of the altitude through A and B.

Sol :



Given vertices are A(– 3, 0), B(10, – 2) and C(12, 3).

$$\text{Slope of BC} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-2-3}{10-12} = \frac{-5}{-2} = \frac{5}{2}$$

Altitude AD is perpendicular to BC and passing through A(– 3, 0)

$\therefore$ Slope of AD $= -\frac{2}{5}$

Equation of AD $\Rightarrow$ $y - y_1 = m(x - x_1)$

$$y - 0 = -\frac{2}{5}(x+3)$$
$$5y = -2x - 6$$
$$2x + 5y + 6 = 0$$

$$\text{Slope of AC} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0-3}{-3-12} = \frac{-3}{-15} = \frac{1}{5}$$

Altitude BE is perpendicular to AC and passing through B(10, – 2). Slope of BE = 5

$\therefore$ Equation of BE $\Rightarrow$ $y - y_1 = m(x - x_1)$

$$y + 2 = -5(x - 10)$$
$$y + 2 = -5x + 50$$
$$5x + y - 48 = 0$$

8. Find the equation of the perpendicular bisector of the line joining the points A(– 4, 2), B(6, – 4).

Sol :

Given points A(– 4, 2) and B(6, – 4)

$$\text{Mid point of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4+6}{2}, \frac{2-4}{2}\right) = (1, -1)$$

$$\text{Slope of AB} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{2+4}{-4-6} = \frac{6}{-10} = \frac{-3}{5}$$

Slope of perpendicular line $= \frac{5}{3}$

Perpendicular bisector is passing through (1, – 1) and having slope $\frac{5}{3}$.

$\therefore$ The equation of perpendicular bisector is

$$y - y_1 = m(x - x_1)$$
$$y + 1 = \frac{5}{3}(x-1)$$
$$3y + 3 = 5x - 5$$
$$5x - 3y - 8 = 0$$

9. Find the equation of a straight line through the intersection of lines 7x + 3y = 10, 5x – 4y = 1 and parallel to the line 13x + 5y + 12 = 0.

Sol :

Given lines are $7x + 3y = 10$ and $5x - 4y = 1$.
Let us solve the equations to get the point of intersection.

$$7x + 3y = 10 \quad \dots (1)$$
$$5x - 4y = 1 \quad \dots (2)$$
$$(1) \times 4 \Rightarrow 28x + 12y = 40 \quad \dots (3)$$
$$(2) \times 3 \Rightarrow 15x - 12y = 3 \quad \dots (4)$$
$$(3) + (4) \Rightarrow 43x = 43$$
$$x = \frac{43}{43} = 1$$

Substituting $x = 1$ in (1)

$$7(1) + 3y = 10 \Rightarrow y = 1$$

The point of intersection is $(1, 1)$

Slope of the line $13x + 5y + 12 = 0$ is $-\frac{a}{b} = \frac{-13}{5}$

$\therefore$ Slope of the parallel line is $\frac{-13}{5}$

Now, equation of the line passing through $(1, 1)$ and having slope $\frac{-13}{5}$ is

$$y - y_1 = m(x - x_1)$$
$$y - 1 = \frac{-13}{5}(x - 1)$$
$$5y - 5 = -13x + 13$$
$$13x + 5y - 18 = 0$$

10. Find the equation of a straight line through the intersection of lines 5x – 6y = 2, 3x + 2y = 10 and perpendicular to the line 4x – 7y + 13 = 0.

Sol :

Given lines are $5x - 6y = 2$ and $3x + 2y = 10$.
Let us solve the equations to get the point of intersection.

$$5x - 6y = 2 \quad \dots (1)$$
$$3x + 2y = 10 \quad \dots (2)$$
$$(2) \times 3 \Rightarrow 9x + 6y = 30 \quad \dots (3)$$
$$5x - 6y = 2 \quad \dots (1)$$
$$(3) + (1) \Rightarrow 14x = 32$$
$$x = \frac{32}{14} = \frac{16}{7}$$

Substituting $x = \frac{16}{7}$ in (2)

$$3\left(\frac{16}{7}\right) + 2y = 10$$
$$2y = 10 - \frac{48}{7} = \frac{22}{7}$$
$$y = \frac{11}{7}$$

$\therefore$ The point of intersection is $\left(\frac{16}{7}, \frac{11}{7}\right)$

Slope of the line $4x - 7y + 13 = 0$ is

$$-\frac{a}{b} = \frac{-4}{-7} = \frac{4}{7}$$

Slope of the perpendicular line is $-\frac{7}{4}$.

Now, equation of the line passing through $\left(\frac{16}{7}, \frac{11}{7}\right)$ and having slope $m = -\frac{7}{4}$ is

$$y - y_1 = m(x - x_1)$$
$$y - \frac{11}{7} = \frac{-7}{4}\left(x - \frac{16}{7}\right)$$
$$\frac{7y - 11}{7} = \frac{-7x}{4} + 4$$
$$28y - 44 = -49x + 112$$
$$49x + 28y - 156 = 0$$

11. Find the equation of a straight line joining the point of intersection of 3x + y + 2 = 0 and x – 2y – 4 = 0 to the point of intersection of 7x – 3y = – 12 and 2y = x + 3.

Sol :

Let us solve $3x + y + 2 = 0$ and $x - 2y - 4 = 0$

$$3x + y = -2 \quad \dots (1)$$
$$x - 2y = 4 \quad \dots (2)$$
$$(1) \times 2 \Rightarrow 6x + 2y = -4 \quad \dots (3)$$
$$x - 2y = 4 \quad \dots (2)$$
$$(3) + (2) \Rightarrow 7x = 0$$
$$x = 0$$

Substituting $x = 0$ in (1)

$$3(0) + y = -2$$
$$y = -2$$

The point of intersection of (1) and (2) is $(0, -2)$

Now, Solving $7x - 3y = -12$ and $2y = x + 3$

$$7x - 3y = -12 \quad \dots (5)$$
$$x - 2y = -3 \quad \dots (2)$$

$$(2)\times 7 \Rightarrow \quad 7x - 14y = -21 \quad \ldots(6)$$
$$(5) \Rightarrow \quad 7x - 3y = -12$$
$$(6)-(5) \Rightarrow \quad -11y = -9$$
$$y = \frac{9}{11}$$

Substituting $y = \frac{9}{11}$ in (2)

$$x - 2\left(\frac{9}{11}\right) = -3$$
$$x = -3 + \frac{18}{11} = \frac{-15}{11}$$

The point of intersection of (5) and (2) is $\left(\frac{-15}{11}, \frac{9}{11}\right)$

Equation of the line joining $(0, -2)$ and $\left(\frac{-15}{11}, \frac{9}{11}\right)$ is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$
$$\Rightarrow \quad \frac{y+2}{\frac{9}{11}+2} = \frac{x-0}{-\frac{15}{11}-0}$$
$$\frac{11(y+2)}{9+22} = \frac{-11x}{15}$$
$$15(y+2) = -31x$$
$$15y + 30 = -31x$$
$$\Rightarrow \quad 31x + 15y + 30 = 0$$

12. Find the equation of a straight line through the point of intersection of the lines $8x + 3y = 18$, $4x + 5y = 9$ and bisecting the line segment joining the points $(5, -4)$ and $(-7, 6)$.

Sol :

Let us solve $8x + 3y = 18$ and $4x + 5y = 9$

$$8x + 3y = 18 \quad \ldots(1)$$
$$4x + 5y = 9 \quad \ldots(2)$$
$$(2)\times 2 \Rightarrow \quad 8x + 10y = 18 \quad \ldots(3)$$
$$(1) \Rightarrow \quad 8x + 3y = 18 \quad \ldots(1)$$
$$(3)-(1) \Rightarrow \quad 7y = 0$$
$$y = 0$$

Substituting in (1)

$$8x + 3(0) = 18$$
$$x = \frac{18}{8} = \frac{9}{4}$$

The point of intersection of (1) and (2) is $\left(\frac{9}{4}, 0\right)$

Mid point of the line segment joining $(5, -4)$ and $(-7, 6)$ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{5-7}{2}, \frac{-4+6}{2}\right) = (-1, 1)$$

Equation of the straight line joining $\left(\frac{9}{4}, 0\right)$ and $(-1, 1)$ is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$
$$\Rightarrow \quad \frac{y-0}{1-0} = \frac{x-\frac{9}{4}}{-1-\frac{9}{4}}$$
$$\Rightarrow \quad \frac{y}{1} = \frac{4x-9}{-13}$$
$$\Rightarrow \quad -13y = 4x - 9$$
$$\Rightarrow \quad 4x + 13y - 9 = 0$$

Exercise 5.5

Multiple Choice Questions:

1. The area of triangle formed by the points $(-5, 0)$, $(0, -5)$ and $(5, 0)$ is

(1) 0 sq. units (2) 25 sq. units
(3) 5 s.q. units (4) none of these

[Ans : (2)]

Sol :

Points $(-5, 0)$, $(0, -5)$ and $(5, 0)$

Area of triangle

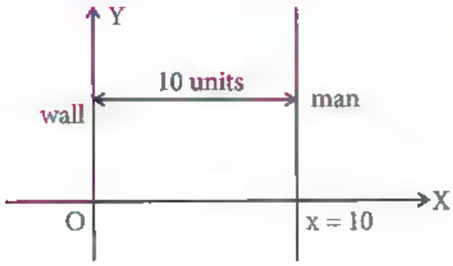
$$= \frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]\ sq.units$$
$$= \frac{1}{2}[-5(-5-0) + 0(0-0) + 5(0+5)]$$
$$= \frac{1}{2}[25+25] = \frac{50}{2} = 25\text{ Sq.units.}$$

2. A man walks near a wall, such that the distance between him and the wall is 10 units. Consider the wall be the Y axis. The path travelled by the man is

(1) $x = 10$ (2) $y = 10$
(3) $x = 0$ (4) $y = 0$ **[Ans : (1)]**

Sol : $x = 10$

A line parallel to Y-axis.



3. The straight line given by the equation x = 11 is

(1) Parallel to X-axis
(2) Parallel to Y-axis
(3) Passing through the origin
(4) Passing through the point (0, 11)

[Ans : (2)]

Sol :

Any line is of the form

x = a is parallel to y-axis

4. If (5, 7), (3, p) and (6,6) are collinear, then the value of p is

(1) 3 (2) 6
(3) 9 (4) 12 **[Ans : (3)]**

Sol :

Given (5, 7), (3, p) and (6, 6) are Collinear.

∴ Area of the triangle formed by the points is zero.

i.e., $\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\} = 0$

$$5(p-6)+3(6-7)+6(7-p) = 0$$
$$5p-30-3+42-6p = 0$$
$$\Rightarrow -p+9=0 \Rightarrow \therefore p = 9$$

5. The point of intersection of 3x – y = 4 and x + y = 8 is

(1) (5, 3) (2) (2, 4)
(3) (3, 5) (4) (4, 4) **[Ans : (3)]**

Sol :

Solving 3x – y = 4 and x + y = 8

$$3x - y = 4 \quad ...(1)$$
$$x + y = 8 \quad ...(2)$$

(1) + (2) ⇒ $4x = 12$

$$x = 3$$

Substituting in (2)

$$3 + y = 8$$
$$y = 5$$

∴ the point of intersection is (3, 5)

6. The slope of the line joining (12, 3), (4, a) is $\frac{1}{8}$. The value of 'a' is

(1) 1 (2) 4
(3) – 5 (4) 2 **[Ans : (4)]**

Sol :

Slope of the line joining (12, 3) and (4, a) is 1/8

$$\frac{y_1-y_2}{x_1-x_2} = \frac{3-a}{12-4} = \frac{1}{8}$$
$$\frac{3-a}{8} = \frac{1}{8}$$
$$3-a = 1 \Rightarrow a = 2$$

7. The slope of the line which is perpendicular to line joining the points (0, 0) and (– 8, 8) is

(1) – 1 (2) 1
(3) $\frac{1}{3}$ (4) – 8 **[Ans : (2)]**

Sol :

Slope of the joining (0, 0) and (– 8, 8) is

$$\frac{y_1-y_2}{x_1-x_2} = \frac{0-8}{0+8} = \frac{-8}{8} = -1$$

∴ Slope of the perpendicular line is 1.

8. If slope of the line PQ is $\frac{1}{\sqrt{3}}$ then the slope of perpendicular bisector of PQ is

(1) $\sqrt{3}$ (2) $-\sqrt{3}$
(3) $\frac{1}{\sqrt{3}}$ (4) 0

[Ans : (2)]

Sol :

Slope of PQ = $\frac{1}{\sqrt{3}}$ = m

Slope of perpendicular bisector = $\frac{-1}{m} = -\sqrt{3}$

9. If A is a point on the Y-axis whose ordinate is 8 and B is a point on the X-axis whose abscissa is 5 then the equation of the line AB is

(1) 8x + 5y = 40 (2) 8x – 5y = 40
(3) x = 8 (4) y = 5 **[Ans : (1)]**

Sol :

Given A is a point on y-axis and its ordinate is 8

B is a point on X-axis and its abscissa is 5

∴ The point A and B are (0, 8) and (5, 0) respectively.

Equation of AB is $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

$$\frac{y-8}{0-8} = \frac{x-0}{5-0}$$
$$5(y-8) = -8x$$
$$8x+5y-40 = 0$$
$$8x+5y = 40$$

10. The equation of a line passing through the origin and perpendicular to the line $7x - 3y + 4 = 0$ is

(1) $7x - 3y + 4 = 0$ (2) $3x - 7y + 4 = 0$
(3) $3x + 7y = 0$ (4) $7x - 3y = 0$

[Ans : (3)]

Sol :

Slope of the line $7x - 3y + 4 = 0$ is

$$\frac{-a}{b} = \frac{-7}{-3} = \frac{7}{3}$$

Slope of the perpendicular line $= \frac{-3}{7}$

equation of the line passing through origin (0, 0) and having slope $\frac{-3}{7}$ is

$$y - y_1 = m(x - x_1)$$
$$y - 0 = \frac{-3}{7}(x-0)$$
$$7y = -3x$$
$$3x + 7y = 0$$

11. Consider four straight line

(i) $l_1 : 3y = 4x + 5$ (ii) $l_2 : 4y = 3x - 1$
(iii) $l_3 : 4y + 3x = 7$ (iv) $l_4 : 4x + 3y = 2$

Which of the following statement is not true?

(1) l_1 and l_2 are perpendicular
(2) l_1 and l_4 are parallel
(3) l_2 and l_4 are perpendicular
(4) l_2 and l_3 are parallel **[Ans : (3)]**

Sol :

Given lines $l_1 : 3y = 4x + 5 \Rightarrow 4x - 3y + 5 = 0$

$l_2 : 4y = 3x - 1 \Rightarrow 3x - 4y - 1 = 0$

$l_3 : 4y + 3x = 7 \Rightarrow 3x + 4y - 7 = 0$

$l_4 : 4x + 3y = 2 \Rightarrow 4x + 3y - 2 = 0$

From l_2, $a_2 = 3, b_2 = -4$ and for l_4, $a_4 = 4, b_4 = 3$

$a_1a_2 + b_1b_2 = 3(4) + (-4)(3) = 0$

$\therefore$ l_2 and l_4 are perpendicular.

12. A straight line has equation $8y = 4x + 21$. Which of the following is true

(1) The slope is 0.5 and the Y-intercept is 2.6
(2) The Slope is 5 and the Y-intercept is 1.6
(3) The Slope is 0.5 and the Y-intercept is 1.6
(4) The Slope is 5 and the Y-intercept is 2.6

[Ans : (1)]

Sol : Given line is $8y = 4x + 21$

$$y = \frac{x}{2} + \frac{21}{8}$$
$$y = \frac{x}{2} + 2.625 \text{ is in the form}$$
$$y = mx + c$$

where $m = \frac{1}{2} = 0.5, c = 2.625$

$$\therefore \text{Slope } m = \frac{1}{2} = 0.5$$

y intercept = 2.6

13. When proving that a quadrilateral is a trapezoid, it is necessary to show

(1) Two lines are parallel.
(2) Two parallel and two non-parallel sides.
(3) Opposite sides are parallel.
(4) All sides are of equal length. **[Ans : (2)]**

Sol :

Two parallel and two non-parallel sides.

14. When proving that a quadrilateral is a parallelogram by using slopes you must find

(1) The slopes of all four sides
(2) The slopes of any one pair of opposite sides
(3) The lengths of all four sides
(4) Both the lengths and slopes of all four sides

[Ans : (1)]

Sol : The slopes of all four sides.

15. (2, 1) is the point of intersection of two lines.

(1) $x - y - 3 = 0; 3x - y - 7 = 0$
(2) $x + y = 3; 3x + y = 7$
(3) $3x + y = 3; x + y = 7$
(4) $x + 3y - 3 = 0; x - y - 7 = 0$ **[Ans : (2)]**

Sol :

Substituting (2, 1) in equation (2)

$$x + y = 3 \text{ and } 3x + y = 7$$

both the equations are satisfied by (2, 1)

UNIT EXERCISE - 5

1. PQRS is a rectangle formed by joining the points P(– 1, –1), Q(–1, 4), R (5, 4) and S(5, – 1). A, B, C and D are the mid-points of PQ , QR , RS and SP respectively. Is the quadrilateral ABCD a square, a rectangle or a rhombus? Justify your answer.

Sol :

Given vertices of a rectangle are P (–1, –1), Q (– 1, 4), R (5, 4) and S (5, –1)

Mid point of PQ $= \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

$= A\left(\dfrac{-1-1}{2}, \dfrac{-1+4}{2}\right) = A\left(-1, \dfrac{3}{2}\right)$

Mid point of QR $= B\left(\dfrac{-1+5}{2}, \dfrac{4+4}{2}\right)$ = B (2, 4)

Midpoint of RS $= C\left(\dfrac{5+5}{2}, \dfrac{4-1}{2}\right) = C\left(5, \dfrac{3}{2}\right)$

Midpoint of SP $= D\left(\dfrac{-1+5}{2}, \dfrac{-1-1}{2}\right)$ = D (2, – 1)

$$AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$= \sqrt{(2+1)^2+(4-3/2)^2} = \sqrt{9+\frac{25}{4}} = \frac{\sqrt{61}}{2}$$

$$BC = \sqrt{(5-2)^2+(3/2-4)^2} = \frac{\sqrt{61}}{2}$$

$$CD = \sqrt{(2-5)^2+(-1-3/2)^2} = \frac{\sqrt{61}}{2}$$

$$DA = \sqrt{(2+1)^2+(-1-3/2)^2} = \frac{\sqrt{61}}{2}$$

$$AC = \sqrt{(5+1)^2+(3/2-3/2)^2} = \sqrt{36} = 6$$

From triangle ABC,

$$AB^2+BC^2 = \frac{61}{4}+\frac{61}{4} = \frac{122}{4} = \frac{61}{2} \neq 36$$

i.e., $AB^2 + BC^2 \neq AC^2$

∴ The points A, B, C and D Cannot be the vertices of a square or Rectangle.

Hence, ABCD is a Rhombus.

2. The area of a triangle is 5. Two of its vertices are (2, 1) and (3, – 2). The third vertex is (x, y) where y = x + 3. Find the coordinates of the third vertex.

Sol :

Given area of triangle is 5

Vertices of triangle are (2, 1), (3, – 2) and (x, y)

Area of triangle =

$$\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)] = 5$$

$$2(-2-y)+3(y-1)+x(1+2) = 10$$
$$-4-2y+3y-3+3x = 10$$
$$3x+y = 17 \quad ...(1)$$

Given that y = x + 3, Substituting in (1)

$$\therefore\ 3x+x+3 = 17$$
$$4x = 14$$
$$x = 7/2$$
$$\text{and } y = 7/2+3$$
$$= 13/2$$

Third vertex is (7/2, 13/2)

3. Find the area of a triangle formed by the lines $3x + y - 2 = 0$, $5x + 2y - 3 = 0$, and $2x - y - 3 = 0$.

Sol : Given sides of a triangle are

$$3x+y-2 = 0 \Rightarrow 3x+y = 2 \quad ...(1)$$
$$5x+2y-3 = 0 \Rightarrow 5x+2y = 3 \quad ...(2)$$
$$2x-y-3 = 0 \Rightarrow 2x-y = 3 \quad ...(3)$$

Solving (1) and (2)

(1) × (2) ⇒ $6x + 2y = 4$ ...(4)

$5x + 2y = 3$ ...(2)

(4) – (2) ⇒ $x = 1$

Substituting in (1)

⇒ $y = 2 - 3 = -1$

Point of intersection of (1) and (2) is (1, – 1)

Now, solving (2) and (3)

(3) × (2) ⇒ $4x - 2y = 6$ ...(5)

$5x + 2y = 3$ ...(2)

(5) + (2) ⇒ $9x = 9$

$x = 1$

Substituting in (3) ⇒ $y = -1$

Point of intersection of (2) and (3) is (1, – 1). Since the point of intersection of (1), (2) and (2), (3) is same, No such triangle is possible.

Hence, area of triangle is zero.

4. If vertices of quadrilateral are at A (– 5, 7), B(– 4, k), C(– 1, – 6) and D(4, 5) and its area is 72 sq.units. Find the value of k.

Sol :

Given vertices of a quadrilateral are A (–5, 7), B (– 4, k), C (– 1, – 6) and D (4, 5)

Area of quadrilateral is 72 sq. units

$$\frac{1}{2}[(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)] = 72$$

$$(-5+1)(k-5) - (-4-4)(7+6) = 144$$

$$-4k + 20 + 104 = 144$$

$$-4k = 144 - 124 = 20$$

$$k = \frac{-20}{4} = -5$$

5. Without using distance formula, show that points (–2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.

Sol :

Given points (–2, –1), (4, 0), (3, 3) and (–3, 2) let the points be A (–2, –1), B (4, 0), C (3, 3) and D (–3, 2)

$$\text{Slope} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$\text{Slope of AB} = \frac{-1-0}{-2-4} = \frac{1}{6}$$

$$\text{Slope of BC} = \frac{0-3}{4-3} = \frac{-3}{1} = -3$$

$$\text{Slope of CD} = \frac{3-2}{3+3} = \frac{1}{6}$$

$$\text{Slope of DA} = \frac{2+1}{-3+2} = \frac{3}{-1} = -3$$

Slope of AB = Slope of CD ⇒ AB is parallel to CD

Slope of BC = Slope of DA ⇒ BC is parallel to DA

Hence, the given points form a parallelogram

6. Find the equations of the lines, whose sum and product of intercepts are 1 and – 6 respectively.

Sol : Let a, b be the intercepts

Given, sum of the intercepts a + b = 1 ...(1)

Product of the intercepts ab = – 6 ...(2)

$$b = \frac{-6}{a}$$

Substituting in (1)

$$a - \frac{6}{a} = 1$$

$$a^2 - a - 6 = 0$$

$$(a-3)(a+2) = 0$$

$$a = -2, 3$$

When $a = -2$, $b = \dfrac{-6}{-2} = 3$

When $a = 3$, $b = \dfrac{-6}{-2} = -2$

Equation of the line in intercepts form is

$$\frac{x}{a} + \frac{y}{b} = 1$$

(i) $a = -2$, $b = 3$ ⇒ $\dfrac{x}{-2} + \dfrac{y}{3} = 1$

$$3x - 2y + 6 = 0$$

(ii) $a = 3, b = -2$ ⇒ $\dfrac{x}{3} + \dfrac{y}{-2} = 1$

$$-2x + 3y + 6 = 0$$

$$\Rightarrow 2x - 3y - 6 = 0$$

7. The owner of a milk store finds that, he can sell 980 litres of milk each week at ₹ 14/litre and 1220 litres of milk each week at ₹16/litre. Assuming a linear relationship between selling price and demand, how many litres could sell weekly at ₹ 17/litre?

Sol :

The relationship between selling price and demand is linear. Taking selling price along x-axis and demand along y-axis. We have two points from the data. (14, 980) and (16, 1220)

Equation of a straight line joining the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 980}{1220 - 980} = \frac{x - 14}{16 - 14}$$

$$\frac{y - 980}{240} = \frac{x - 14}{2}$$

$$y - 980 = 120x - 1680$$

$$120x - y - 700 = 0$$

(or) $y = 120x - 700$

when x = 17, $y = 120(17) - 700$

$= 2040 - 700 = 1340$

Hence the owner could sell 1340 litres of milk weekly at Rs.17/litre.

8. Find the image of the point (3, 8) with respect to the line x + 3y = 7 assuming the line to be a plane mirror.

Sol :

Let P be (3, 8) and Q (a, b) be the image of P.

$$\text{Slope of PQ} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{8-b}{3-a} = m_1$$

Slope of the line x + 3y = 7 is $\frac{-1}{3} = m_2$

Since PQ is perpendicular to the given line,

then $m_1 \times m_2 = -1$

$$\frac{8-b}{3-a} \times \left(\frac{-1}{3}\right) = -1$$

$$8 - b = 9 - 3a$$

$$3a - b = 1 \quad ...(1)$$

$$\text{Mid point of PQ} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{3+a}{2}, \frac{8+b}{2}\right)$$

Since the line x + 3y = 7 is the perpendicular bisector of PQ, the midpoint of PQ lies on the line.

$$\therefore \quad \left(\frac{3+a}{2}\right) + 3\left(\frac{8+b}{2}\right) = 7$$

$$3 + a + 24 + 3b = 14$$

$$a + 3b = -13 \quad ...(2)$$

Solving (1) and (2)

$(1) \times (3) \Rightarrow \quad 9a - 3b = 3 \quad ...(3)$

$a + 3b = -13 \quad ...(2)$

$(3) + (2) \Rightarrow \quad 10a = -10$

$a = -1$

Substituting in (1)

$b = -4$

$\therefore$ Image of (3, 8) is (– 1, – 4).

9. Find the equation of a line passing through the point of intersection of the lines 4x + 7y – 3 = 0 and 2x – 3y + 1 = 0 that has equal intercepts on the axes.

Sol :

Given lines $4x + 7y - 3 = 0 \Rightarrow 4x + 7y = 3 \quad ...(1)$

$2x - 3y + 1 = 0 \Rightarrow 2x - 3y = -1 \quad ...(2)$

Solving (1) and (2)

$4x + 7y = 3 \quad ...(1)$

$(2) \times 2 \Rightarrow \quad 4x - 6y = -2 \quad ...(3)$

$(1) - (3) \Rightarrow \quad 13y = 5$

$$y = \frac{5}{13}$$

Substituting in (1) $\Rightarrow \quad 4x + 7\left(\frac{5}{13}\right) = 3$

$$\Rightarrow \quad x = \frac{1}{13}$$

The point intersection of (1) and (2) is $\left(\frac{1}{13}, \frac{5}{13}\right)$

Given that the line has equal intercepts on the axes.

i.e., a = b

Now,intercept form of the line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a$$

This passes through $\left(\frac{1}{13}, \frac{5}{13}\right)$

$$\Rightarrow \quad \frac{1}{13} + \frac{5}{13} = a \Rightarrow a = \frac{6}{13}$$

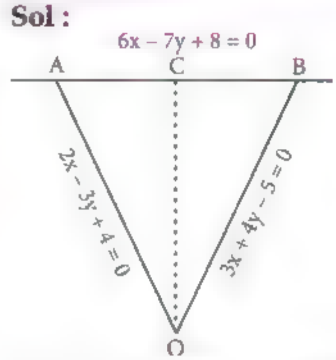
$$\therefore \quad b = \frac{6}{13}$$

Hence the equation of a line is $\dfrac{x}{\frac{6}{13}} + \dfrac{y}{\frac{6}{13}} = 1$

$\Rightarrow \quad 13x + 13y - 6 = 0.$

10. A person standing at a junction (crossing) of two straight paths represented by the equations 2x – 3y + 4 = 0 and 3x + 4y – 5 = 0 seek to reach the path whose equation is 6x – 7y + 8 = 0 in the least time. Find the equation of the path that he should follow.

Sol :



Let the person standing at 'O' and the two straight paths are OA and OB

Where OA: $2x - 3y + 4 = 0 \quad ...(1)$

and OB: $3x + 4y - 5 = 0$ ... (2)

Given that the person seeks to reach the path

AB: $6x - 7y = 8 = 0$

To reach AB in least time, he needs to choose the least distance (perpendicular distance) OC.

Let as find out equation of OC

Solving OA and OB

$$\begin{aligned}
(1)\times(3) \Rightarrow \quad 6x - 9y &= -12 \quad \ldots (3)\\
(2)\times(2) \Rightarrow \quad 6x + 8y &= 10 \quad \ldots (4)\\
(3)-(4) \Rightarrow \quad -17y &= -22\\
y &= \frac{22}{17}
\end{aligned}$$

Substituting in (2)

$$3x + 4\left(\frac{22}{17}\right) = 5$$

$$3x = 5 - \frac{88}{17} = -\frac{3}{17}$$

$$x = -\frac{1}{17}$$

The point of intersection of OA and OB is

$$O\left(-\frac{1}{17}, \frac{22}{17}\right)$$

$$\text{Slope of AB} = \frac{-6}{-7} = \frac{6}{7}$$

Now, OC is perpendicular to AB, $\therefore$ Slope of OC $= -\frac{7}{6}$

OC is passing through $\left(-\frac{1}{17}, \frac{22}{17}\right)$ and having slope $-\frac{7}{6}$

Equation of OC $\Rightarrow \quad y - y_1 = m(x - x_1)$

$$\Rightarrow \quad y - \frac{22}{17} = -\frac{7}{6}\left(x + \frac{1}{17}\right)$$

$$\Rightarrow \quad \frac{17y - 22}{17} = -\frac{7}{6}\left(\frac{17x+1}{17}\right)$$

$$\Rightarrow \quad 102y - 132 = -119x - 7$$

$$\Rightarrow \quad 119x + 102y - 125 = 0$$

CREATIVE QUESTIONS

I. Multiple Choice Questions

Area of Triangle and Area of parallelogram

1. The area of the triangle whose vertices are (2, – 3), (3, 2) and (– 2, 5) is

(1) 11 (2) 12

(3) 14 (4) 13 **[Ans: (3)]**

Sol :

Area of triangle

$$= \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} \text{ sq. units}$$

$$= \frac{1}{2}[2(2-5) + 3(5+3) - 2(-3-2)]$$

$$= \frac{1}{2}(28) = 14 \text{ sq. units}$$

2. AD is the median of triangle ABC with vertices A (– 3, 2), B (5, – 2) and C (1, 3) The area of triangle ABD is

(1) 5 (2) 6

(3) 7 (4) 8 **[Ans: (2)]**

Sol :

$$\text{Area of } \Delta ABC = \frac{1}{2}[-3(-2-3) + 5(3-2) + 1(2+2)]$$

$$= \frac{1}{2}(24) = 12$$

$$\text{Area of } \Delta ABD = \frac{1}{2}(Area\ of\ \Delta ABC) = \frac{12}{2}$$

$$= 6 \text{ sq.units}$$

3. If the points (2, 1), (3, – 2) and (a, b) are collinear, then

(1) a + b = 7 (2) 3a + b = 7

(3) a – b = 7 (4) 3a – b = 7 **[Ans: (2)]**

Sol : Since the points are collinear, Area of triangle is zero.

$$\therefore \frac{1}{2}[2(-2-b) + 3(b-1) + a(1+2] = 0$$

$$3a + b = 7$$

4. If (a, b), (c, d) and (a – c, b – d) are collinear, then

(1) $\frac{a}{b} = \frac{c}{d}$ (2) $\frac{a}{d} = \frac{b}{c}$

(3) $\frac{a}{c} = \frac{d}{b}$ (4) $\frac{a}{b} = \frac{b}{c}$ **[Ans: (1)]**

Sol : Area of triangle is zero, when the points are collinear.

$\therefore a(d-b+d)+c(b-d-b)+(a-c)(b-d) = 0$

$2ad - ab - cd + ab - ad - bc + cd = 0$

$ad - bc = 0$

$\frac{a}{b} = \frac{c}{d}$

5. If the area of the triangle formed by the points (– 2, 3), (4, – 5) and (– 3, y) is 10 square units, then y =

(1) 1 (2) – 1

(3) $\frac{23}{3}$ (4) $\frac{-22}{3}$ **[Ans: (3)]**

Sol : Area of triangle = 10

$\frac{1}{2}\left[-2(-5-y)+4(y-3)-3(3+5)\right] = 10$

$\frac{1}{2}(6y-26) = 10$

$3y - 13 = 10$

$y = \frac{23}{3}$

6. The area of quadrilateral formed by the points (0, 0), (1, 0), (1, 4) and (0, 2) is

(1) 4 (2) 8

(3) 12 (4) 16 **[Ans: (3)]**

Sol : Area of quadrilateral

$= \frac{1}{2}\left[(x_1-x_3)(y_2-y_4)-(x_2-x_4)(y_1-y_3)\right]$ sq. units

$= 8$ sq. units

(or) Plotting the points on the graph, we get the rectangle with length 4 and breadth 2.

$\therefore$ Area = 8 sq.units

7. The area of the rhombus formed by the points (3, 0), (0, 4), (– 3, 0) and (0, – 4) is

(1) 24 (2) 30

(3) 32 (4) 36 **[Ans: (1)]**

Sol : Area of Quadrilateral

$= \frac{1}{2}[(x_1-x_3)(y_2-y_4)-(x_2-x_4)(y_1-y_3)]$

$= \frac{1}{2}\left[(3+3)(4+4)-(0-0)(0-0)\right]$

$= \frac{1}{2}[48] = 24$ sq. units

8. The point (x, y) lies on the line joining (3, 4) and (– 5, – 6) if

(1) $4x - 5y = 1$ (2) $5x - 4y = 1$

(3) $5x - 4y + 1 = 0$ (4) $4x + 5y = 1$

[Ans: (3)]

Sol :

Points are collinear, $\therefore$ Area of triangle is zero.

$\frac{1}{2}\left[x(4+6)+3(-6-y)-5(y-4)\right] = 0$

$10x - 8y + 2 = 0$

$5x - 4y + 1 = 0$

Angle of Inclination and Slope of a Straight line

9. What can be said regarding a line if its slope is negative?

(1) acute (2) obtuse

(3) zero (4) None of these

[Ans: (2)]

Sol :

If slope is negative i.e., $\tan\theta$ is negative then 'θ' is obtuse.

10. What is the slope of a line whose inclination is 45°?

(1) 1 (2) 2

(3) 0 (4) $\frac{1}{2}$ **[Ans: (1)]**

Sol :

Slope $m = \tan 45° = 1$

11. Find the inclination whose slope is $\frac{1}{\sqrt{3}}$

(1) 30° (2) 60°

(3) 90° (4) 45° **[Ans: (1)]**

Sol : Slope $m = \frac{1}{\sqrt{3}}$

i.e., $\tan\theta = \frac{1}{\sqrt{3}} = \tan 30°$

$\theta = 30°$

12. Slope of the line joining the points (4, – 6) and (– 2, – 5) is

(1) $\frac{1}{6}$ (2) $\frac{-1}{6}$

(3) 6 (4) – 6 **[Ans: (2)]**

Sol : Slope $= \frac{y_1-y_2}{x_1-x_2} = \frac{-6+5}{4+2} = \frac{-1}{6}$

13. The points A (1, – 2), B (3, 4) and C (4, 7)

(1) form a right triangle
(2) form an isosceles triangle
(3) form an equilateral triangle
(4) collinear **[Ans: (4)]**

Sol :

$$\text{Slope of AB} = \frac{-2-4}{1-3} = \frac{-6}{-2} = 3$$

$$\text{Slope of BC} = \frac{4-7}{3-4} = \frac{-3}{-1} = 3$$

Slope of AB = Slope of BC

∴ Points are collinear.

14. The points A (4, 4), B (3, 5) and C (– 1, – 1) form

(1) right triangle
(2) isosceles triangle
(3) equilateral triangle
(4) None of these **[Ans: (1)]**

Sol :

Slope of AB = – 1

$$\text{Slope of BC} = \frac{3}{2}$$

Slope of AC = 1

Here, Slope of AB × Slope of AC = – 1 × 1 = – 1

∴ Points form a right triangle.

15. The value of 'x' if the slope of the line joining (2, 5) and (x, 3) is 2

(1) 4 (2) 3
(3) 2 (4) 1 **[Ans: (4)]**

Sol :

$$\text{Slope} = 2$$

$$\frac{5-3}{2-x} = 2$$

$$2 = 4 - 2x$$

$$2x = 2$$

$$x = 1$$

16. Slope of the median through B if the vertices of ΔABC are A (2, 4), B (– 3, 1) and C (4, – 7) is

(1) $\frac{12}{5}$ (2) $\frac{-12}{5}$
(3) $\frac{5}{12}$ (4) $\frac{-5}{12}$ [Ans: (4)]

Sol :

$$\text{Mid-point of AC} = \left(\frac{2+4}{2}, \frac{4-7}{2}\right) = \left(3, \frac{-3}{2}\right)$$

$$\therefore \text{Slope of median through B} = \frac{1+\frac{3}{2}}{-3-3} = \frac{-5}{12}$$

17. The slopes of two line segments are equal. Which of the following is correct?

(1) The line segments are parallel
(2) The end points of the line segments are collinear
(3) The line segments are perpendicular
(4) The end points of line segments are non-collinear [Ans: (1) and (2)]

18. What does a linear equation with a slope of zero look like on a plane?

(1) Vertical line
(2) Horizontal line
(3) line passing through the origin
(4) y-axis [Ans: (2)]

Equations of Straight lines

19. The equation of straight line which passes through the point (2, – 3) and parallel to x-axis is

(1) x = – 2 (2) x = 2
(3) y = – 3 (4) y = 3 **[Ans: (3)]**

Sol :

Equation of any line parallel to X-axis y = b

It passes through (2, – 3)

$$y = -3$$

Hence the equation of straight line is y = – 3

20. The equation of straight line parallel to y-axis and at a distance 3 units to the right is

(1) x = 1 (2) x = 2
(3) x = – 3 (4) x = 3 **[Ans: (4)]**

Sol :

Equation of any line parallel to y-axis at a distance of 'a' units to the right is x = a

∴ Required equation is x = 3

21. The equation of straight line having slope 3 and making intercept 4 on the y-axis is

(1) 3x – y – 4 = 0 (2) 3x – y + 4 = 0
(3) 3x + y – 4 =0 (4) 3x + y + 4 = 0
[Ans: (2)]

Sol :

Given slope 'm' = 3, y intercept 'c' = 4

∴ equation of straight line is

$$y = mx + c$$

$$y = 3x + 4$$

$$3x - y + 4 = 0$$

22. The equation of straight line which passes through (– 4, 3) and having slope $\frac{1}{2}$ is

(1) $x - 2y + 10 = 0$ (2) $x - 2y - 10 = 0$
(3) $x + 2y + 10 = 0$ (4) $x + 2y - 10 = 0$

[Ans: (1)]

Sol :
Equation of straight line in slope-point form is

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{1}{2}(x+4)$$
$$x - 2y + 10 = 0$$

23. Equation of straight line passes through the points (0, – a) and (b, 0) is

(1) $bx - ay = ab$ (2) $ax - by = ab$
(3) $x - y = ab$ (4) $ax + by = 1$

[Ans: (2)]

Sol : Equation of straight line in two points form is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\frac{y + a}{0 + a} = \frac{x - 0}{b - 0}$$
$$ax - by = ab$$

24. Slope of the line $\frac{x}{a} + \frac{y}{b} = 1$ is

(1) $\frac{b}{a}$ (2) $\frac{a}{b}$
(3) $\frac{-b}{a}$ (4) $\frac{-a}{b}$ **[Ans: (3)]**

Sol : Equation is $\frac{x}{a} + \frac{y}{b} = 1$

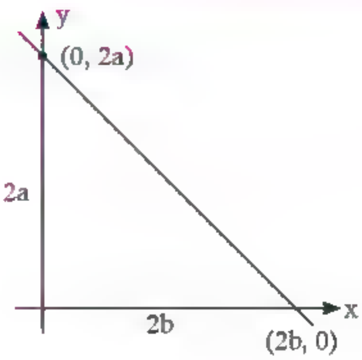
$$\text{Slope} = \frac{-\textit{Coefficient of 'x'}}{\textit{Coefficient of 'y'}} = \frac{\frac{-1}{a}}{\frac{1}{b}} = \frac{-b}{a}$$

25. Area of the triangle formed by the co-ordinate axes and the line ax + by = 2ab is

(1) ab (2) 2ab
(3) $\frac{ab}{2}$ (4) 4ab **[Ans: (2)]**

Sol : Equation of straight line is $ax + by = 2ab$
$(\div 2ab)$

$\frac{x}{2b} + \frac{y}{2a} = 1$ which is in intercept form



x intercept is '2b'
y intercept is '2a'

Area of triangle $= \frac{1}{2} \times 2b \times 2a$

$= 2ab$

26. If the line y = mx meets the lines x + 2y – 1 = 0 and 2x – y + 3 = 0 at the same point, then m is

(1) 1 (2) – 1
(3) 2 (4) – 2 **[Ans: (2)]**

Sol :
Solving $x + 2y - 1 = 0$ and $2x - y + 3 = 0$, the point of intersection is $(-1, 1)$

$y = mx$ passes through $(-1, 1)$

$\therefore$ $1 = m(-1) \Rightarrow m = -1$

27. Equation of the line perpendicular to x = 2 and passing through the point (2, – 8) is

(1) $y = 8$ (2) $y = -8$
(3) $x = 8$ (4) $x = -2$ **[Ans: (2)]**

Sol :
Slope of any line perpendicular to the line of the form x = a is '0'

$\therefore$ Equation is
$$y - y_1 = m(x - x_1)$$
$$y + 8 = 0(x - 2)$$
$$y + 8 = 0 \Rightarrow y = -8$$

28. Equation of straight line which cuts off intercepts 2 and 3 from the co-ordinate axes is

(1) $2x - 3y - 6 = 0$ (2) $2x + 3y - 6 = 0$
(3) $3x - 2y - 6 = 0$ (4) $3x + 2y - 6 = 0$

[Ans: (4)]

Sol : Equation of line in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1$$
$$\frac{x}{2} + \frac{y}{3} = 1$$
$$3x + 2y = 6$$
$$3x + 2y - 6 = 0$$

General form of Straight line

29. General equation of a straight line is

(1) $\frac{-a}{b}+by+\frac{c}{b}=0$

(2) $ax^2 + by^2 + c = 0$

(3) $y = mx + c$

(4) $ax + by + c = 0$ [Ans: (4)]

30. Equation of line parallel to $ax + by + c = 0$ is

(1) $x + y + k = 0$

(2) $ax + by + k = 0$

(3) $x + y = -c$

(4) $bx + ay = c$ [Ans: (2)]

31. $ax + by + c = 0$ represents a line parallel to x-axis if

(1) $a = 0, b = 0$ (2) $a=0, b\neq 0$

(3) $a\neq 0, b=0$ (4) $c = 0$ **[Ans: (2)]**

32. The condition for the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ to be perpendicular is

(1) $a_1 a_2 + b_1 b_2 = 0$ (2) $a_1 b_1 + a_2 b_2 = 0$

(3) $a_1 a_2 - b_1 b_2 = 0$ (4) $a_1 b_1 - a_2 b_2 = 0$

[Ans: (1)]

33. The lines $3x + 4y + 7 = 0$ and $4x - 3y + 5 = 0$ are

(1) Parallel

(2) Perpendicular

(3) Neither parallel nor perpendicular

(4) Parallel and Perpendicular **[Ans: (2)]**

Sol :

$3x + 4y + 7 = 0$ and $4x - 3y + 5 = 0$

$a_1 = 3, b_1 = 4, a_2 = 4, b_2 = -3$

Now $a_1 a_2 + b_1 b_2 = (3)(4) + (4)(-3)$

$= 12 - 12 = 0$

$\therefore$ The lines are perpendicular.

34. Equation of line perpendicular to $2x + 5y = 7$ and passing through the point $(-1, 4)$ is

(1) $x - y + 13 = 0$ (2) $x + y + 13 = 0$

(3) $2x + 5y + 13 = 0$ (4) $5x - 2y + 13 = 0$

[Ans: (4)]

Sol :

Slope of the line $2x + 5y = 7$ is $-\frac{2}{5}$

$\therefore$ Slope of perpendicular line is $\frac{5}{2}$

Required equation is

$$y - 4 = \frac{5}{2}(x+1)$$

$$2y - 8 = 5x + 5$$

$$\Rightarrow 5x - 2y + 13 = 0.$$

35. Find the value of k if the staight lines $(2 + 6k)x + (3 - k)y + (4 + 12k) = 0$ and $7x + 5y - 4 = 0$ are perpendicular.

(1) $\frac{29}{37}$ (2) $-\frac{29}{37}$

(3) $\frac{37}{29}$ (4) $-\frac{37}{29}$ [Ans: (2)]

Sol :

Since the lines are perpendicular

$$a_1 a_2 + b_1 b_2 = 0$$

$$(2 + 6k)(7) + (3 - k)(5) = 0$$

$$14 + 42k + 15 - 5k = 0$$

$$29 + 37k = 0 \Rightarrow k = -\frac{29}{37}$$

36. The value of k if the lines $4x + ky = 8$ and $4x + 3y = 5$ are parallel is

(1) 3 (2) 5

(3) 4 (4) 2 [Ans: (1)]

Sol :

When two lines are parallel, the Co-efficients of 'x' are equal and Co-efficients of 'y' are equal.

II. Very Short Answer Questions

1. Find the area of triangle whose vertices are (1, 1), (2, 3) and (4, 5).

Sol :

$$\text{Area of triangle} = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \text{ sq. units}$$

$$= \frac{1}{2}[1(3-5) + 2(5-1) + 4(1-3)]$$

$$= \frac{1}{2}[-2 + 8 - 8]$$

$$= \frac{-2}{2} = -1$$

$= 1$ sq. unit [$\because$ area cannot be negative].

2. Find the area of triangle ABC, where A (0, 0), B(3, 4) and C (0, 3).

Sol :

$$\text{Area of triangle} = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \text{ sq. units}$$

$$= \frac{1}{2}[0(4-3)+3(3-0)+0(0-4)]$$

$$= \frac{1}{2}(9) = \frac{9}{2} \text{ sq. units}$$

3. Find the value of k, for which the points (7, – 2), (5, 1) and (3, – k) are collinear?

Sol : For collinear points, Area of triangle is zero.

i.e., $\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)] = 0$

$$7(1+k)+5(-k+2)+3(-2-1) = 0$$

$$7+7k-5k+10-9 = 0$$

$$2k = -8$$

$$k = -4$$

4. Find the area of quadrilateral whose vertices are (– 1, – 1), (– 1, 4), (5, 4) and (5, – 1)

Sol :

Area of Quadrilateral $= \frac{1}{2}[(x_1-x_3)(y_2-y_4)-(x_2-x_4)(y_1-y_3)]$ sq. units

$$= \frac{1}{2}[(-1-5)(4+1)-(-1-5)(-1-4)$$

$$= \frac{1}{2}[-30-30]$$

$$= -\frac{60}{2} = -30$$

Area $= 30$ sq. units

[∵ area cannot be negative.]

5. Find the Slope or Gradient of a line whose angle of inclination is (i) 45° (ii) 60°.

Sol :

(i) Angle of inclination $\theta = 45°$

Slope $m = \tan\theta$

$m = \tan 45° = 1$

(ii) When $\theta = 60°$,

slope $m = \tan 60° = \sqrt{3}$

6. Find the slope of the line that passes through the points (2, 0) and (3, 4).

Sol : Slope of the line joining two points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_1-y_2}{x_1-x_2} = \frac{0-4}{2-3} = \frac{-4}{-1} = 4 = m.$$

7. What is the slope of the line parallel to the line whose slope is 2 ?

Sol :

When the lines are parallel, their slopes are equal.

Slope of the required line = 2.

8. Are the three points A (2, 3), B (5, 6) and C (0, –2) collinear?

Sol :

$$\text{Slope of AB} = \frac{y_1-y_2}{x_1-x_2} = \frac{3-6}{2-5} = \frac{-3}{-3} = 1$$

$$\text{Slope of BC} = \frac{y_1-y_2}{x_1-x_2} = \frac{6+2}{5-0} = \frac{8}{5}$$

Slope of AB ≠ Slope of BC

∴ The points are not collinear.

9. Find the equation of the straight line passing through the points (a, b) and (a + b, a – b).

Sol :

Equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\Rightarrow \frac{y-b}{a-b-b} = \frac{x-a}{a+b-a}$$

$$\Rightarrow \frac{y-b}{a-2b} = \frac{x-a}{b}$$

$$\Rightarrow b(y-b) = (x-a)(a-2b)$$

$$\Rightarrow by-b^2 = ax-2bx-a^2+2ab$$

$$\Rightarrow (a-2b)x-by-a^2+2ab+b^2 = 0$$

10. Find the equation of the line passing through (1, 2) and making an angle of 30° with Y-axis.

Sol :

Given that the line makes an angle 30° with Y-axis. i.e., the line makes 60° with the positive direction of X-axis.

$\therefore$ Slope of the line $= m = \tan 60° = \sqrt{3}$

Equation of the line passing through the point (x_1, y_1) and having slope 'm' is $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 2 = \sqrt{3}(x-1)$$

$$\Rightarrow y - 2 = \sqrt{3}x - \sqrt{3}$$

$$\Rightarrow \sqrt{3}x - y + 2 - \sqrt{3} = 0.$$

11. Find the equation of the line whose x intercept is 4 and y intercept is $-\frac{3}{2}$.

Sol :

Equation of straight line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow \frac{x}{4} + \frac{y}{-\frac{3}{2}} = 1$$

$$\Rightarrow \frac{x}{4} - \frac{2y}{3} = 1$$

$$\Rightarrow 3x - 8y - 12 = 0.$$

12. Find the values of k if the straight line $2x + 3y + 4 + k(6x - y + 12) = 0$ is perpendicular to the line $7x + 5y - 4 = 0$.

Sol :

The two lines are $x(2 + 6k) + y(3 - k) + 4 + 12k = 0$ and $7x + 5y - 4 = 0$

Slope of the first line $= m_1 = -\frac{(2+6k)}{3-k}$

Slope of the second line $= m_2 = -\frac{7}{5}$

The lines are perpendicular, $\therefore$ $m_1 \times m_2 = -1$

$$-\frac{(2+6k)}{3-k} \times -\frac{7}{5} = -1$$

$$14 + 42k = -15 + 5k$$

$$k = -\frac{29}{37}$$

13. A line passing through the points (a, 2a) and (–2, 3) is perpendicular to the line $4x + 3y + 5 = 0$, find the values of 'a'.

Sol :

Slope of the line $4x + 3y + 5 = 0$ is $-\frac{4}{3} = m_1$

Slope of the line joining (a, 2a) and (– 2, 3) is

$\frac{2a-3}{a+2} = m_2$

Since, the lines are perpendicular, $m_1 \times m_2 = -1$

$$\Rightarrow -\frac{4}{3} \times \frac{2a-3}{a+2} = -1$$

$$\Rightarrow 8a - 12 = 3a + 6$$

$$\Rightarrow a = \frac{18}{5}$$

14. Find the angle between the lines $x = a$ and $by + c = 0$.

Sol :

$x = a$ is parallel to Y-axis and

$by + c = 0 \Rightarrow y = -\frac{c}{b}$ is parallel to X-axis

$\therefore$ The angle between the two lines is 90°.

III. Short Answer Questions:

1. Find the area of the triangle formed by the points (a, c + a), (a, c) and (– a, c – a).

Sol :

Area of triangle $= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ sq. units

$= \frac{1}{2}[a(c - c + a) + a(c - a - c - a) - a(c + a - c)]$

$= \frac{1}{2}[a(a) + a(-2a) - a(a)]$

$= \frac{1}{2}(-2a^2) = -a^2$

Area $= a^2$ sq. units. [$\because$ Area cannot be negative]

2. Find the value of p if the points (p + 1, 1), (2p + 1, 3) and (2p + 2, 2p) are collinear.

Sol :

Area of triangle is zero, Since the points are collinear.

i.e., $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$

$(p + 1)(3 - 2p) + (2p + 1)(2p - 1) + (2p + 2)(1 - 3) = 0$

$3p - 2p^2 + 3 - 2p + 4p^2 - 1 - 4p - 4 = 0$

$$2p^2 - 3p - 2 = 0$$
$$(2p + 1)(p - 2) = 0$$
$$2p + 1 = 0, \quad p - 2 = 0$$
$$p = -\frac{1}{2}, \quad p = 2$$

3. Prove that the points A (0, – 1), B (2, 1) and C(– 4, 3) form a right angled triangle.

Sol :

$$\text{Slope of AB} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-1-1}{0-2} = \frac{-2}{-2} = 1$$

$$\text{Slope of BC} = \frac{1-3}{2+4} = -\frac{2}{6} = -\frac{1}{3}$$

$$\text{Slope of AC} = \frac{-1-3}{0+4} = -\frac{4}{4} = -1$$

(Slope of AB) × (Slope of AC) = 1 × – 1 = – 1

[∴ $m_1 \times m_2 = -1$]

∴ The given points form a right triangle.

4. Show that the points A (– 2, 0), B (2, 4) , C (4, 1) and D (0, – 3) form a parallelogram.

Sol :

$$\text{Slope of AB} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{0-4}{-2-2} = \frac{-4}{-4} = 1$$

$$\text{Slope of BC} = \frac{4-1}{2-4} = -\frac{3}{2}$$

$$\text{Slope of CD} = \frac{1+3}{4-0} = \frac{4}{4} = 1$$

$$\text{Slope of AD} = \frac{0+3}{-2-0} = -\frac{3}{2}$$

Slope of AB = Slope of CD,
Slope of BC = Slope of AD

∴ The given points form a parallelogram.

5. The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F when K = 0.

Sol : Assuming F along X-axis and K along Y-axis.

The two points are (32, 273) and (212, 373).

The equation of straight line passing through the points (32, 273) and (212, 373) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\Rightarrow \frac{K - 273}{373 - 273} = \frac{F - 32}{212 - 32}$$

$$K = \frac{5}{9}(F - 32) + 273 \quad \text{.......(1)}$$

Putting K = 0 in (1), we get

$$0 = \frac{5}{9}(F - 32) + 273$$

$$\Rightarrow F = 32 - 491.4 = -459.4$$

6. If the intercept of a line between the co-ordinate axes is divided by point (– 5, 4) in the ratio 1 : 2, then find the equation of a line.

Sol :

Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$

Which meets X-axis at A (a, 0) and Y-axis at B (0, b).

Given that P (– 5, 4) dividing AB in the ratio 1 : 2

Using section formula, $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$$\Rightarrow \left(\frac{1(0) + 2(a)}{1+2}, \frac{1(b) + 2(0)}{1+2}\right) = (-5, 4)$$

$$\Rightarrow \left(\frac{2a}{3}, \frac{b}{3}\right) = (-5, 4)$$

$$\frac{2a}{3} = -5, \quad \frac{b}{3} = 4$$

$$a = -\frac{15}{2}, \quad b = 12$$

∴ The equation of line is $\frac{x}{a} + \frac{y}{b} = 1$.

$$\text{i.e., } -\frac{2x}{15} + \frac{y}{12} = 1$$

$$\Rightarrow 8x - 5y + 60 = 0.$$

7. Find the equations of the straight lines which pass through (4, 3) and are respectively parallel and perpendicular to the x-axis.

Sol :

(i) Slope of a line parallel to X-axis is '0'

i.e., $m = 0$

Equation of a line passing through (x_1, y_1) and having slope 'm' is

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow \quad y - 3 = 0(x - 4)$$
$$\Rightarrow \quad y - 3 = 0$$
$$\Rightarrow \quad y = 3$$

(ii) Slope of a line perpendicular to X-axis is undefined i. e., $\frac{1}{0} = m$

Equation of a straight line is $y - 3 = \frac{1}{0}(x - 4)$

$$\Rightarrow \quad x - 4 = 0$$
$$\Rightarrow \quad x = 4$$

8. If the straight line y = mx + c passes through the points (2, 4) and (– 3, 6). Find the values of m and c.

Sol :

$y = mx + c$ passes through (2, 4)

$$\therefore \quad 4 = 2m + c \qquad \ldots (1)$$

Again $y = mx + c$ passes through (– 3, 6)

$$\text{Then,} \quad 6 = -3m + c \qquad \ldots (2)$$

Solving (1) and (2)

$$\Rightarrow \quad m = -\frac{2}{5}$$

Substituting in (1)

$$\Rightarrow \quad c = \frac{24}{5}$$

9. Find the equation of the line passing through the point (2, 1) and parallel to join of the points (1, 3) and (– 3, 1).

Sol : Slope of the line joining the points (1, 3) and (– 3, 1) is

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{3-1}{1+3} = \frac{2}{4} = \frac{1}{2}$$

Required line is parallel to the line joining the points (1, 3) and (– 3, 1).

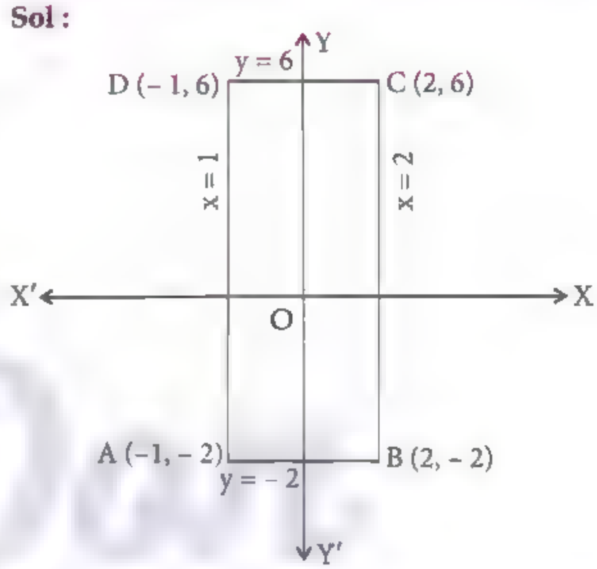
$\therefore$ Slope of the required line $= \frac{1}{2}$ as slopes are equal Since the lines are parallel.

Equation of line passing through the point (x_1, y_1) and having slope 'm' is

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow \quad y - 1 = \frac{1}{2}(x - 2)$$
$$\Rightarrow \quad 2(y - 1) = x - 2$$
$$\Rightarrow \quad x - 2y = 0.$$

10. Find the equation of the diagonals of the rectangle whose sides are x = 2, x = – 1, y = 6 and y = – 2.

Sol :



$$\text{Equation of AC} = \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$
$$\Rightarrow \quad \frac{y+2}{6+2} = \frac{x+1}{2+1}$$
$$\Rightarrow \quad 3(y + 2) = 8(x + 1)$$
$$\Rightarrow \quad 8x - 3y + 2 = 0$$

$$\text{Equation of BD} \Rightarrow \frac{y+2}{6+2} = \frac{x-2}{-1-2}$$
$$\Rightarrow \quad -3(y + 2) = 8(x - 2)$$
$$\Rightarrow \quad 8x + 3y - 10 = 0.$$

11. Show that the line $a^2x + ay + 1 = 0$ is perpendicular to the line $x - ay = 0$ for all non-zero real values of 'a'.

Sol :

$$\text{Slope of } a^2x + ay + 1 = 0 \text{ is } -\frac{a^2}{a} = -a = m_1$$

$$\text{Slope of } x - ay = 1 \text{ is } \frac{-1}{-a} = \frac{1}{a} = m_2$$

$$\text{Now, } m_1 \times m_2 = -a \times \left(\frac{1}{a}\right) = -1$$

$\therefore$ The lines are perpendicular.

12. Find the equation of the perpendicular bisector of the line segment joining the points (1, 1) and (2, 3).

Sol :

Let P be the mid point of the line segment joining A(1, 1) and B (2, 3)

$$\Rightarrow \quad P\left(\frac{3}{2}, 2\right)$$

Let 'm' be the slope of perpendicular bisector of AB .

$$m \times \text{Slope of AB} = -1$$

$$m \times \left(\frac{3-1}{2-1}\right) = -1$$

$$m = -\frac{1}{2}$$

Perpendicular bisector passes through $\left(\frac{3}{2}, 2\right)$ and having slope $m = -\frac{1}{2}$

Equation of perpendicular bisector is

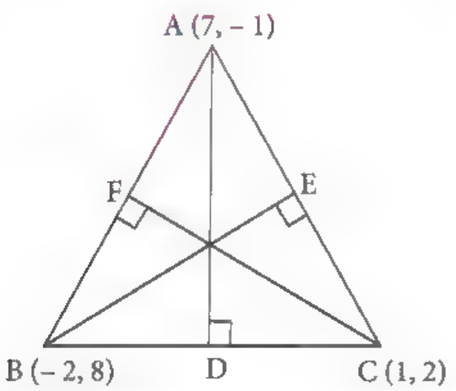
$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - 2 = -\frac{1}{2}\left(x - \frac{3}{2}\right)$$

$$\Rightarrow \quad 2x + 4y - 11 = 0.$$

13. Find the equations of the altitudes of the triangle whose vertices are A (7, – 1), B (– 2, 8) and C (1, 2).

Sol :



Let AD, BE and CF be three altitudes of triangle ABC. Let m_1, m_2 and m_3 be the slopes of AD, BE and CF respectively.

AD perpendicular to BC

$$\Rightarrow \quad \text{Slope of AD} \times \text{Slope of BC} = -1$$

$$m_1 \times \left(\frac{2-8}{1+2}\right) = -1$$

$$\therefore \quad m_1 = \frac{1}{2}$$

BE perpendicular to AC $\Rightarrow$

$$\text{Slope of BE} \times \text{Slope of AC} = -1$$

$$m_2 \times \left(\frac{-1-2}{7-1}\right) = -1$$

$$\therefore \quad m_2 = 2$$

and CF perpendicular to AB $\Rightarrow$

$$\text{Slope of CF} \times \text{Slope of AB} = -1$$

$$m_3 \times \left(\frac{-1-8}{7+2}\right) = -1$$

$$m_3 = 1$$

Since AD passes through A (7, –1) and has slope $m_1 = \frac{1}{2}$, then its equation is $y - y_1 = m(x - x_1)$

$$y + 1 = \frac{1}{2}(x - 7)$$

$$\Rightarrow \quad x - 2y - 9 = 0$$

BE passes through B (– 2, 8) and having slope $m_2 = 2$,

$$\text{Its equation is } y - 8 = 2(x + 2)$$

$$\Rightarrow \quad 2x - y + 12 = 0$$

CE passes through C (1, 2) and has slope $m_3 = 1$

$$\text{Its equation is } y - 2 = 1(x - 1)$$

$$\Rightarrow \quad x - y + 1 = 0.$$

IV. Long Answer Questions

1. If the points A (0, 1), [redacted] (x, y), C (5, – 2) and D (2, – 1) form a parallelogram, then find the values of x, y.

Sol :

$$\text{Slope of AB} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{1-y}{0-x}$$

$$= \frac{1-y}{-x}$$

$$\text{Slope of BC} = \frac{y+2}{x-5}$$

$$\text{Slope of CD} = \frac{-2+1}{5-2} = -\frac{1}{3}$$

$$\text{Slope of DA} = \frac{-1-1}{2-0} = -\frac{2}{2} = -1$$

In parallelogram ABCD, AB is parallel to CD

$\therefore$ Slope of AB = Slope of CD

$$\frac{1-y}{-x} = -\frac{1}{3}$$

$$\Rightarrow 3 - 3y = x$$

$$\Rightarrow x + 3y = 3 \quad \ldots (1)$$

BC is parallel to AD

Slope of BC = Slope of AD

$$\frac{y+2}{x-5} = -1$$

$$\Rightarrow y + 2 = -x + 5$$

$$\Rightarrow x + y = 3 \quad \ldots (2)$$

Solving (1) and (2)

$$x + 3y = 3 \quad \ldots (1)$$

$$x + y = 3 \quad \ldots (2)$$

$$(1) - (2) \Rightarrow 2y = 0$$

$$y = 0$$

Substituting in (2)

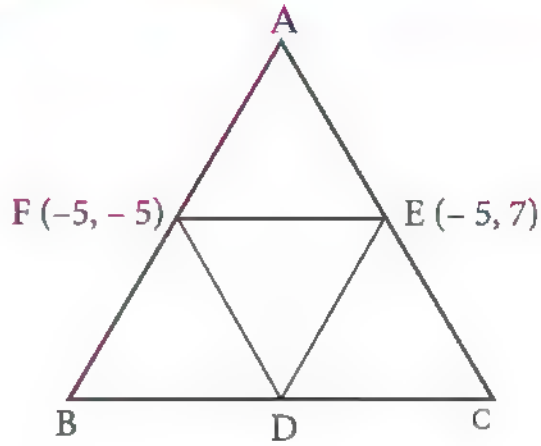
$$x + 0 = 3$$

$$x = 3$$

$\therefore$ x = 3 and y = 0 and the point B is (3, 0).

2. The mid points of the sides of a triangle are (2, 1), (– 5, 7) and (– 5, – 5). Find the equation of the sides.

Sol :



Let D (2, 1), E (– 5, 7) and F (– 5, – 5) be the mid points of the sides BC, CA and AB of a ΔABC.

$$\text{Slope of EF} = \frac{-5-7}{-5+5} = -\frac{12}{0} \quad \text{(undefined.)}$$

$\therefore$ EF is parallel to Y-axis

But EF parallel to BC [$\because$ line joining mid points of two sides is parallel to third side and half of it]

Equation of any line parallel to Y-axis at a distance of 'a' units is x = a.

If it passes through (2, 1), then 2 = a

$\therefore$ Equation of side BC is x = 2

$$\text{Now, Slope of FD} = \frac{1+5}{2+5} = \frac{6}{7}$$

By Geometry, CA is parallel to FD

$$\therefore \text{Slope of CA} = \frac{6}{7}$$

CA passes through the point E (–5, 7)

$\therefore$ Equation of side CA is $y - 7 = \frac{6}{7}(x+5)$

$$\Rightarrow 6x - 7y + 79 = 0$$

$$\text{Now, slope of DE} = \frac{7-1}{-5-2} = \frac{-6}{7}$$

AB is parallel to DE

$$\text{Slope of AB} = -\frac{6}{7}$$

AB passes through (– 5, – 5)

$\therefore$ Equation of AB is $y + 5 = -\frac{6}{7}(x+5)$

$$\Rightarrow 6x + 7y + 65 = 0.$$

3. Find the equation to the straight line which passes through the points (3, 4) and have intercepts on the axes.

(i) equal in magnitude but opposite in sign

(ii) such that their sum is 14.

Sol :

(i) Let the intercepts on the axes be 'a' and '– a' respectively.

Equation of line in intercept from is $\frac{x}{a}+\frac{y}{-a}=1$

$$\Rightarrow x - y = a$$

Since, this passes through (3, 4) then

$$a = 3 - 4 = -1$$

$\therefore$ The equation is $x - y + 1 = 0$

(ii) Given sum of the intercepts is 14.

i.e., $a + b = 14 \Rightarrow b = 14 - a$

Equation of straight line is $\frac{x}{a}+\frac{y}{b} = 1$

$$\Rightarrow \frac{x}{a}+\frac{y}{14-a} = 1$$

Since, this passes through (3, 4) then

$$\frac{3}{a}+\frac{4}{14-a} = 1$$

$$\Rightarrow a^2 - 13a + 42 = 0$$

$$\Rightarrow (a-6)(a-7) = 0$$

$$\Rightarrow a = 6, 7$$

When $a = 6, b = 8$

Equation of a straight line is $\frac{x}{6}+\frac{y}{8} = 1$

$$\Rightarrow 4x + 3y = 24$$

When $a = 7, b = 7$

Equation of a straight line is $\frac{x}{7}+\frac{y}{7} = 1$

$$\Rightarrow x + y = 7$$

UNIT 6 TRIGONOMETRY

MIND MAP

- **TRIGONOMETRY**
  - **Basic of Trigonometry**
    - Trigonometric Ratios
      - $\sin\theta = \dfrac{Opp.side}{Hypotenuse}$
      - $\cos\theta = \dfrac{Adj.side}{Hypotenuse}$
      - $\tan\theta = \dfrac{opp.side}{Adjacent\ side}$
      - $\operatorname{cosec}\theta, \sec\theta, \cot\theta$
    - Complementary Angles
      - $\sin(90^\circ - \theta) = \cos\theta$
      - $\cos(90^\circ - \theta) = \sin\theta$
      - $\tan(90^\circ - \theta) = \cot\theta$
      - $\operatorname{cosec}(90^\circ - \theta) = \sec\theta$
      - $\sec(90^\circ - \theta) = \operatorname{cosec}\theta$
      - $\cot(90^\circ - \theta) = \tan\theta$
    - Trigonometric Identities
      - $\sin^2\theta + \cos^2\theta = 1$
      - $1 + \tan^2\theta = \sec^2\theta$
      - $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
      - Problems and proof based on Trigonometric Identities
    - Values of Trigonometric ratios for 0°, 30°, 45°, 60° and 90°
  - **Applications of Trigonometry**
    - Clinometer
    - Theodolite
    - Height and Distances
      - Angle of Elevation
      - Angle of Depression
      - Problems on Angle of elevation and Depression

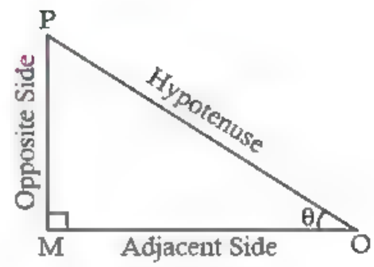
TRIGONOMETRIC RATIOS

Key Points

(i) Trigonometric Ratios

Let $0^\circ < \theta < 90^\circ$

1. $\sin\theta = \dfrac{\text{Opposite side}}{\text{Hypotenuse}}$
2. $\cos\theta = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}}$
3. $\tan\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}}$
4. $\operatorname{cosec}\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite side}}$
5. $\sec\theta = \dfrac{\text{Hypotenuse}}{\text{Adjacent side}}$
6. $\cot\theta = \dfrac{\text{Adjacent side}}{\text{Opposite side}}$



Again

7. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
8. $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
9. $\sec\theta = \dfrac{1}{\cos\theta}$
10. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(ii) Table For Trigonometric Ratios For 0°, 30°, 45°, 60°, 90°

θ Trigonometric Ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
$\operatorname{cosec}\theta$	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

(iii) Complementary Angle

1. $\sin(90-\theta) = \cos\theta$
2. $\cos(90-\theta) = \sin\theta$
3. $\tan(90-\theta) = \cot\theta$
4. $\operatorname{cosec}(90-\theta) = \sec\theta$
5. $\sec(90-\theta) = \operatorname{cosec}\theta$
6. $\cot(90-\theta) = \tan\theta$

We can write

1. $(\sin\theta)^2 = \sin^2\theta$
2. $(\cos\theta)^2 = \cos^2\theta$
3. $(\tan\theta)^2 = \tan^2\theta$
4. $(\text{cosec}\,\theta)^2 = \text{cosec}^2\theta$
5. $(\sec\theta)^2 = \sec^2\theta$
6. $(\cot\theta)^2 = \cot^2\theta$

(iv) Trigonometric Identities

For all real values of θ

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \text{cosec}^2\theta$

Examples

These identities are termed as fundamental identities of trigonometry.

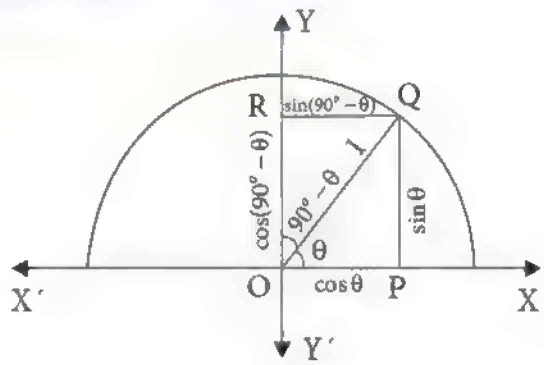
Though the above identities are true for any angle θ, we will consider the following equal forms for $0° < \theta < 90°$

1. $\sin^2\theta = 1 - \cos^2\theta$ (or) $\cos^2\theta = 1 - \sin^2\theta$
2. $\tan^2\theta = \sec^2\theta - 1$ (or) $\sec^2\theta - \tan^2\theta = 1$
3. $\cot^2\theta = \text{cosec}^2\theta - 1$ (or) $\text{cosec}^2\theta - \cot^2\theta = 1$

1. **Prove that (i)** $\sin(90 - \theta) = \cos\theta$ **(ii)** $\cos(90 - \theta) = \sin\theta$.

Visual proof of Trigonometric complementary angle:

Consider a semicircle of radius 1 as shown in the figure.



Let $\angle QOP = \theta$.

Then $\angle QOR = 90° - \theta$, so that OPQR forms a rectangle.

From triangle OPQ, $\dfrac{OP}{OQ} = \cos\theta$

But $OQ = \text{radius} = 1$

Therefore $OP = OQ\cos\theta = \cos\theta$

Similarly, $\dfrac{PQ}{OQ} = \sin\theta$

$\Rightarrow PQ = OQ\sin\theta = \sin\theta$ (since $OQ = 1$)

$OP = \cos\theta,\ PQ = \sin\theta$ ...(1)

Now, from triangle QOR,

We have $\dfrac{OR}{OQ} = \cos(90° - \theta)$

Therefore, $OR = OQ\cos(90° - \theta)$

$\Rightarrow OR = \cos(90° - \theta)$

Similarly, $\dfrac{RQ}{OQ} = \sin(90° - \theta)$

$$\Rightarrow RQ = \sin(90^\circ - \theta)$$

$$OR = \cos(90^\circ - \theta),\ RQ = \sin(90^\circ - \theta) \qquad ...(2)$$

Since OPQR is a rectangle,

$$OP = RQ \text{ and } OR = PQ$$

Therefore From (1) and (2)

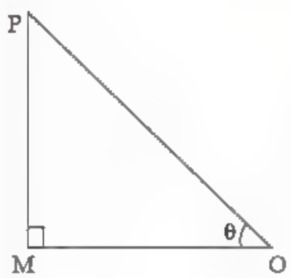
We get, $\sin(90^\circ - \theta) = \cos\theta$ and $\cos(90^\circ - \theta) = \sin\theta$

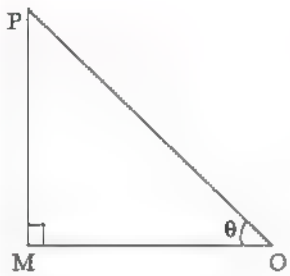
2. Prove the Trigonometric Identities

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \text{cosec}^2\theta$

Picture	Identity	Proof
	$\sin^2\theta + \cos^2\theta = 1$	In right triangle OMP, We have $\frac{OM}{OP} = \cos\theta, \frac{PM}{OP} = \sin\theta$ ... (1) By Pythagoras theorem $MP^2 + OM^2 = OP^2$ ... (2) Dividing each term on both sides of (2) by OP^2, (since $OP \neq 0$) We get, $\frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{OP^2}{OP^2}$ $\Rightarrow \left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = \left(\frac{OP}{OP}\right)^2$ From (1), $(\sin\theta)^2 + (\cos\theta)^2 = 1^2$ Hence $\sin^2\theta + \cos^2\theta = 1$
	$1 + \tan^2\theta = \sec^2\theta$	In right triangle OMP, we have $\frac{MP}{OM} = \tan\theta, \frac{OP}{OM} = \sec\theta$ ... (3) From (2), $MP^2 + OM^2 = OP^2$ Dividing each term on both sides of (2) by OM^2, (since $OM \neq O$) we get, $\frac{MP^2}{OM^2} + \frac{OM^2}{OM^2} = \frac{OP^2}{OM^2}$ $\Rightarrow \left(\frac{MP}{OM}\right)^2 + \left(\frac{OM}{OM}\right)^2 = \left(\frac{OP}{OM}\right)^2$ From (3), $(\tan\theta)^2 + 1^2 = (\sec\theta)^2$ Hence $1 + \tan^2\theta = \sec^2\theta$

	$1+\cot^2\theta = \mathrm{cosec}^2\theta$	In right triangle OMP, we have $\frac{OM}{MP} = \cot\theta, \frac{OP}{MP} = \mathrm{cosec}\,\theta$ ... (4) From (2), $MP^2 + OM^2 = OP^2$ Dividing each term on both sides of (2) by MP^2, (since $MP \neq 0$) We get, $\frac{MP^2}{MP^2} + \frac{OM^2}{MP^2} = \frac{OP^2}{MP^2}$ $\Rightarrow \left(\frac{MP}{MP}\right)^2 + \left(\frac{OM}{MP}\right)^2 = \left(\frac{OP}{MP}\right)^2$ From (4), $1^2 + (\cot\theta)^2 = (\mathrm{cosec}\,\theta)^2$ Hence $1+\cot^2\theta = \mathrm{cosec}^2\theta$

Worked Examples

6.1 Prove that $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$.

Sol :

$$\tan^2\theta - \sin^2\theta = \tan^2\theta - \frac{\sin^2\theta}{\cos^2\theta}\cos^2\theta$$

$$= \tan^2\theta\,(1-\cos^2\theta) = \tan^2\theta\sin^2\theta$$

6.2 Prove that $\frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$.

Sol :

$$\frac{\sin A}{1+\cos A} = \frac{\sin A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}$$

[multiply numerator and denominator by the conjugate of 1 + cos A]

$$= \frac{\sin A\,(1-\cos A)}{(1+\cos A)(1-\cos A)}$$

$$= \frac{\sin A\,(1-\cos A)}{1-\cos^2 A}$$

$$= \frac{\sin A\,(1-\cos A)}{\sin^2 A} = \frac{1-\cos A}{\sin A}$$

6.3 Prove that $1+\frac{\cot^2\theta}{1+\mathrm{cosec}\,\theta} = \mathrm{cosec}\,\theta$.

Sol : $1+\frac{\cot^2\theta}{1+\mathrm{cosec}\,\theta}$

$$= 1+\frac{\mathrm{cosec}^2\theta - 1}{\mathrm{cosec}\,\theta+1} \quad [\text{since } \mathrm{cosec}^2\theta - 1 = \cot^2\theta]$$

$$= 1+\frac{(\mathrm{cosec}\,\theta+1)(\mathrm{cosec}\,\theta-1)}{\mathrm{cosec}\,\theta+1}$$

$$= 1+(\mathrm{cosec}\,\theta-1) = \mathrm{cosec}\,\theta$$

6.4 Prove that $\sec\theta - \cos\theta = \tan\theta\sin\theta$.

Sol :

$$\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1-\cos^2\theta}{\cos\theta}$$

$$= \frac{\sin^2\theta}{\cos\theta} \quad [\text{since } 1-\cos^2\theta = \sin^2\theta]$$

$$= \frac{\sin\theta}{\cos\theta}\times\sin\theta = \tan\theta\sin\theta$$

6.5 Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \mathrm{cosec}\,\theta + \cot\theta$.

Sol :

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}\times\frac{1+\cos\theta}{1+\cos\theta}}$$

[multiply numerator and denominator by the conjugate of $1-\cos\theta$]

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}} \quad [\text{since } \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{1+\cos\theta}{\sin\theta} = \mathrm{cosec}\,\theta + \cot\theta$$

6.6 **Prove that** $\dfrac{\sec\theta}{\sin\theta}-\dfrac{\sin\theta}{\cos\theta}=\cot\theta$.

Sol :

$$\frac{\sec\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\cos\theta}}{\sin\theta}-\frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta}{\sin\theta\cos\theta} = \cot\theta$$

6.7 **Prove that** $\sin^2 A\cos^2 B+\cos^2 A\sin^2 B+\cos^2 A\cos^2 B+\sin^2 A\sin^2 B=1$

Sol :

$\sin^2 A\cos^2 B+\cos^2 A\sin^2 B+\cos^2 A\cos^2 B+\sin^2 A\sin^2 B$

$=\sin^2 A\cos^2 B+\sin^2 A\sin^2 B+\cos^2 A\sin^2 B+\cos^2 A\cos^2 B$

$=\sin^2 A(\cos^2 B+\sin^2 B)+\cos^2 A(\sin^2 B+\cos^2 B)$

$=\sin^2 A(1)+\cos^2 A(1)$ [since $\sin^2 B+\cos^2 B=1$]

$=\sin^2 A+\cos^2 A=1$

6.8 **If** $\cos\theta+\sin\theta=\sqrt{2}\cos\theta$, **then prove that** $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$.

Sol :

Now, $\cos\theta+\sin\theta = \sqrt{2}\cos\theta$

Squaring both sides

$$(\cos\theta+\sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$

$$\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta = 2\cos^2\theta$$

$$2\cos^2\theta-\cos^2\theta-\sin^2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta-\sin^2\theta = 2\sin\theta\cos\theta$$

$$(\cos\theta+\sin\theta)(\cos\theta-\sin\theta) = 2\sin\theta\cos\theta$$

$$\cos\theta-\sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta+\sin\theta}$$

$$=\frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta}\quad[\text{since }\cos\theta+\sin\theta=\sqrt{2}\cos\theta]$$

$$=\sqrt{2}\sin\theta$$

Therefore $\cos\theta-\sin\theta = \sqrt{2}\sin\theta$

6.9 **Prove that** $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$.

Sol :

$(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)$

$$=\left(\frac{1}{\sin\theta}-\sin\theta\right)\left(\frac{1}{\cos\theta}-\cos\theta\right)\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)$$

$$=\frac{1-\sin^2\theta}{\sin\theta}\times\frac{1-\cos^2\theta}{\cos\theta}\times\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{\cos^2\theta\sin^2\theta\times 1}{\sin^2\theta\cos^2\theta}=1.$$

6.10 **Prove that** $\dfrac{\sin A}{1+\cos A}+\dfrac{\sin A}{1-\cos A}=2\operatorname{cosec}A$.

Sol :

$$\frac{\sin A}{1+\cos A}+\frac{\sin A}{1-\cos A}$$

$$=\frac{\sin A(1-\cos A)+\sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$$

$$=\frac{\sin A-\sin A\cos A+\sin A+\sin A\cos A}{1-\cos^2 A}$$

$$=\frac{2\sin A}{1-\cos^2 A}=\frac{2\sin A}{\sin^2 A}$$

$$=2\operatorname{cosec}A$$

6.11 **If** $\operatorname{cosec}\theta+\cot\theta=P$ **then prove that** $\cos\theta=\dfrac{P^2-1}{P^2+1}$.

Sol :

Given $\operatorname{cosec}\theta+\cot\theta = P$ ...(1)

$\operatorname{cosec}^2\theta-\cot^2\theta = 1$ (identity)

$$\operatorname{cosec}\theta-\cot\theta = \frac{1}{\operatorname{cosec}\theta+\cot\theta}$$

$$\operatorname{cosec}\theta-\cot\theta = \frac{1}{P}\qquad\text{...(2)}$$

Adding (1) and (2) we get,

$$2\operatorname{cosec}\theta = P+\frac{1}{P}$$

$$2\operatorname{cosec}\theta = \frac{P^2+1}{P}\qquad\text{...(3)}$$

Subtracting (2) from (1), we get,

$$2\cot\theta = P-\frac{1}{P}$$

$$2\cot\theta = \frac{P^2-1}{P} \qquad \ldots(4)$$

Dividing (4) by (3) we get,

$$\frac{2\cot\theta}{2\,\text{cosec}\,\theta} = \frac{P^2-1}{P}\times\frac{P}{P^2+1} \Rightarrow \cos\theta = \frac{P^2-1}{P^2+1}.$$

6.12 **Prove that $\tan^2 A - \tan^2 B = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$**

Sol :

$$\tan^2 A - \tan^2 B = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$$

$$= \frac{\sin^2 A\cos^2 B - \sin^2 B\cos^2 A}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A(1-\sin^2 B) - \sin^2 B(1-\sin^2 A)}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B}{\cos^2 A\cos^2 B}$$

$$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A\cos^2 B}$$

6.13 **Prove that $\left(\dfrac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\dfrac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right) = 2\sin A\cos A$**

Sol :

$$\left(\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right) - \left(\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right)$$

$$= \left(\frac{(\cos A - \sin A)(\cos^2 A + \sin^2 A + \cos A\sin A)}{\cos A - \sin A}\right) - \left(\frac{(\cos A + \sin A)(\cos^2 A + \sin^2 A - \cos A\sin A)}{\cos A + \sin A}\right)$$

$$\left[\begin{array}{r}\text{since } a^3 - b^3 = (a-b)(a^2+b^2+ab) \\ a^3+b^3 = (a+b)(a^2+b^2-ab)\end{array}\right]$$

$$= (1 + \cos A\sin A) - (1 - \cos A\sin A) = 2\cos A\sin A$$

6.14 **Prove that**

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\text{cosec}\,A + \cot A - 1} = 1$$

Sol :

$$\frac{\sin A}{\sec A + \tan A - 1} + \frac{\cos A}{\text{cosec}\,A + \cot A - 1}$$

$$= \frac{\sin A(\text{cosec}\,A + \cot A - 1) + \cos A(\sec A + \tan A - 1)}{(\sec A + \tan A - 1)(\text{cosec}\,A + \cot A - 1)}$$

$$= \frac{\sin A\,\text{cosec}\,A + \sin A\cot A - \sin A + \cos A\sec A + \cos A\tan A - \cos A}{(\sec A + \tan A - 1)(\text{cosec}\,A + \cot A - 1)}$$

$$= \frac{1 + \cos A - \sin A + 1 + \sin A - \cos A}{\left(\dfrac{1}{\cos A} + \dfrac{\sin A}{\cos A} - 1\right)\left(\dfrac{1}{\sin A} + \dfrac{\cos A}{\sin A} - 1\right)}$$

$$= \frac{2}{\left(\dfrac{1 + \sin A - \cos A}{\cos A}\right)\left(\dfrac{1 + \cos A - \sin A}{\sin A}\right)}$$

$$= \frac{2\sin A\cos A}{(1 + \sin A - \cos A)(1 + \cos A - \sin A)}$$

$$= \frac{2\sin A\cos A}{[1 + (\sin A - \cos A)][1 - (\sin A - \cos A)]}$$

$$= \frac{2\sin A\cos A}{1 - (\sin A - \cos A)^2}$$

$$= \frac{2\sin A\cos A}{1 - (\sin^2 A + \cos^2 A - 2\sin A\cos A)}$$

$$= \frac{2\sin A\cos A}{1 - (1 - 2\sin A\cos A)}$$

$$= \frac{2\sin A\cos A}{1 - 1 + 2\sin A\cos A} = \frac{2\sin A\cos A}{2\sin A\cos A} = 1.$$

6.15 **Show that $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1-\tan A}{1-\cot A}\right)^2$.**

Sol :

$$\text{LHS} = \left(\frac{1+\tan^2 A}{1+\cot^2 A}\right) = \frac{1+\tan^2 A}{1+\dfrac{1}{\tan^2 A}}$$

$$= \frac{1+\tan^2 A}{\dfrac{\tan^2 A + 1}{\tan^2 A}} = \tan^2 A \qquad \ldots(1)$$

$$\text{RHS} = \left(\frac{1-\tan A}{1-\cot A}\right)^2 = \left(\frac{1-\tan A}{1-\dfrac{1}{\tan A}}\right)^2$$

$$= \left(\frac{1-\tan A}{\dfrac{\tan A - 1}{\tan A}}\right)^2 = (-\tan A)^2 = \tan^2 A \qquad \ldots(2)$$

From (1) and (2), $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1-\tan A}{1-\cot A}\right)^2$

6.16 Prove that $\dfrac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}=\sin^2 A\cos^2 A.$

Sol :

$$\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3 A-\operatorname{cosec}^3 A}$$

$$=\frac{\left(1+\dfrac{\cos A}{\sin A}+\dfrac{\sin A}{\cos A}\right)(\sin A-\cos A)}{(\sec A-\operatorname{cosec} A)(\sec^2 A+\sec A\operatorname{cosec} A+\operatorname{cosec}^2 A)}$$

$$=\frac{\dfrac{(\sin A\cos A+\cos^2 A+\sin^2 A)(\sin A-\cos A)}{\sin A\cos A}}{(\sec A-\operatorname{cosec} A)\left(\dfrac{1}{\cos^2 A}+\dfrac{1}{\cos A\sin A}+\dfrac{1}{\sin^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)\left(\dfrac{\sin A}{\sin A\cos A}-\dfrac{\cos A}{\sin A\cos A}\right)}{(\sec A-\operatorname{cosec} A)\left(\dfrac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin^2 A\cos^2 A}\right)}$$

$$=\frac{(\sin A\cos A+1)(\sec A-\operatorname{cosec} A)}{(\sec A-\operatorname{cosec} A)(1+\sin A\cos A)}\times\sin^2 A\cos^2 A$$

$$=\sin^2 A\cos^2 A$$

6.17 If $\dfrac{\cos^2\theta}{\sin\theta}=p$ and $\dfrac{\sin^2\theta}{\cos\theta}=q$, then prove that $p^2q^2(p^2+q^2+3)=1$.

Sol :

We have $\dfrac{\cos^2\theta}{\sin\theta}=p$ ...(1)

and $\dfrac{\sin^2\theta}{\cos\theta}=q$ ... (2)

$p^2q^2(p^2+q^2+3)=$

$$\left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\times\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2+\left(\frac{\sin^2\theta}{\cos\theta}\right)^2+3\right]$$

[from (1) and (2)]

$$=\left(\frac{\cos^4\theta}{\sin^2\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right)\times\left[\frac{\cos^4\theta}{\sin^2\theta}+\frac{\sin^4\theta}{\cos^2\theta}+3\right]$$

$$=(\cos^2\theta\times\sin^2\theta)\times\left[\left(\frac{\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)\right]$$

$$=\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta$$

$$=(\cos^2\theta)^3+(\sin^2\theta)^3+3\sin^2\theta\cos^2\theta$$

$$=[(\cos^2\theta+\sin^2\theta)^3-3\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)]+3\sin^2\theta\cos^2\theta$$

$$=1-3\cos^2\theta\sin^2\theta(1)+3\cos^2\theta\sin^2\theta=1.$$

Progress Check

1. The number of trigonometric ratios is _____
   Ans : 6

2. $1-\cos^2\theta$ is _____
   Ans : $\sin^2\theta$

3. $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)$ is _____
   Ans : 1

4. $(\cot\theta+\operatorname{cosec}\theta)(\cot\theta-\operatorname{cosec}\theta)$ is _____
   Ans : -1

5. $\cos 60^\circ\sin 30^\circ+\cos 30^\circ\sin 60^\circ$ is _____
   Ans : 1

6. $\tan 60^\circ\cos 60^\circ+\cot 60^\circ\sin 60^\circ$ is _____
   Ans : $\dfrac{\sqrt{3}+1}{2}$

7. $(\tan 45^\circ\cot 45^\circ)+(\sec 45^\circ\operatorname{cosec} 45^\circ)$ is _____
   Ans : 4

8. (i) $\sec\theta=\operatorname{cosec}\theta$ if θ is _____
   (ii) $\cot\theta=\tan\theta$ if θ is _____
   Ans : 45°, 45°

Thinking Corner

1. When will the values of $\sin\theta$ and $\cos\theta$ be equal?
   Ans : When $\theta=45^\circ$

   $\sin 45^\circ=\cos 45^\circ=\dfrac{1}{\sqrt{2}}$

   $\therefore\ \sin\theta=\cos\theta$ for $\theta=45^\circ$

2. For what values of θ, $\sin\theta=2$?
   Ans : since $\sin\theta=\dfrac{\textit{Opposite side}}{\textit{Hypotenuse}}$,

   it takes values From -1 to 1.

   $\therefore$ For no real value, $\sin\theta$ equal to 2.

3. Among the six trigonometric quantities, as the value of angle θ increase from 0° to 90°, which of the six trigonometric quantities has undefined values?

Ans : tan 90° is undefined
sec 90° is undefined
cosec 0° is undefined
cot 0° is undefined

4. Is it possible to have eight trigonometric ratios?

Ans : No. Since trigonometric ratios are relation between two of three sides of triangles only 6 combinations are there.

5. Let $0° \le \theta \le 90°$. For what values of θ does

(i) $\sin\theta > \cos\theta$ (ii) $\cos\theta > \sin\theta$
(iii) $\sec\theta = 2\tan\theta$ (iv) $\csc\theta = 2\cot\theta$

Ans :

(i) $\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$

$\Rightarrow \sin 60° > \cos 60°$
$\sin 90° = 1$; $\cos 90° = 0$
$\Rightarrow \sin 90° > \cos 90°$
$\sin\theta > \cos\theta$ for $\theta = 60°$ and $\theta = 90°$

(ii) $\sin 0° = 0$; $\cos 0° = 1$
$\Rightarrow \cos 0° > \sin 0°$
$\sin 30° = \frac{1}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$
$\Rightarrow \cos 30° > \sin 30°$
$\therefore \cos\theta > \sin\theta$ for $\theta = 0°$ and $\theta = 30°$

(iii) Given $\sec\theta = 2\tan\theta$

$$\frac{\sec\theta}{\tan\theta} = 2$$

$$\frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} = 2$$

$$\frac{1}{\cos\theta} \times \frac{\cos\theta}{\sin\theta} = 2$$

$$\frac{1}{\sin\theta} = 2$$

$\csc\theta = 2$
$\therefore \csc\theta = 2$ for $\theta = 30°$
$\therefore \sec\theta = 2\tan\theta$ for $\theta = 30°$

Also $\sec 30° = \frac{2}{\sqrt{3}}$
$\tan 30° = \frac{1}{\sqrt{3}}$
$2\tan 30° = \frac{2}{\sqrt{3}} = \sec 30°$
$\therefore \theta = 30°$

(iv) $\csc\theta = 2\cot\theta$

$$\frac{1}{\sin\theta} = 2\frac{\cos\theta}{\sin\theta}$$

$$\frac{\sin\theta}{\cos\theta} \times \frac{1}{\sin\theta} = 2$$

$$\frac{1}{\cos\theta} = 2$$

$\sec\theta = 2$
$\sec\theta = 2$ for $\theta = 60°$
$\therefore \csc\theta = 2\cot\theta$ for $\theta = 60°$

Exercise 6.1

1. Prove the following identities.

(i) $\cot\theta + \tan\theta = \sec\theta\csc\theta$
(ii) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

Sol :

(i) $\cot\theta + \tan\theta = \sec\theta\csc\theta$

$$\begin{aligned} \text{LHS} &= \cot\theta + \tan\theta \\ &= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \\ &= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta} \\ &\qquad [\because \sin^2\theta + \cos^2\theta = 1] \\ &= \frac{1}{\sin\theta} \times \frac{1}{\cos\theta} \\ &= \csc\theta\sec\theta \\ &= \sec\theta\csc\theta \\ &= \text{RHS} \end{aligned}$$

(ii) $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$

$$\begin{aligned} \text{LHS} &= \tan^4\theta + \tan^2\theta \\ &= \tan^2\theta(\tan^2\theta + 1) \\ &= \tan^2\theta . \sec^2\theta \quad [\because 1 + \tan^2\theta = \sec^2\theta] \\ &= \sec^4\theta - \sec^2\theta \\ &\qquad [\because \tan^2\theta = \sec^2\theta - 1] \\ &= \text{RHS} \end{aligned}$$

2. Prove the following identities.

(i) $\dfrac{1-\tan^2\theta}{\cot^2\theta-1}=\tan^2\theta$ **(ii)** $\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta-\tan\theta$

Sol :

(i) $\dfrac{1-\tan^2\theta}{\cot^2\theta-1}=\tan^2\theta$

$$\text{LHS} = \frac{1-\tan^2\theta}{\cot^2\theta-1}$$

$$= \frac{1-\dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta}{\sin^2\theta}-1}$$

$$= \frac{\dfrac{(\cos^2\theta-\sin^2\theta)}{\cos^2\theta}}{\dfrac{(\cos^2\theta-\sin^2\theta)}{\sin^2\theta}}$$

$$= \frac{(\cos^2\theta-\sin^2\theta)}{\cos^2\theta}\times\frac{\sin^2\theta}{(\cos^2\theta-\sin^2\theta)}$$

$$= \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta = \text{RHS}$$

(ii) $\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta-\tan\theta$

$$\text{LHS} = \frac{\cos\theta}{1+\sin\theta}$$

$$= \frac{\cos\theta}{1+\sin\theta}\times\frac{1-\sin\theta}{1-\sin\theta}$$

[Multiplying the Numerator and Denominator by $1-\sin\theta$]

$$= \frac{\cos\theta(1-\sin\theta)}{1^2-\sin^2\theta}$$

$[\because (a+b)(a-b)=a^2-b^2]$

$$= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$$

$[\because 1-\sin^2\theta=\cos^2\theta]$

$$= \frac{(1-\sin\theta)}{\cos\theta}$$

$$= \frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta-\tan\theta$$

$$= \text{RHS}$$

3. Prove the following identities.

(i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}=\sec\theta+\tan\theta$

(ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}=2\sec\theta$

Sol :

(i) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}=\sec\theta+\tan\theta$

$$\text{LHS} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1-\sin\theta}{1-\sin\theta}}$$

[Multiplying the Numerator and denominator by $\sqrt{1-\sin\theta}$]

$$= \sqrt{\frac{1^2-\sin^2\theta}{(1-\sin\theta)^2}} \quad [\because (a+b)(a-b)=a^2-b^2]$$

$$= \sqrt{\frac{\cos^2\theta}{(1-\sin\theta)^2}} \quad [\because 1-\sin^2\theta=\cos^2\theta]$$

$$= \frac{\cos\theta}{1-\sin\theta}$$

$$= \frac{\cos\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}$$

[Multiplying Numerator and denominator by $1+\sin\theta$]

$$= \frac{\cos\theta(1+\sin\theta)}{1^2-\sin^2\theta} = \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$$

$[\because (a+b)(a-b)=a^2-b^2]$ $[1-\sin^2\theta=\cos^2\theta]$

$$= \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta+\tan\theta = \text{RHS}$$

(ii) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}=2\sec\theta$

$$\text{LHS} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1^2-\sin^2\theta}} + \sqrt{\frac{(1-\sin\theta)^2}{1^2-\sin^2\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}} + \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1+\sin\theta}{\cos\theta} + \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1+\sin\theta+1-\sin\theta}{\cos\theta}$$

$$= 2 \times \frac{1}{\cos\theta}$$

$$= 2\sec\theta$$

$$= \text{RHS}$$

4. Prove the following identities.

(i) $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1$

(ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2 = 1 + (\sec\theta + \operatorname{cosec}\theta)^2$.

Sol :

(i) $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1$

LHS $= \sec^6\theta$

$= (\sec^2\theta)^3$

$= (1+\tan^2\theta)^3$ $\quad [\because 1+\tan^2\theta = \sec^2\theta)$

$= 1^3 + (\tan^2\theta)^3 + 3(1)(\tan^2\theta)(1+\tan^2\theta)$

$[\because (a+b)^3 = a^3 + b^3 + 3ab(a+b)]$

$= 1 + \tan^6\theta + 3\tan^2\theta(\sec^2\theta)$

$[\because 1+\tan^2\theta = \sec^2\theta]$

$= \tan^6\theta + 3\tan^2\theta\sec^2\theta + 1 = \text{RHS}$

(ii) $(\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$= 1 + (\sec\theta + \operatorname{cosec}\theta)^2$

LHS $= (\sin\theta + \sec\theta)^2 + (\cos\theta + \operatorname{cosec}\theta)^2$

$= \sin^2\theta + \sec^2\theta + 2\sin\theta\sec\theta + \cos^2\theta$

$+ \operatorname{cosec}^2\theta + 2\cos\theta\operatorname{cosec}\theta$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$= (\sin^2\theta + \cos^2\theta) + (\sec^2\theta + \operatorname{cosec}^2\theta)$

$+ 2(\sin\theta\sec\theta + \cos\theta\operatorname{cosec}\theta)$

$$= 1 + \sec^2\theta + \operatorname{cosec}^2\theta + 2\left(\sin\theta\frac{1}{\cos\theta} + \cos\theta\frac{1}{\sin\theta}\right)$$

$$= 1 + \sec^2\theta + \operatorname{cosec}^2\theta + 2\left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)$$

$$= 1 + \sec^2\theta + \operatorname{cosec}^2\theta + 2\left(\frac{1}{\sin\theta\cos\theta}\right)$$

$$= 1 + \sec^2\theta + \operatorname{cosec}^2\theta + 2(\operatorname{cosec}\theta\sec\theta)$$

$$= 1 + [\sec^2\theta + \operatorname{cosec}^2\theta + 2\sec\theta\operatorname{cosec}\theta]$$

$$= 1 + (\sec\theta + \operatorname{cosec}\theta)^2 = \text{RHS}$$

5. Prove the following identities.

(i) $\sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta = 1$

(ii) $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1}$

Sol :

(i) $\sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta = 1$

LHS $= \sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta$

$= \sec^4\theta(1^2 - (\sin^2\theta)^2) - 2\tan^2\theta$

$= \sec^4\theta(1+\sin^2\theta)(1-\sin^2\theta) - 2\tan^2\theta$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$= \sec^4\theta(1+\sin^2\theta)\cos^2\theta - 2\tan^2\theta$

$[\because 1-\sin^2\theta = \cos^2\theta]$

$$= \frac{1}{\cos^4\theta}(1+\sin^2\theta)\cos^2\theta - 2\tan^2\theta$$

$$= \frac{1}{\cos^2\theta}(1+\sin^2\theta) - 2\frac{\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1+\sin^2\theta}{\cos^2\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1^2 + \sin^2\theta - 2\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1-\sin^2\theta}{\cos^2\theta}$$

$$= \frac{(\cos^2\theta)}{\cos^2\theta} = 1 = \text{RHS}$$

(ii) $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1}$

$$\text{LHS} = \frac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta}$$

$$= \frac{\dfrac{\cos\theta}{\sin\theta} - \cos\theta}{\dfrac{\cos\theta}{\sin\theta} + \cos\theta}$$

$$= \frac{\frac{\cos\theta - \sin\theta\cos\theta}{\sin\theta}}{\frac{\cos\theta + \sin\theta\cos\theta}{\sin\theta}}$$

$$= \frac{(\cos\theta - \sin\theta\cos\theta)}{\sin\theta} \times \frac{\sin\theta}{(\cos\theta + \sin\theta\cos\theta)}$$

$$= \frac{\cos\theta\,(1 - \sin\theta)}{\cos\theta\,(1 + \sin\theta)}$$

$$\text{LHS} = \frac{1 - \sin\theta}{1 + \sin\theta} \qquad ...(1)$$

$$\text{RHS} = \frac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1}$$

$$= \frac{\frac{1}{\sin\theta} - 1}{\frac{1}{\sin\theta} + 1} = \frac{\frac{1 - \sin\theta}{\sin\theta}}{\frac{1 + \sin\theta}{\sin\theta}} = \frac{1 - \sin\theta}{\sin\theta} \times \frac{\sin\theta}{1 + \sin\theta}$$

$$\text{RHS} = \frac{1 - \sin\theta}{1 + \sin\theta} \qquad ..(2)$$

From (1) and (2)

$$\text{LHS} = \text{RHS}$$

6. Prove the following identities.

(i) $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$

(ii) $\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2.$

Sol :

(i) $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$

$$\text{LHS} = \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$= \frac{(\sin A - \sin B)(\sin A + \sin B) + (\cos A - \cos B)(\cos A + \cos B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{(\sin^2 A - \sin^2 B) + (\cos^2 A - \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{1 - 1}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{0}{(\cos A + \cos B)(\sin A + \sin B)} = 0 = \text{RHS}$$

(ii) $\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$

$$\text{LHS} = \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A}$$

$$= \frac{(\sin A + \cos A)^3 - 3\sin A\cos A\,(\sin A + \cos A)}{\sin A + \cos A}$$
$$+ \frac{(\sin A - \cos A)^3 + 3\sin A\cos A\,(\sin A - \cos A)}{\sin A - \cos A}$$

$$= \frac{(\sin A + \cos A)[(\sin A + \cos A)^2 - 3\sin A\cos A]}{(\sin A + \cos A)}$$
$$+ \frac{(\sin A - \cos A)[(\sin A - \cos A)^2 + 3\sin A\cos A]}{\sin A - \cos A}$$

$$= \sin^2 A + \cos^2 A + 2\sin A\cos A - 3\sin A\cos A$$
$$+ \sin^2 A + \cos^2 A - 2\sin A\cos A + 3\sin A\cos A$$
$$= 2\sin^2 A + 2\cos^2 A$$
$$= 2(\sin^2 A + \cos^2 A) = 2 = \text{RHS}$$

7. (i) If $\sin\theta + \cos\theta = \sqrt{3}$, then prove that $\tan\theta + \cot\theta = 1$.

(ii) If $\sqrt{3}\sin\theta - \cos\theta = 0$, then show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Sol :

(i) We have $\sin\theta + \cos\theta = \sqrt{3}$

Squaring on both the sides,

$$(\sin\theta + \cos\theta)^2 = \left(\sqrt{3}\right)^2$$

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 3$$

$$1 + 2\sin\theta\cos\theta = 3$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$2\sin\theta\cos\theta = 3 - 1$$

$$2\sin\theta\cos\theta = 2$$

$$\sin\theta\cos\theta = 2/2$$

$$\sin\theta\cos\theta = 1 \qquad ...(1)$$

Now to prove $\tan\theta + \cot\theta = 1$

$$\text{LHS} = \tan\theta + \cot\theta$$
$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$
$$= \frac{1}{1} \quad [\because \textit{From } (1) \sin\theta\cos\theta = 1]$$
$$= 1$$
$$\therefore \tan\theta + \cot\theta = 1$$

(ii) Given $\sqrt{3}\sin\theta - \cos\theta = 0$

$$\sqrt{3}\sin\theta = \cos\theta$$
$$\frac{\sqrt{3}\sin\theta}{\cos\theta} = 1$$
$$\frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}}$$
$$\tan\theta = \frac{1}{\sqrt{3}}$$
$$\therefore \theta = 30^\circ$$
$$\text{LHS} = \tan 3\theta$$
$$= \tan 3(30^\circ)$$
$$= \tan 90^\circ$$
$$= \text{undefined} \quad \dots(1)$$
$$\text{RHS} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$
$$= \frac{3\tan 30^\circ - \tan^3 30^\circ}{1 - 3\tan^2 30^\circ}$$
$$= \frac{3\frac{1}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}\right)^3}{1 - 3\left(\frac{1}{\sqrt{3}}\right)^2}$$
$$= \frac{\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{1 - 3 \times \frac{1}{3}}$$
$$= \frac{\frac{3 \times 3 - 1}{3\sqrt{3}}}{1 - 1}$$
$$= \frac{9 - 1}{3\sqrt{3}(0)} = \frac{8}{0} = \text{undefined} \quad \dots(2)$$

From (1) and (2), LHS = RHS.

8. (i) If $\frac{\cos\alpha}{\cos\beta} = m$ and $\frac{\cos\alpha}{\sin\beta} = n$, prove that $(m^2 + n^2)\cos^2\beta = n^2$.

(ii) If $\cot\theta + \tan\theta = x$ and $\sec\theta - \cos\theta = y$, then prove that $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$.

Sol :

Given $\frac{\cos\alpha}{\cos\beta} = m$

$$\frac{\cos\alpha}{\sin\beta} = n$$
$$\text{LHS} = (m^2 + n^2)\cos^2\beta$$
$$= \left(\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right)\cos^2\beta$$
$$= \frac{(\cos^2\alpha\sin^2\beta + \cos^2\alpha\cos^2\beta)}{\cos^2\beta\sin^2\beta}\cos^2\beta$$
$$= \frac{\cos^2\alpha(\sin^2\beta + \cos^2\beta)}{\sin^2\beta}$$
$$= \frac{\cos^2\alpha}{\sin^2\beta}(1) \quad [\because \sin^2\theta + \cos^2\theta = 1]$$
$$= \left(\frac{\cos\alpha}{\sin\beta}\right)^2$$
$$= n^2 = \text{RHS}$$

(ii) We have $\cot\theta + \tan\theta = x$

$$\sec\theta - \cos\theta = y$$

Taking $\cot\theta + \tan\theta = x$

$$\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} = x$$
$$\frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta} = x$$
$$\frac{1}{\sin\theta\cos\theta} = x \quad \dots(1)$$

Also $\sec\theta - \cos\theta = y$

$$\frac{1}{\cos\theta} - \cos\theta = y$$
$$\Rightarrow \frac{1 - \cos^2\theta}{\cos\theta} = y$$
$$\Rightarrow \frac{\sin^2\theta}{\cos\theta} = y \quad \dots(2)$$

Now LHS $= (x^2y)^{2/3} - (xy^2)^{2/3}$

$$= \left[\left(\frac{1}{\cos\theta\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)\right]^{\frac{2}{3}} - \left[\left(\frac{1}{\cos\theta\sin\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right)\right]^{\frac{2}{3}}$$

[$\because$ using (1) *and* (2)]

$$= \left[\frac{1}{\cos^2\theta\sin^2\theta}\times\frac{\sin^2\theta}{\cos\theta}\right]^{\frac{2}{3}} - \left[\frac{1}{\cos\theta\sin\theta}\times\frac{\sin^4\theta}{\cos^2\theta}\right]^{\frac{2}{3}}$$

$$= \left(\frac{1}{\cos^3\theta}\right)^{\frac{2}{3}} - \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{\frac{2}{3}}$$

$$= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta - \tan^2\theta$$

$= 1 =$ RHS [$\because \sec^2\theta - \tan^2\theta = 1$]

9. (i) If $\sin\theta + \cos\theta = p$ *and* $\sec\theta + \operatorname{cosec}\theta = q$, **then prove that q ($p^2 - 1$) = 2p.**

(ii) If $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$, **then prove that** $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

Sol :

(i) $\sin\theta + \cos\theta = p$ and $\sec\theta + \operatorname{cosec}\theta = q$

LHS $= q(p^2 - 1)$

$= (\sec\theta + \operatorname{cosec}\theta)[(\sin\theta + \cos\theta)^2 - 1]$

$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)(\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1)$

$= \left(\frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta}\right)(1 + 2\sin\theta\cos\theta - 1)$

$= \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} \times 2\sin\theta\cos\theta$

$= 2(\sin\theta + \cos\theta) = 2p =$ RHS

(ii) Given $\sin\theta(1 + \sin^2\theta) = \cos^2\theta$

Squaring on both the sides.

$\sin^2\theta(1 + \sin^2\theta)^2 = \cos^4\theta$

$\Rightarrow (1 - \cos^2\theta)\{1 + (1 - \cos^2\theta)\}^2 = \cos^4\theta$

$\left[\begin{array}{l}\because \sin^2\theta = 1 - \cos^2\theta \\ \cos^2\theta = 1 - \sin^2\theta\end{array}\right]$

$\Rightarrow (1 - \cos^2\theta)\{1 + (1 - \cos^2\theta)^2 + 2(1)(1 - \cos^2\theta)\}$

$= \cos^4\theta$

$\Rightarrow (1 - \cos^2\theta)(1 + 1 + \cos^4\theta - 2(1)\cos^2\theta$

$+ 2 - 2\cos^2\theta) = \cos^4\theta$

$\Rightarrow (1 - \cos^2\theta)(4 - 4\cos^2\theta + \cos^4\theta) = \cos^4\theta$

$\Rightarrow 4 - 4\cos^2\theta + \cos^4\theta - 4\cos^2\theta + 4\cos^4\theta - \cos^6\theta$

$= \cos^4\theta$

$\Rightarrow 4 - 8\cos^2\theta + 5\cos^4\theta - \cos^6\theta = \cos^4\theta$

$4 - 8\cos^2\theta + 4\cos^4\theta - \cos^6\theta = 0$

$4 = 8\cos^2\theta - 4\cos^4\theta + \cos^6\theta$

$\Rightarrow \cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$

10. If $\frac{\cos\theta}{1 + \sin\theta} = \frac{1}{a}$, **then prove that** $\frac{a^2 - 1}{a^2 + 1} = \sin\theta$.

Sol : Given $\frac{\cos\theta}{1 + \sin\theta} = \frac{1}{a}$

$\therefore a = \frac{1 + \sin\theta}{\cos\theta}$

Now LHS $= \frac{a^2 - 1}{a^2 + 1}$

$$= \frac{\left(\frac{1 + \sin\theta}{\cos\theta}\right)^2 - 1}{\left(\frac{1 + \sin\theta}{\cos\theta}\right)^2 + 1}$$

$$= \frac{\frac{1^2 + \sin^2\theta + 2\sin\theta}{\cos^2\theta} - 1}{\frac{1^2 + \sin^2\theta + 2\sin\theta}{\cos^2\theta} + 1}$$

$$= \frac{\frac{1 + \sin^2\theta + 2\sin\theta - \cos^2\theta}{\cos^2\theta}}{\frac{1 + \sin^2\theta + 2\sin\theta + \cos^2\theta}{\cos^2\theta}}$$

$$= \frac{(1 - \cos^2\theta) + \sin^2\theta + 2\sin\theta}{\cos^2\theta} \times \frac{\cos^2\theta}{1 + (\sin^2\theta + \cos^2\theta) + 2\sin\theta}$$

$$= \frac{\sin^2\theta + \sin^2\theta + 2\sin\theta}{1 + 1 + 2\sin\theta}$$

$\left[\begin{array}{l}\because \sin^2\theta + \cos^2\theta = 1 \\ 1 - \cos^2\theta = \sin^2\theta\end{array}\right]$

$$= \frac{2\sin^2\theta + 2\sin\theta}{2 + 2\sin\theta}$$

$$= \frac{2\sin\theta(\sin\theta + 1)}{2(1 + \sin\theta)}$$

$= \sin\theta =$ RHS

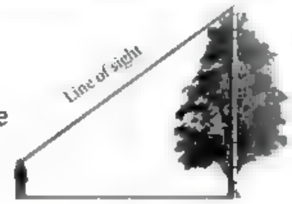
HEIGHT AND DISTANCES

Key Points

Trigonometry is used for finding the heights and distances of various objects without actually measuring them.

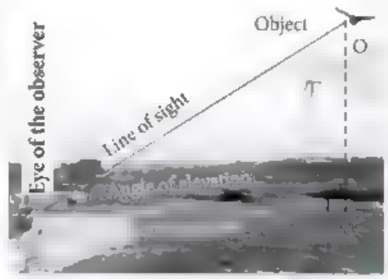
(i) Line of Sight

The line of sight is the line drawn from the eye of an observer to the point in the object viewed by the observer.



(ii) Theodolite

1. Theodolite is an instrument which is used in measuring the angle between an object and the eye of the observer.
2. It has two wheels placed right angles to each other used for measuring horizontal and vertical angles.

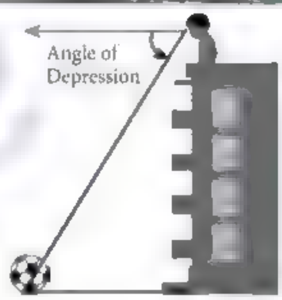


(iii) Angle of Elevation

The angle of elevation is an angle formed by the line of sight with the horizontal when the point being viewed is above the horizontal level.

(iv) Angle of Depression

The angle of depression is an angle formed by the line of sight with the horizontal when the point is below the horizontal level.

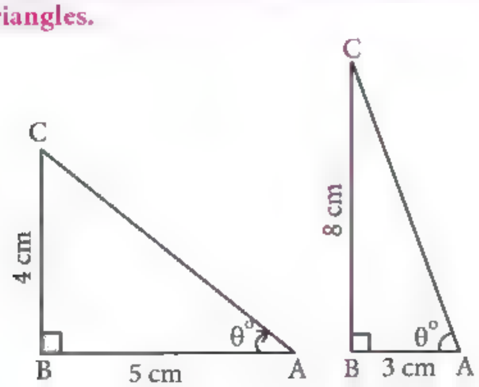


(v) Clinometer

1. The angle of elevation and angle of depression are usually measured by a device called inclinometer or clinometer.
2. From a given point when height of an object increases, the angle of elevation increases.
3. The angle of elevation increases as we move towards the foot of the vertical like building.

Worked Examples

6.18 **Calculate the size of $\angle BAC$ in the given triangles.**



(a) (b)

Sol :

(i) In right triangle ABC [see fig (a)]

$$\tan\theta = \frac{\textit{opposite side}}{\textit{adjacent side}} = \frac{4}{5}$$

$$\theta = \tan^{-1}\left(\frac{4}{5}\right) = \tan^{-1}(0.8)$$

$$\theta = 38.7^\circ \text{ (since } \tan 38.7^\circ = 0.8011)$$

$$\angle BAC = 38.7^\circ$$

(ii) In right triangle ABC [see fig (b)]

$$\tan\theta = \frac{8}{3}$$

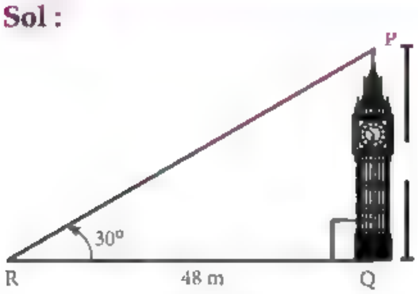
$$\theta = \tan^{-1}\left(\frac{8}{3}\right) = \tan^{-1}(2.66)$$

$$\theta = 69.4^\circ \text{ (since } \tan 69.4^\circ = 2.6604)$$

$$\angle BAC = 69.4^\circ$$

6.19 **A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower.**

Sol :



Let PQ be the height of the tower.

Take PQ = h and QR is the distance between the tower and the point R. In right triangle PQR,

$$\angle PRQ = 30°$$

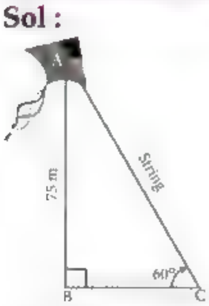
$$\tan\theta = \frac{PQ}{QR}$$

$$\tan 30° = \frac{h}{48} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{48} \Rightarrow h = 16\sqrt{3}$$

Therefore the height of the tower is $16\sqrt{3}$ m.

6.20 **A kite is flying at a height of 75 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.**

Sol :



Let AB be the height of the kite above the ground. Then, AB = 75.

Let AC be the length of the string.

In right triangle ABC, $\angle ACB = 60°$

$$\sin\theta = \frac{AB}{AC} \Rightarrow \sin 60° = \frac{75}{AC}$$

$$\frac{\sqrt{3}}{2} = \frac{75}{AC} \Rightarrow AC = \frac{150}{\sqrt{3}} = 50\sqrt{3}$$

Hence, the length of the string is $50\sqrt{3}$ m.

6.21 **Two ships are sailing in the sea on either sides of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are 30° and 45° respectively. If the lighthouse is 200 m high, find the distance between the two ships. $(\sqrt{3} = 1.732)$**

Sol :

Let AB be the lighthouse and C and D the positions of the two ships.

Then, $AB = 200$ m,

$$\angle ACB = 30°, \ \angle ADB = 45°$$

In right triangle BAC, $\tan 30° = \dfrac{AB}{AC}$

$$\frac{1}{\sqrt{3}} = \frac{200}{AC} \Rightarrow AC = 200\sqrt{3} \qquad ...(1)$$

In right triangle BAD, $\tan 45° = \dfrac{AB}{AD}$

$$1 = \frac{200}{AD} \Rightarrow AD = 200 \qquad ...(2)$$

Now, $CD = AC + AD = 200\sqrt{3} + 200$ [by (1) and (2)]

$$CD = 200\left(\sqrt{3}+1\right) = 200 \times 2.732 = 546.4$$

Distance between two ships is 546.4 m.

6.22 **From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 30 m high building are 45° and 60° respectively. Find the height of the tower $\left(\sqrt{3} = 1.732\right)$**

Sol :

Let AC be the height of the tower.

Let AB be the height of the building
Then, AC = h metres, AB = 30 m.
In right triangle CBP, $\angle CPB = 60^\circ$

$$\tan\theta = \frac{BC}{BP}$$

$$\tan 60^\circ = \frac{AB+AC}{BP} \Rightarrow \sqrt{3} = \frac{30+h}{BP} \quad \text{... (1)}$$

In right triangle ABP, $\angle APB = 45^\circ$

$$\tan\theta = \frac{AB}{BP}$$

$$\tan 45^\circ = \frac{30}{BP} \Rightarrow BP = 30 \quad \text{... (2)}$$

Substituting (2) in (1), we get $\sqrt{3} = \dfrac{30+h}{30}$

$$h = 30\left(\sqrt{3}-1\right)$$
$$= 30\,(1.732-1) = 30(0.732) = 21.96$$

Hence, the height of the tower is 21.96 m.

6.23 **A TV tower stands vertically on a bank of a canal. The tower is watched from a point on the other bank directly opposite to it. The angle of elevation of the top of the tower is 58°. From another point 20 m away from this point on the line joining this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal. ($\tan 58^\circ = 1.6003$)**

Sol :

Let AB be the height of the TV tower.
CD = 20 m
Let BC be the width of the canal.

In right triangle ABC, $\tan 58^\circ = \dfrac{AB}{BC}$

$$1.6003 = \frac{AB}{BC} \quad \text{...(1)}$$

In right triangle ABD, $\tan 30^\circ = \dfrac{AB}{BD} = \dfrac{AB}{BC+CD}$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC+20} \quad \text{...(2)}$$

Dividing (1) by (2) we get, $\dfrac{1.6003}{\frac{1}{\sqrt{3}}} = \dfrac{BC+20}{BC}$

$$BC = \frac{20}{1.7717} = 11.24 \text{ m} \quad \text{...(3)}$$

$$1.6003 = \frac{AB}{11.29} \text{ [from (1) and (3)]}$$

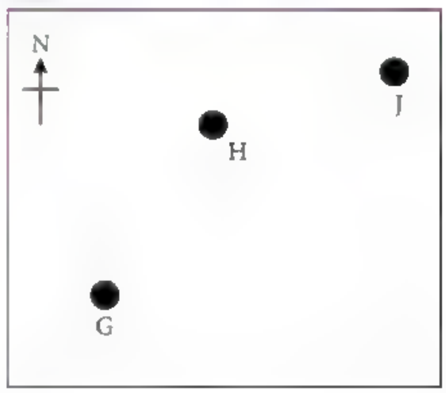
$$AB = 17.99$$

Hence, the height of the tower is 17.99 m and the width of the canal is 11.24 m.

6.24 **An aeroplane sets off from G on a bearing of 24° towards H, a point 250 km away. At H it changes course and heads towards J on a bearing of 55° and a distance of 180 km away.**
(i) How far is H to the North of G?
(ii) How far is H to the East of G?
(iii) How far is J to the North of H?
(iv) How far is J to the East of H?

$$\begin{pmatrix} \sin 24^\circ = 0.4067, & \sin 11^\circ = 0.1908 \\ \cos 24^\circ = 0.9135, & \cos 11^\circ = 0.9816 \end{pmatrix}$$

Sol :



(i) In right triangle GOH, $\cos 24^\circ = \dfrac{OG}{GH}$

$$0.9135 = \frac{OG}{250};\ OG = 228.38 \text{ km}$$

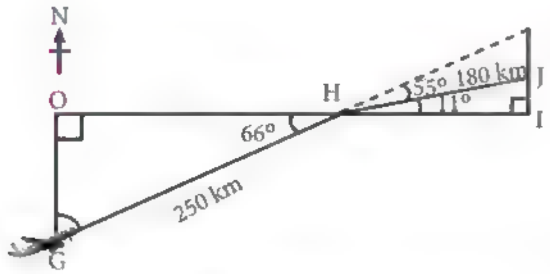
Distance of H to the North of G = 228.38 km

(ii) In right triangle GOH,

$$\sin 24^\circ = \frac{OH}{GH}$$

$$0.4067 = \frac{OH}{250};\ OH = 101.68$$

Distance of H to the East of G = 101.68 km



(iii) In right triangle HIJ,

$$\sin 11^\circ = \frac{IJ}{HJ}$$

$$0.1908 = \frac{IJ}{180};\ IJ = 34.34 \text{ km}$$

Distance of J to the North of H = 34.34 km

(iv) In right triangle HIJ,

$$\cos 11^\circ = \frac{HI}{HJ}$$

$$0.9816 = \frac{HI}{180};\ HI = 176.69 \text{ km}$$

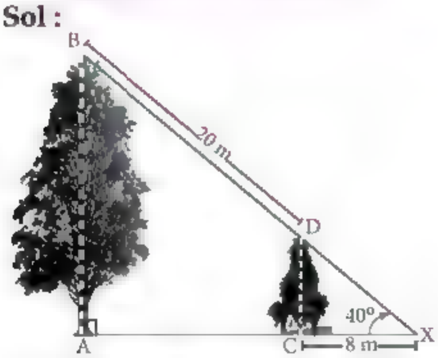
Distance of J to the East of H = 176.69 km

6.25 **Two trees are standing on flat ground. The angle of elevation of their tops from a point X on the ground is 40°. If the horizontal distance between X and the smaller tree is 8 m and the distance the tops of the two trees is 20 m, calculate**

(i) the distance between the point X and the top of the smaller tree.

(ii) the horizontal distance between the two trees. (cos 40° = 0.7660)

Sol :



Let AB be the height of the bigger tree and CD be the height of the smaller tree and X is the point on the ground.

(i) In right triangle XCD, $\cos 40^\circ = \frac{CX}{XD}$

$$XD = \frac{8}{0.7660} = 10.44 \text{ m}$$

Therefore the distance between X and top of the smaller tree = XD = 10.44 m

(ii) In right triangle XAB,

$$\cos 40^\circ = \frac{AX}{BX} = \frac{AC+CX}{BD+DX}$$

$$0.7660 = \frac{AC+8}{20+10.44}$$

$$AC = 23.32 - 8 = 15.32 \text{ m}$$

Therefore the horizontal distance between two trees = AC = 15.32 m

Thinking Corner

1. What type of triangle is used to calculate heights and distances?

Ans : Right angled triangle is used to calculate heights and distances.

2. When the height of the building and distance from the foot of the building are given, which trigonometric ratio is used to find the angle of elevation?

Ans :

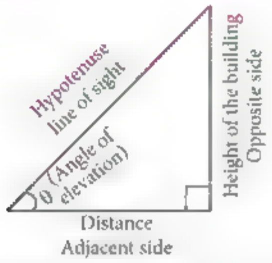
If θ is the angle of elevation then the known measures are opposite side and adjacent side.

$\therefore$ tan θ is used to find the angle of elevation.

$$\text{i.e.,}\quad \tan\theta = \frac{\textit{Opposite side}}{\textit{Adjacent side}} = \frac{\textit{Height of the building}}{\textit{Distance}}$$

3. **If the line of sight and angle of elevation is given, then which trigonometric ratio, is used.**
   (i) to find the height of the building
   (ii) to find the distance from the foot of the building.

Ans :



(i) To find the height of the building

$$\sin\theta = \frac{\textit{Opposite side}}{\textit{Hypotenuse}} \text{ is used.}$$

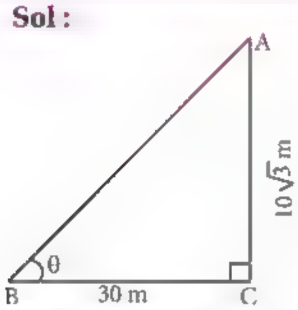
(ii) To find the distance from the foot of the building.

$$\cos\theta = \frac{\textit{Adjacent side}}{\textit{Hypotenuse}} \text{ is used.}$$

Exercise 6. 2

1. **Find the angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of a tower of height $10\sqrt{3}\,m$.**

Sol :



From the right ΔABC

$$\tan\theta = \frac{\textit{Opposite side}}{\textit{Adjacent side}} = \frac{AC}{BC}$$

$$= \frac{10\sqrt{3}\,m}{30\,m} = \frac{\sqrt{3}}{3}$$

$$= \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}}$$

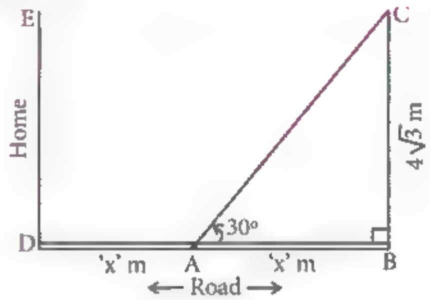
$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = 30^\circ$$

$\therefore$ Angle of elevation is 30°

2. **A road is flanked on either side by continuous rows of houses of height $4\sqrt{3}$ m with no space in between them. A pedestrian is standing on the median of the road facing a row house. The angle of elevation from the pedestrian to the top of the house is 30°. Find the width of the road.**

Sol :



Let AB = x be the distance between foot of the house and the observer at the median of the road.

$\therefore$ DB = 2x is the width of the road.

Height of the house BC = $4\sqrt{3}\,m$

From the right triangle $\Delta\, ABC$

$$\therefore \quad \tan 30^\circ = \frac{BC}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{4\sqrt{3}}{x}$$

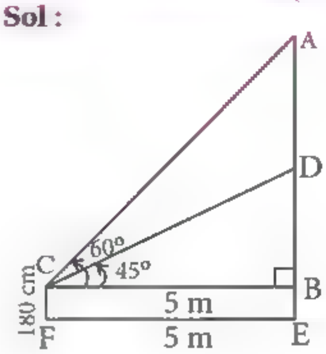
$$x = 4\sqrt{3}\times\sqrt{3} = 4\times 3 = 12 \text{ m}$$

Width of the road $= 2\times x = 2\times 12 = 24$ m

$\therefore$ Width of the road = 24 m.

3. **To a man standing outside his house, the angles of elevation of the top and bottom of a window are 60° and 45° respectively. If the height of the man is 180 cm and if he is 5 m away from the wall, what is the height of the window? $\left(\sqrt{3} = 1.732\right)$**

Sol :



Let CF be the height of the man; AD be the height of the window; BC is the distance between the observer and the house.

From the right triangle ΔCBD

$$\tan 45^\circ = \frac{DB}{BC}$$

$$1 = \frac{DB}{5}$$

$$DB = 5 \text{ m} \quad \text{... (1)}$$

From the right triangle CBA

$$\tan 60^\circ = \frac{AB}{CB}$$

$$\sqrt{3} = \frac{AD+DB}{5}$$

$$5\sqrt{3} = AD + 5 \quad [\because \textit{from (1) } DB = 5m]$$

$$AD = 5\sqrt{3} - 5 = 5\left(\sqrt{3}-1\right)$$

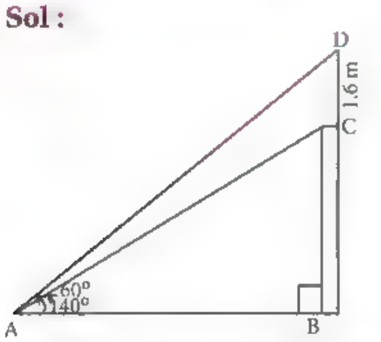
$$AD = 5\,(1.732 - 1) \quad \left[\textit{Given } \sqrt{3} = 1.732\right]$$

$$= 5 \times 0.732 = 3.660$$

$\therefore$ Height of the window = 3.66 m

4. A statue 1.6 m tall stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 40°. Find the height of the pedestal. $\left(\tan 40^\circ = 0.8391, \sqrt{3} = 1.732\right)$

Sol :



Let CD be the statue of tall 1.6 m.

BC be the pedestal.

From the right triangle $\Delta\ ABC$

$$\tan 40^\circ = \frac{BC}{AB}$$

$$0.8391 = \frac{BC}{AB}$$

$$AB = \frac{BC}{0.8391} \quad \text{... (1)}$$

From the right triangle $\Delta\ ABD$

$$\tan 60^\circ = \frac{BD}{AB}$$

$$\sqrt{3} = \frac{BC+CD}{AB}$$

$$1.732 = \frac{BC+1.6}{AB}$$

$$AB = \frac{BC+1.6}{1.732} \quad \text{... (2)}$$

From (1) and (2)

$$\frac{BC}{0.8391} = \frac{BC+1.6}{1.732}$$

$$1.732\ BC = 0.8391\ (BC + 1.6)$$

$$1.732\ BC = 0.8391\ BC + (0.8391)\ (1.6)$$

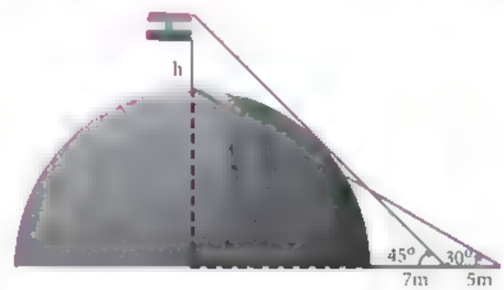
$$1.732\ BC - 0.8391\ BC = 1.34256$$

$$0.8929\ BC = 1.34256$$

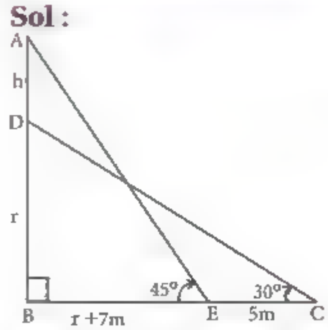
$$BC = \frac{1.34256}{0.8929} = \frac{13425.6}{8929} = 1.5 \text{ m}$$

$\therefore$ Height of the pedestal = 1.5 m

5. A flag pole 'h' metres is on the top of the hemispherical dome of radius 'r' metres. A man is standing 7 m away from the dome. Seeing the top of the pole at an angle 45° and moving 5 m away from the dome and seeing the bottom of the pole at an angle 30°. Find (i) the height of the pole (ii) radius of the dome. $\left(\sqrt{3} = 1.732\right)$



Sol :



Let BD be the radius of the dome AD is the flag

pole of height 'h' m.

(i) From the right triangle $\Delta\, ABE$

$$\tan 45^\circ = \frac{AB}{BE}$$
$$1 = \frac{r+h}{r+7}$$
$$r+7 = r+h$$
$$r+h-r = 7$$
$$h = 7 \text{ m}$$

$\therefore$ Height of the flag pole = 7 m

(ii) From the right triangle $\Delta\, BDC$

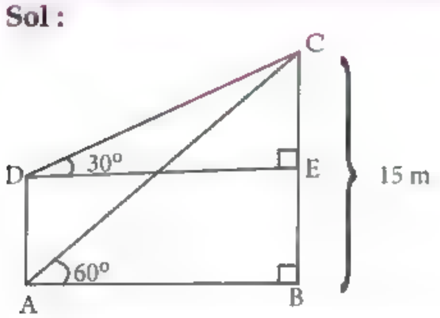
$$\tan 30^\circ = \frac{BD}{BC} = \frac{r}{r+7+5}$$
$$\frac{1}{\sqrt{3}} = \frac{r}{r+12}$$
$$r+12 = \sqrt{3}\,r$$
$$12 = \sqrt{3}\,r - r$$
$$12 = r(\sqrt{3}-1)$$
$$r = \frac{12}{1.732-1} = \frac{12}{0.732} = 16.39 \text{ m}$$

Height of pole = 7 m

$\therefore$ Radius of dome = 16.39 m.

6. The top of a 15 m high tower makes an angle of elevation of 60° with the bottom of an electronic pole and angle of elevation of 30° with the top of the pole. What is the height of the electric pole?

Sol :



Let AD be the electronic pole and BC be the tower.

From the right triangle $\Delta\, ABC$

$$\tan 60^\circ = \frac{BC}{AB}$$
$$\sqrt{3} = \frac{15}{AB}$$
$$AB = \frac{15}{\sqrt{3}} \quad \ldots (1)$$

From the right triangle $\Delta\, DEC$

$$\tan 30^\circ = \frac{CE}{DE}$$
$$\frac{1}{\sqrt{3}} = \frac{CB-EB}{AB} \quad [\because \text{ DE} = \text{AB and CE} = \text{CB} - \text{EB}]$$
$$\frac{1}{\sqrt{3}} = \frac{15-DA}{AB} \quad [\because \text{ EB} = \text{DA}]$$
$$AB = (15-AD)\sqrt{3} \quad \ldots (2)$$

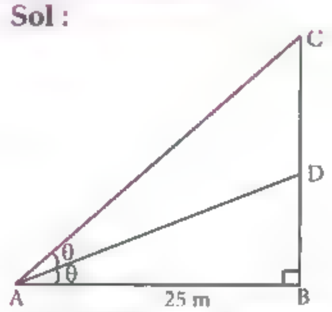
From (1) and (2)

$$\frac{15}{\sqrt{3}} = (15-AD)\sqrt{3}.$$
$$15 = (15-AD)\sqrt{3}.\sqrt{3}$$
$$= (15-AD)\,3$$
$$15 = 45-3AD$$
$$3AD = 45-15$$
$$3AD = 30$$
$$AD = \frac{30}{3} = 10 \text{ m}$$

$\therefore$ Height of the electric pole = 10 m.

7. A vertical pole fixed to the ground is divided in the ratio 1 : 9 by a mark on it with lower part shorter than the upper part. If the two parts subtend equal angles at a place on the ground, 25 m away from the base of the pole, what is the height of the pole?

Sol :



Let CB be the pole and point D divides it such that

$$BD : DC = 1:9$$

Given that $AB = 25$m

Let the two parts subtend equal angles at point A such that $CAD = \lfloor BAD = \theta$

By angle Bisector theorem, we have

$$\frac{BD}{DC} = \frac{AB}{AC}$$
$$\Rightarrow \frac{1}{9} = \frac{25}{AC}$$

$[\because BD = DC = 1:9 \text{ and } AB = 25m]$

$$AC = 9 \times 25 \text{ m} \qquad \ldots(1)$$

From the right triangle ΔABC

$$CB = \sqrt{AC^2 - AB^2} \qquad [\because \textit{Pythagorus Theorem}]$$

$$= \sqrt{(25 \times 9)^2 - 25^2} \qquad [\because \textit{From } (1)]$$

$$= \sqrt{25^2 \times 9^2 - 25^2}$$

$$= \sqrt{25^2(9^2 - 1)} = 25\sqrt{81 - 1}$$

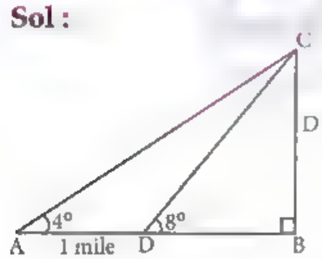
$$= 25 \times \sqrt{80} = 25 \times 4\sqrt{5}$$

$$= 100\sqrt{5}\, m$$

$\therefore$ Height of the pole $= 100\sqrt{5}\, m$.

8. A traveler approaches a mountain on highway. He measures the angle of elevation to the peak at each milestone. At two consecutive milestones the angles measured are 4° and 8°. What is the height of the peak if the distance between consecutive milestones is 1 mile (tan 4° = 0.0699, tan 8° = 0.1405)

Sol :



Let BC be the mountain

$$AD = 1 \text{ mile}$$

From the right triangle ΔABC

$$\tan 4^\circ = \frac{BC}{AB}$$

$$0.0699 = \frac{BC}{AD + DB} \qquad [\text{Given } \tan 4^\circ = 0.0699]$$

$$0.0699 = \frac{BC}{1 + DB}$$

$$0.0699\,(1 + BD) = BC \qquad \ldots(1)$$

From the right triangle ΔDBC

$$\tan 8^\circ = \frac{BC}{BD}$$

$$0.1405 = \frac{BC}{BD}$$

$$BC = 0.1405\, BD \qquad \ldots(2)$$

From (1) and (2)

$$0.0699\,(1 + BD) = 0.1405\, BD$$

$$0.0699 + 0.0699\, BD = 0.1405\, BD$$

$$0.0699 = 0.1405\, BD - 0.0699\, BD$$

$$0.0699 = 0.0706\, BD$$

$$\therefore BD = \frac{0.0699}{0.0706} = 0.99 \text{ mile.}$$

From (2)

$$BC = 0.1405 \times 0.99 = 0.1390$$

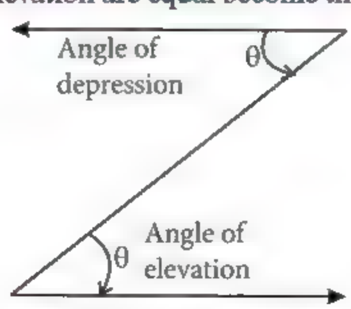
$$= 0.14 \text{ miles}$$

$\therefore$ Height of the peak = 0.14 miles approximately.

PROBLEMS INVOLVING ANGLE OF DEPRESSION

Key Points

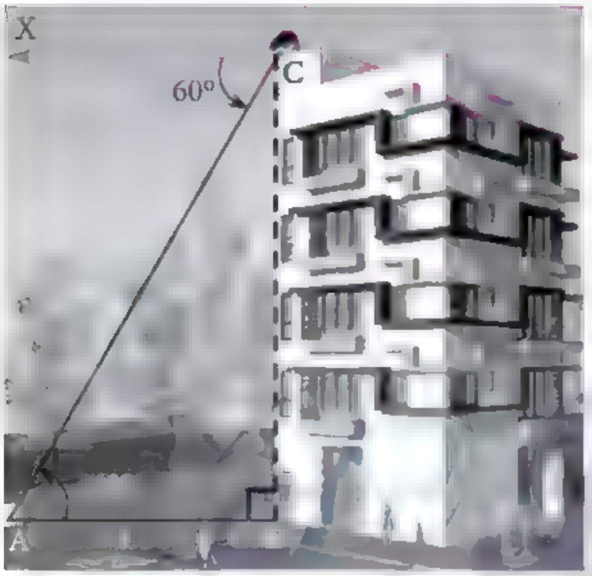
Angle of Depression and Angle of Elevation are equal become they are alternative Angles.



Worked Examples

6.26 **A player sitting on the top of a tower of height 20 m observes the angle of depression of a ball lying on the ground as 60°. Find the distance between the foot of the tower and the ball. ($\sqrt{3} = 1.732$)**

Sol :



Let BC be the height of the tower and A be the position of the ball lying on the ground. Then,
BC = 20 m and $\angle XCA = 60° = \angle CAB$

Let AB = x metres.

In right triangle ABC,

$$\tan 60° = \frac{BC}{AB}$$

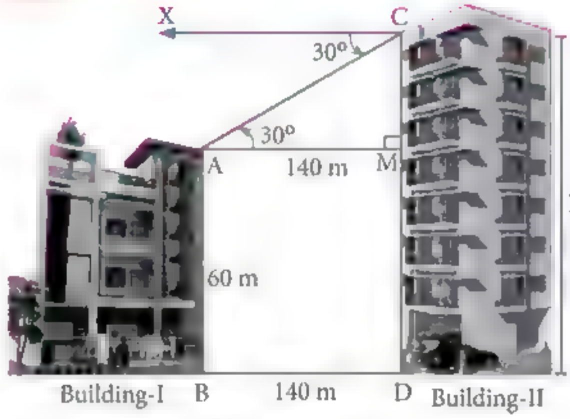
$$\sqrt{3} = \frac{20}{x}$$

$$x = \frac{20 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{20 \times 1.732}{3} = 11.54 \text{ m}$$

Hence, the distance between the foot of the tower and the ball is 11.54 m.

6.27 **The horizontal distance between two buildings is 140 m. The angle of depression of the top of the first building when seen from the top of the second building is 30°. If the height of the first building is 60 m, find the height of the second building. ($\sqrt{3} = 1.732$)**

Sol :



The height of the first building AB = 60 m. Now, AB = MD = 60 m.

Let the height of the second building CD = h.

Distance BD = 140 m

Now, AM = BD = 140 m

From the diagram,

$$\angle XCA = 30° = \angle CAM$$

In right triangle AMC, $\tan 30° = \dfrac{CM}{AM}$

$$\frac{1}{\sqrt{3}} = \frac{CM}{140}$$

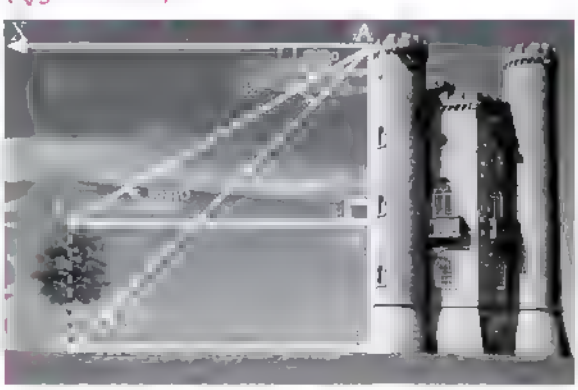
$$CM = \frac{140}{\sqrt{3}} = \frac{140\sqrt{3}}{3} = \frac{140 \times 1.732}{3}$$

$$CM = 80.78$$

Now, h = CD = CM + MD = 80.78 + 60 = 140.78

Therefore the height of the second building is 140.78 m

6.28 **From the top of a tower 50 m high, the angles of depression of the top and bottom of a tree are observed to be 30° and 45° respectively. Find the height of the tree. ($\sqrt{3} = 1.732$)**



Sol :

The height of the tower AB = 50 m

Let the height of the tree CD = y and BD = x

From the diagram, $\angle XAC = 30° = \angle ACM$ and $\angle XAD = 45° = \angle ADB$

In right triangle ABD,

$$\tan 45° = \frac{AB}{BD} \Rightarrow 1 = \frac{50}{x} \Rightarrow x = 50 \text{ m}$$

In right triangle AMC,

$$\tan 30° = \frac{AM}{CM}$$

$$\frac{1}{\sqrt{3}} = \frac{AM}{50} \qquad [\text{since DB} = \text{CM}]$$

$$\text{AM} = \frac{50}{\sqrt{3}} = \frac{50\sqrt{3}}{3} = \frac{50 \times 1.732}{3} = 28.85 \text{ m}$$

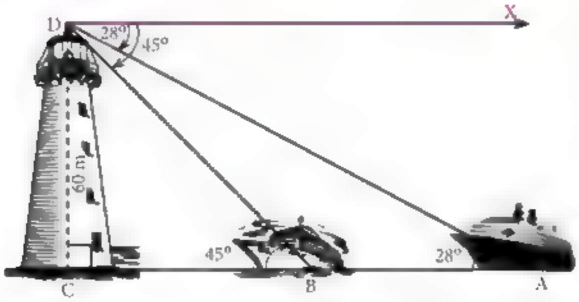
Therefore,

Height of the tree = CD = MB = AB − AM

= 50 − 28.85 = 21.15 m

6.29 As observed from the top of a 60 m high light house from the sea level, the angles of depression of two ships are 28° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships. (tan 28° = 0.5317)

Sol :



Let the observer on the lighthouse CD be at D.

Height of the lighthouse CD = 60 m

From the diagram,

$$\angle XDA = 28° = \angle DAC \text{ and}$$
$$\angle XDB = 45° = \angle DBC$$

In right triangle DCB,

$$\tan 45° = \frac{DC}{BC}$$

$$1 = \frac{60}{BC} \Rightarrow \text{BC} = 60 \text{ m}$$

In right triangle DCA, $\tan 28° = \frac{DC}{AC}$

$$0.5317 = \frac{60}{AC}$$

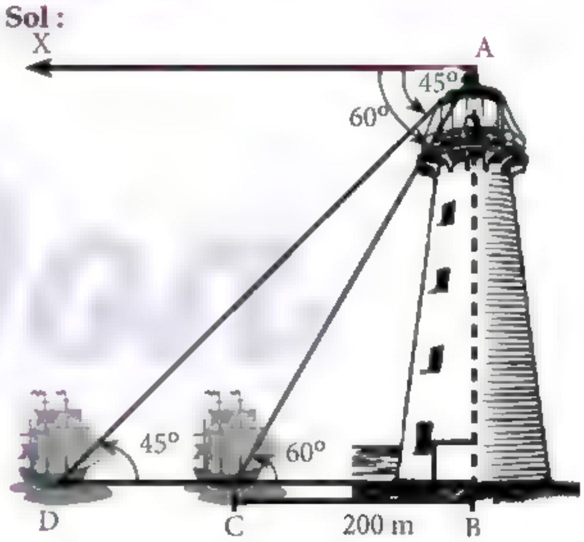
$$\text{AC} = \frac{60}{0.5317} = 112.85$$

Distance between the two ships

AB = AC − BC = 52.85 m

6.30 A man is watching a boat speeding away from the top of a tower. The boat makes an angle of depression of 60° with the man's eye when at a distance of 200 m from the tower. After 10 seconds, the angle of depression becomes 45°. What is the approximate speed of the boat (in km/hr), assuming that it is sailing in still water? ($\sqrt{3}$ = 1.732)

Sol :



Let AB be the tower.

Let C and D be the positions of the boat.

From the diagram,

$$\angle XAC = 60° = \angle ACB \text{ and}$$
$$\angle XAD = 45° = \angle ADB, \text{ BC} = 200 \text{ m}$$

In right triangle ABC, $\tan 60° = \frac{AB}{BC}$

$$\sqrt{3} = \frac{AB}{200}$$

$$\Rightarrow \quad \text{AB} = 200\sqrt{3} \qquad \ldots (1)$$

In right triangle ABD, $\tan 45° = \frac{AB}{BD}$

$$\Rightarrow \quad 1 = \frac{200\sqrt{3}}{BD} \qquad [\text{by (1)}]$$

$$\Rightarrow \quad \text{BD} = 200\sqrt{3}$$

Now, CD = BD − BC

$CD = 200\sqrt{3} - 200 = 200(\sqrt{3}-1) = 146.4$

It is given that the distance CD is covered in 10 seconds.

That is, the distance of 146.4 m is covered in 10 seconds.

Therefore, speed of the boat $= \dfrac{distance}{time}$

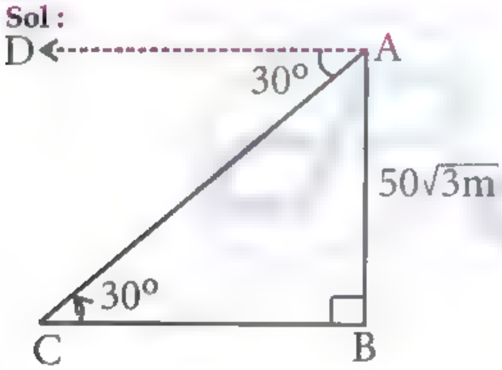
$$= \frac{146.4}{10} = 14.64 \text{ m/s} \Rightarrow 14.64 \times \frac{3600}{1000} \text{ km/hr}$$

$$= 52.704 \text{ km/hr}$$

Exercise 6. 3

1. From the top of a rock $50\sqrt{3}$ *m* high, the angle of depression of a car on the ground is observed to be 30°. Find the distance of the car from the rock.

Sol :



Let AB be the rock.

C be the position of the car.

$$\angle DAC = \angle ACB = 30°$$

In right triangle $\Delta\ ABC$

$$\tan 30° = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{50\sqrt{3}}{BC}$$

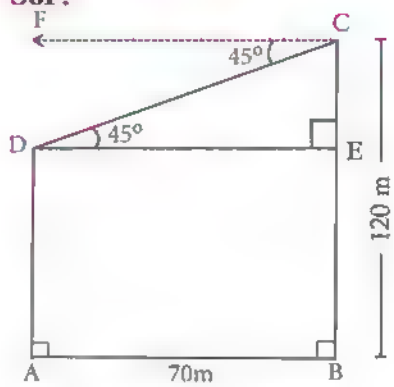
$$BC = 50\sqrt{3} \times \sqrt{3}$$

$$= 50 \times 3 = 150 \text{ m}$$

$\therefore$ Distance of the car from the rock = 150 m.

2. The horizontal distance between two buildings is 70 m. The angle of depression of the top of the first building when seen from the top of the second building is 45°. If the height of the second building is 120 m, find the height of the first building.

Sol :



Let AD is the first building.

BC is the second building.

$$AD = BE = BC - CE$$

$$\angle FCD = \angle CDE = 45°$$

From the right triangle ΔCED

$$\tan 45° = \frac{CE}{DE}$$

$$1 = \frac{CE}{AB} = \frac{CE}{70\,m}$$

$$CE = 70 \text{ m}$$

$\therefore$ $BE = BC - EC$

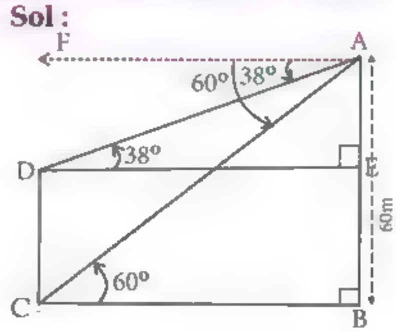
$= 120 \text{ m} - 70 \text{ m} = 50 \text{ m}$

$\therefore$ $AD = 50 \text{ m}$

$\therefore$ Height of the first building is 50 m.

3. From the top of the tower 60 m high the angles of depression of the top and bottom of a vertical lamp post are observed to be 38° and 60° respectively. Find the height of the lamp post. ($\tan 38° = 0.7813$, $\sqrt{3} = 1.732$)

Sol :



Let AB be the building of height 60 m.

DC be the lamp post.

$$DC = BE$$

$$\angle FAD = \angle ADE = 38°$$

$$\angle FAC = \angle ACB = 60°$$

In the right triangle ΔADE

$$\tan 38° = \frac{AE}{DE}$$

$$0.7813 = \frac{AE}{CB}$$

$$CB = \frac{AE}{0.7813} \quad ...(1)$$

From the right triangle ΔACB

$$\tan 60^\circ = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{60}{BC}$$

$$BC = \frac{60}{\sqrt{3}} = \frac{60\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$$

$$= \frac{60\sqrt{3}}{3}$$

$$CB = 20\sqrt{3} \quad ...(2)$$

From (1) and (2)

$$\frac{AE}{0.7813} = 20\sqrt{3}$$

$$AE = 20 \times 1.732 \times 0.7813$$

$$= 34.64 \times 0.7813 = 27.064232 = 27.06 \text{ m}$$

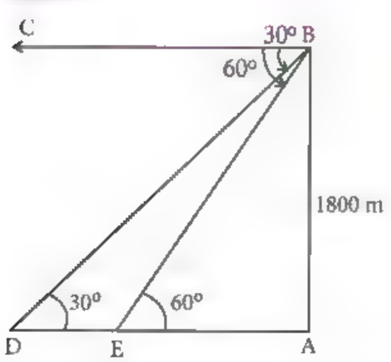
Now height of the lamp post

= DC = EB = AB – AE = 60 – 27.06 = 32.93 m

∴ Height of the lamp post = 32.93 m

4. An aeroplane at an altitude of 1800 m finds that two boats are sailing towards it in the same direction. The angles of depression of the boats as observed from the aeroplane are 60° and 30° respectively. Find the distance between the two boats. ($\sqrt{3}$ = 1.732)

Sol :



Let AB = 1800 m be the height where the aeroplane is flying D and E are positions of two boats.

$$\angle CBD = \angle BDA = 30^\circ$$

$$\angle CBE = \angle BEA = 60^\circ$$

In right triangle ΔBAE

$$\tan 60^\circ = \frac{BA}{EA}$$

$$\sqrt{3} = \frac{1800}{EA}$$

$$EA = \frac{1800}{\sqrt{3}} \quad ...(1)$$

In right triangle ΔBDA

$$\tan 30^\circ = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{1800}{DE+EA}$$

$$DE + EA = 1800\sqrt{3}$$

$$DE = 1800\sqrt{3} - EA$$

$$DE = 1800\sqrt{3} - \frac{1800}{\sqrt{3}}$$

[∵ From (1)]

$$= \frac{1800\sqrt{3}\sqrt{3}-1800}{\sqrt{3}}$$

$$= \frac{1800\times 3-1800}{\sqrt{3}}$$

$$= \frac{5400-1800}{\sqrt{3}} = \frac{3600}{\sqrt{3}}$$

$$= \frac{3600\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} = \frac{3600\sqrt{3}}{3} = 1200\sqrt{3}$$

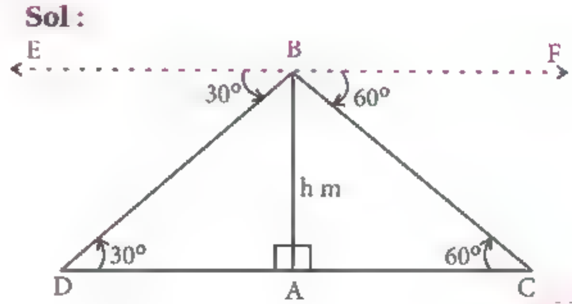
$$= 1200 \times 1.732$$

$$DE = 2078.4 \text{ m}$$

∴ Distance between the boats = 2078.4 m

5. From the top of a lighthouse, the angles of depression of two ships on the opposite sides of it are observed to be 30° and 60°. If the height of the lighthouse is h meters and the line joining the ships passes through the foot of the lighthouse, show that the distance between the ships is $\frac{4h}{\sqrt{3}}$ m.

Sol :



Let D and C be the positions of two ships AB be the light house of height 'h' m.

$$\angle EBD = \angle BDA = 30^\circ$$
$$\angle FBC = \angle BCA = 60^\circ$$

In right triangle BAC

$$\tan 60^\circ = \frac{AB}{AC}$$
$$\sqrt{3} = \frac{h}{AC}$$
$$AC = \frac{h}{\sqrt{3}} \quad \text{... (1)}$$

In right triangle ΔBAD

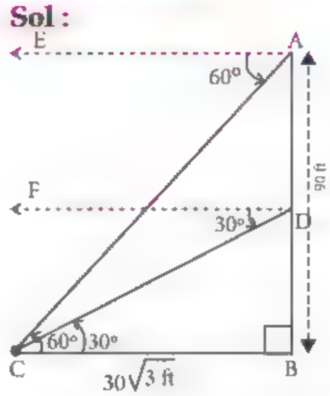
$$\tan 30^\circ = \frac{AB}{AD}$$
$$\frac{1}{\sqrt{3}} = \frac{h}{AD}$$
$$AD = h\sqrt{3} \quad \text{... (2)}$$

$$(1) + (2) \Rightarrow AC + AD = \frac{h}{\sqrt{3}} + h\sqrt{3}$$
$$DC = \frac{h + h\sqrt{3}\sqrt{3}}{\sqrt{3}}$$
$$DC = \frac{h+3h}{\sqrt{3}} = \frac{4h}{\sqrt{3}}\, m$$

$\therefore$ Distance between the ships is $\frac{4h}{\sqrt{3}}\, m$.

6. A lift in a building of height 90 feet with transparent glass walls is descending from the top of the building. At the top of the building, the angle of depression to a fountain in the garden is 60°. Two minutes later, the angle of depression reduces to 30°. If the fountain is $30\sqrt{3}$ feet from the entrance of the lift, find the speed of the lift which is descending.

Sol :



Let AB be the building of height 90 ft.
AD is the distance descending by the lift in 2 minutes.

$$\angle EAC = \angle ACB = 60^\circ$$
$$\angle FDC = \angle DCB = 30^\circ$$

In right triangle ΔDCB

$$\tan 30^\circ = \frac{DB}{CB}$$
$$\frac{1}{\sqrt{3}} = \frac{DB}{30\sqrt{3}}$$
$$\frac{30\sqrt{3}}{\sqrt{3}} = DB$$

1 feet	= 30.5 cm
	= 0.305 m

$$DB = 30 \text{ ft}$$
$$\therefore \quad AD = AB - DB = 90 - 30 = 60 \text{ ft}$$

Distance covered=60 × 0.305 m

Time taken= 2 min = 2 × 60 sec

$$\therefore \text{ Speed of the lift} = \frac{Distance}{Time}$$
$$= \frac{60 \times 0.305}{2 \times 60}\, m/s = 0.1525 \text{ m/s} = 0.15 \text{ m/s}$$

Speed of the lift = 0.15 m/s.

Another Method

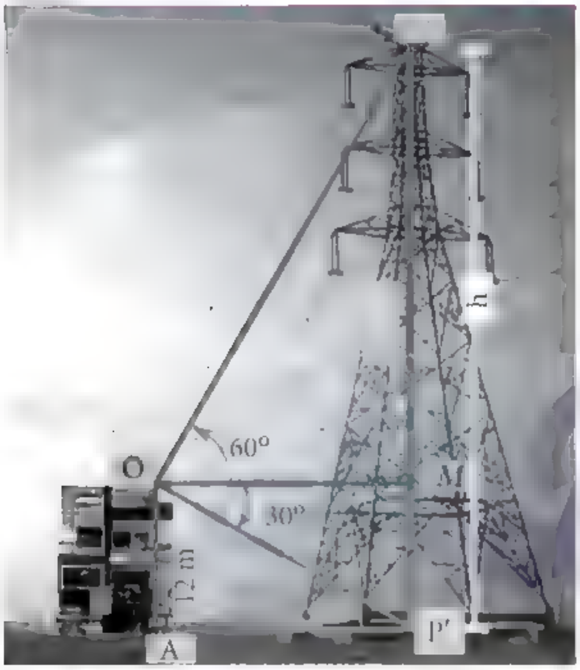
$$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{60}{2 \times 60} \text{ ft/sec}$$
$$= \frac{1}{2} \text{ ft/sec}$$
$$= 0.5 \text{ ft/sec}$$

PROBLEMS INVOLVING ANGLE OF ELEVATION AND DEPRESSION

Worked Examples

6.31 **From the top of a 12 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 30°. Determine the height of the tower.**

Sol :



As shown in Figure, OA is the building, O is the point of observation on the top of the building OA. PP' is the cable tower with P as the top and P' as the bottom.

Then the angle of elevation of P, $\angle MOP = 60°$

And the angle of depression of P', $\angle MOP' = 30°$

Suppose, height of the cable tower

PP' = h metres.

Through O, draw $OM \perp PP'$

$$MP = PP' - MP' = h - OA = h - 12$$

In right triangle OMP, $\frac{MP}{OM} = \tan 60°$

$$\Rightarrow \quad \frac{h-12}{OM} = \sqrt{3}$$

$$\Rightarrow \quad OM = \frac{h-12}{\sqrt{3}} \quad \ldots (1)$$

Similarly in right triangle OMP', $\frac{MP'}{OM} = \tan 30°$

$$\Rightarrow \quad \frac{12}{OM} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad OM = 12\sqrt{3} \quad \ldots (2)$$

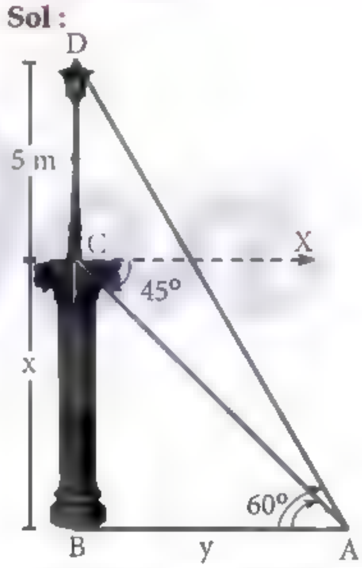
From (1) and (2) we have, $\frac{h-12}{\sqrt{3}} = 12\sqrt{3}$

$$\Rightarrow \quad h - 12 = 12\sqrt{3} \times \sqrt{3} \Rightarrow h = 48$$

Hence, the required height of the cable tower is 48 m.

6.32 **A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point 'A' on the ground is 60° and the angle of depression to the point 'A' from the top of the tower is 45°. Find the height of the tower. ($\sqrt{3}$ = 1.732)**

Sol :



Let BC be the height of the tower and CD be the height of the pole.

Let 'A' be the point of observation.

Let BC = x and AB = y.

From the diagram,

$\angle BAD = 60°$ and $\angle XCA = 45° = \angle BAC$

In right triangle ABC, $\tan 45° = \frac{BC}{AB}$

$$\Rightarrow \quad 1 = \frac{x}{y} \Rightarrow x = y \quad \ldots (1)$$

In right triangle ABD,

$$\tan 60° = \frac{BD}{AB} = \frac{BC+CD}{AB}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x+5}{y} \Rightarrow \sqrt{3}\,y = x + 5$$

$$\Rightarrow \quad \sqrt{3}\,x = x+5 \quad \text{[From (1)]}$$

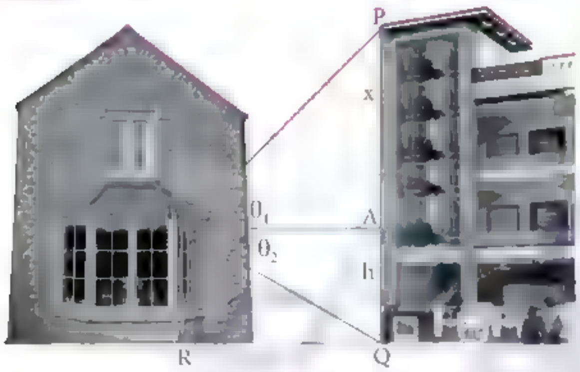
$$\Rightarrow x = \frac{5}{\sqrt{3}-1} = \frac{5}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{5(1.732+1)}{2}$$

$$= 6.83$$

Hence, height of the tower is 6.83 m.

6.33 From a window (h meters high above the ground) of a house in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are θ_1 and θ_2 respectively. Show that the height of the opposite house is $h\left(1+\frac{\cot\theta_2}{\cot\theta_1}\right)$.

Sol :



Let W be the point on the window where the angles of elevation and depression are measured. Let PQ be the house on the opposite side.

Then WA is the width of the street.

Height of the window = h metres = AQ

(WR = AQ)

Let PA = x metres.

In right triangle PAW,

$$\tan\theta_1 = \frac{AP}{AW}$$

$$\Rightarrow \quad \tan\theta_1 = \frac{x}{AW} \Rightarrow AW = \frac{x}{\tan\theta_1}$$

$$\Rightarrow \quad AW = x\cot\theta \qquad \text{... (1)}$$

In right triangle QAW,

$$\tan\theta_2 = \frac{AQ}{AW}$$

$$\Rightarrow \quad \tan\theta_2 = \frac{h}{AW}$$

$$\Rightarrow \quad AW = h\cot\theta_2 \qquad \text{... (2)}$$

From (1) and (2) we get, $x\cot\theta_1 = h\cot\theta_2$

$$\Rightarrow \quad x = h\frac{\cot\theta_2}{\cot\theta_1}$$

Therefore, height of the opposite house

$$= PA + AQ = x + h = h\frac{\cot\theta_2}{\cot\theta_1} + h$$

$$= h\left(1+\frac{\cot\theta_2}{\cot\theta_1}\right)$$

Hence Proved.

Progress Check

1. **The line drawn from the eye of an observer to the point of object is ____**

   Ans : Line of sight

2. **Which instrument is used in measuring the angle between an object and the eye of the observer?**

   Ans : Theodolite

3. **When the line of sight is above the horizontal level, the angle formed is _____**

   Ans : Angle of elevation

4. **The angle of elevation ____ as we move towards the foot of the vertical object (tower).**

   Ans : Increases

5. **When the line of sight is below the horizontal level, the angle formed is ____**

   Ans : Angle of Depression.

Thinking Corner

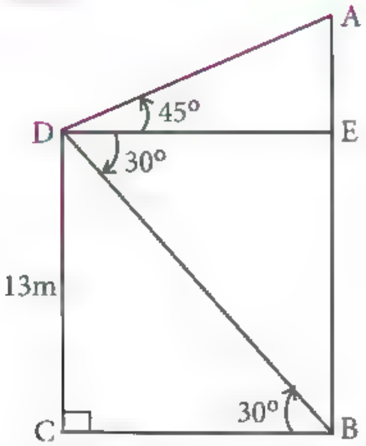
1. **What is the minimum number of measurements required to determine the height or distance or angle of elevation?**

   Ans : Any two measurements are needed in minimum to find the other.

Exercise 6. 4

1. **From the top of a tree of height 13 m the angle of elevation and depression of the top and bottom of another tree are 45° and 30° respectively. Find the height of the second tree. ($\sqrt{3}$ = 1.732)**

Sol :



Let CD is the tree of height 13 m.
AB is another tree.

$\angle ADE = 45°$

$\angle EDB = \angle DBC = 30°$

CB = DE and CD = EB = 13 m

In right triangle ΔAED

$$\tan 45° = \frac{AE}{DE}$$

$$1 = \frac{AE}{DE}$$

$$AE = DE \qquad \ldots (1)$$

In the right triangle ΔDBC

$$\tan 30° = \frac{DC}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{13}{BC}$$

$$BC = 13\sqrt{3}\, m$$

$\therefore$ From (1) $AE = 13\sqrt{3}\, m$

[$\because$ BC = DE = AE]

Height of the tree = AB = AE + EB

$= (13\sqrt{3} + 13)\, m$

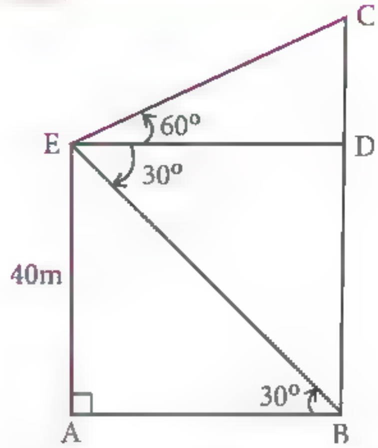
$= (13 \times 1.732 + 13)$ m

$= (22.516 + 13)$ m = 35.516 m = 35.52 m

$\therefore$ Height of the second tree = 35.52 m

2. A man is standing on the deck of a ship, which is 40 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill. $(\sqrt{3} = 1.732)$

Sol :



Let a man is standing on the deck of a ship at a point E

Such that AE = 40 m.

$\therefore$ AE = BD = 40 m

Also AB = ED

Let BC be the height of the hill.

(i) In right triangle ΔABE

$$\tan 30° = \frac{AE}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{40}{AB}$$

$$AB = 40\sqrt{3}\, m \qquad \ldots (1)$$

$$AB = 40 \times 1.732 = 69.28 \text{ m}$$

$\therefore$ Distance of the hill from the ship = 69.28 m

(ii) In the right triangle ΔCDE

$$\tan 60° = \frac{CD}{ED}$$

$$\sqrt{3} = \frac{CD}{40\sqrt{3}} \qquad [\because AB = ED = 40\sqrt{3}\, m]$$

$CD = 40\sqrt{3} \times \sqrt{3} = 40 \times 3 = 120$ m

Now height of the hill = BC = BD + DC

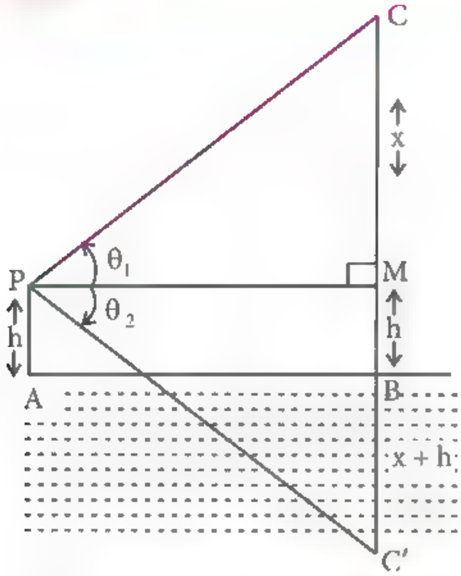
= 40 + 120 = 160 m

$\therefore$ Height of the hill = 160 m

Distance of the hill from ship = 69.28 m.

3. If the angle of elevation of a cloud from a point 'h' metres above a lake is θ_1 and the angle of depression of its reflection in the lake is θ_2. Prove that the height that the cloud is located from the ground is $\frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$.

Sol :



Let θ_1 be the angle of elevation of the cloud from P and θ_2 be the angle of depression.

PA = MB = 'h' m

BC be the height of the cloud from earth.

C' be the reflection of the cloud.

$\therefore \quad BC' = $ 'x + h' m

In right triangle ΔCPM

$$\tan\theta_1 = \frac{CM}{PM}$$

$$\tan\theta_1 = \frac{x}{AB}$$

$$AB = x\cot\theta_1 \qquad \ldots (1)$$

In right triangle $\Delta PMC'$

$$\tan\theta_2 = \frac{C'M}{PM}$$

$$= \frac{x+h+h}{AB}$$

$$\tan\theta_2 = \frac{x+2h}{AB}$$

$$AB = (x+2h)\cot\theta_2 \qquad \ldots (2)$$

From (1) and (2), we have

$$x\cot\theta_1 = (x+2h)\cot\theta_2$$

$$x\cot\theta_1 = x\cot\theta_2 + 2h\cot\theta_2$$

$$x\cot\theta_1 - x\cot\theta_2 = 2h\cot\theta_2$$

$$x(\cot\theta_1 - \cot\theta_2) = 2h\cot\theta_2$$

$$x\left(\frac{1}{\tan\theta_1} - \frac{1}{\tan\theta_2}\right) = \frac{2h}{\tan\theta_2} \quad \left[\because \frac{1}{\tan\theta} = \cot\theta\right]$$

$$x\left(\frac{\tan\theta_2 - \tan\theta_1}{\tan\theta_1\tan\theta_2}\right) = \frac{2h}{\tan\theta_2}$$

$$x = \frac{2h\tan\theta_1\tan\theta_2}{\tan\theta_2(\tan\theta_2 - \tan\theta_1)}$$

$$x = \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

Hence the height CB of the cloud is given by

$$CB = x + h$$

$$= \frac{2h\tan\theta_1}{\tan\theta_2 - \tan\theta_1} + h$$

$$= \frac{2h\tan\theta_1 + h(\tan\theta_2 - \tan\theta_1)}{\tan\theta_2 - \tan\theta_1}$$

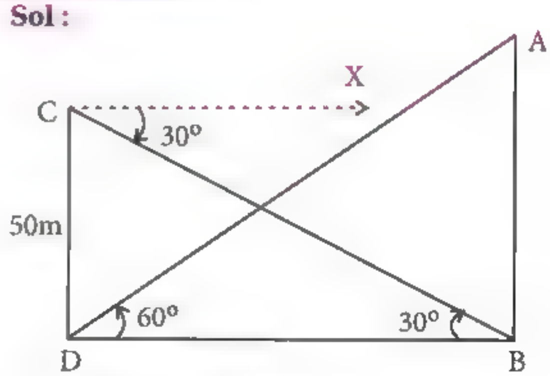
$$= \frac{2h\tan\theta_1 + h\tan\theta_2 - h\tan\theta_1}{\tan\theta_2 - \tan\theta_1}$$

$$= \frac{h\tan\theta_1 + h\tan\theta_2}{\tan\theta_2 - \tan\theta_1} = \frac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$$

$\therefore$ The required height $= \dfrac{h(\tan\theta_1 + \tan\theta_2)}{\tan\theta_2 - \tan\theta_1}$

4. The angle of elevation of the top of a cell phone tower from the foot of a high apartment is 60° and the angle of depression of the foot of the tower from the top of the apartment is 30°. If the height of the apartment is 50 m, find the height of the cell phone tower. According to Radiations control norms, the minimum height of a cell phone tower should be 120 m. State if the height of the above mentioned cell phone tower meets the radiation norms.

Sol :



Let AB be the cell phone tower.

CD be the apartment.

$$\angle XCB = \angle CBD = 30^\circ$$

$$\angle ADB = 60^\circ$$

In right triangle ABD

$$\tan 60^\circ = \frac{AB}{BD}$$

$$\sqrt{3} = \frac{AB}{BD}$$

$$BD = \frac{AB}{\sqrt{3}} \quad \text{... (1)}$$

In the right triangle ΔCDB

$$\tan 30^\circ = \frac{CD}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{50}{BD}$$

$$BD = 50\sqrt{3} \quad \text{... (2)}$$

From (1) and (2)

$$\frac{AB}{\sqrt{3}} = 50\sqrt{3}$$

$AB = 50 \times \sqrt{3} \times \sqrt{3} = 50 \times 3 = 150$ m

$\therefore$ Height of the cell phone tower = 150 m.

Since height of the tower > 120 m, yes, the tower meets the radiation norms.

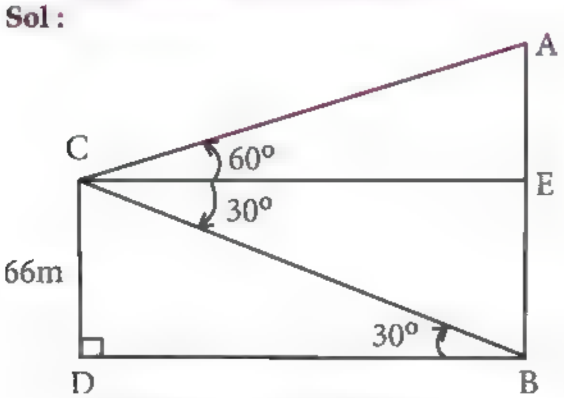
5. The angles of elevation and depression of the top and bottom of a Lamp post from the top of a 66 m high apartment are 60° and 30° respectively. Find

(i) The height of the Lamp post.

(ii) The difference between height of the Lamp post and the apartment.

(iii) The distance between the Lamp post and the apartment. ($\sqrt{3}$ = 1.732)

Sol :



Let AB be the lamp post and CD be the apartment given CD = 66 m = EB.

$$\angle ACE = 60^\circ$$

$$\angle ECB = \angle CBD = 30^\circ$$

(i) In the right triangle ΔBDC

$$\tan 30^\circ = \frac{CD}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{66}{BD}$$

$$BD = 66\sqrt{3} \quad \text{... (1)}$$

$$= 66 \times 1.732 = 114.312 \text{ m}$$

$\therefore$ The distance between the lamp post and the apartment = 114.31 m

Now $BD = EC = 114.31$ m

In the right triangle ΔACE

$$\tan 60^\circ = \frac{AE}{CE}$$

$$\sqrt{3} = \frac{AE}{66\sqrt{3}}$$

$$AE = 66\sqrt{3} \times \sqrt{3} \quad \text{[From (1)]}$$

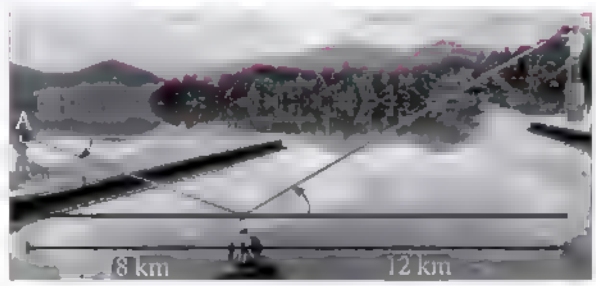
$$= 66 \times 3 = 198 \text{ m}$$

(i) Height of the lamp post = AB = AE + EB = 198 + 66 = 264 m

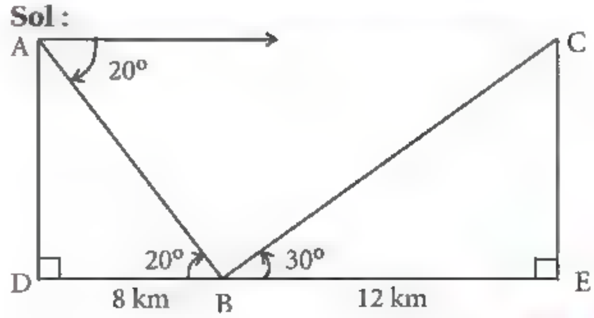
(ii) The difference between lamp post and apartment = 198 m.

(iii) The distance between the lamp post and apartment = 114.31 m.

6. Three villagers A, B and C can see each other across a valley. The horizontal distance between A and B is 8 km and the horizontal distance between B and C is 12 km. The angle of depression of B from A is 20° and the angle of elevation of C from B is 30°. Calculate: (i) the vertical height between A and B. (ii) the vertical height between B and C. (tan 20° = 0.3640, $\sqrt{3}$ = 1.732)



Sol :



In the right ΔADB

$$\tan 20^\circ = \frac{AD}{DB}$$

$$0.3640 = \frac{AD}{8}$$

$$AD = 8 \times 0.3640 = 2.91 \text{ km}$$

Vertical height between A and B = 2.91 km.

(ii) In the right ΔCEB

$$\tan 30^\circ = \frac{CE}{BE}$$

$$\frac{1}{\sqrt{3}} = \frac{CE}{12}$$

$$CE = \frac{12}{\sqrt{3}} = \frac{12\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{12 \times 1.732}{3} = 4 \times 1.732 = 6.928 = 6.93 \text{ km}$$

Vertical height between B and C = 6.93 km.

Exercise 6. 5

Multiple Choice Questions:

1. The value of $\sin^2\theta + \frac{1}{1+\tan^2\theta}$ is equal to

(1) $\tan^2\theta$ (2) 1
(3) $\cot^2\theta$ (4) 0 **[Ans : (2)]**

Sol :

$$\sin^2\theta + \frac{1}{1+\tan^2\theta} = \sin^2\theta + \frac{1}{\sec^2\theta} = \sin^2\theta + \cos^2\theta = 1$$

2. $\tan\theta\, \mathrm{cosec}^2\theta - \tan\theta$ is equal to

(1) $\sec\theta$ (2) $\cot^2\theta$
(3) $\sin\theta$ (4) $\cot\theta$ **[Ans : (4)]**

Sol :

$$\tan\theta\,\mathrm{cosec}^2\theta - \tan\theta = \tan\theta(\mathrm{cosec}^2\theta - 1) = \tan\theta\cot^2\theta = \tan\theta \times \frac{1}{\tan^2\theta} = \frac{1}{\tan\theta} = \cot\theta$$

3. If $(\sin\alpha + \mathrm{cosec}\,\alpha)^2 + (\cos\alpha + \sec\alpha)^2 = k + \tan^2\alpha + \cot^2\alpha$, then the value of k is equal to

(1) 9 (2) 7
(3) 5 (4) 3 **[Ans : (2)]**

Sol : $(\sin\alpha + \mathrm{cosec}\,\alpha)^2 + (\cos\alpha + \sec\alpha)^2$

$$= \sin^2\alpha + \mathrm{cosec}^2\alpha + 2\sin\alpha\,\mathrm{cosec}\,\alpha + \cos^2\alpha + \sec^2\alpha + 2\cos\alpha\sec\alpha$$

$$= (\sin^2\alpha + \cos^2\alpha) + \mathrm{cosec}^2\alpha + \sec^2\alpha + 2\sin\alpha\frac{1}{\sin\alpha} + 2\cos\alpha\frac{1}{\cos\alpha}$$

$$= 1 + \mathrm{cosec}^2\alpha + \sec^2\alpha + 2 + 2$$

$$= 5 + 1 + \cot^2\alpha + 1 + \tan^2\alpha$$

$$= 7 + \tan^2\alpha + \cot^2\alpha$$

comparing with $k + \tan^2\alpha + \cot^2\alpha$

$k = 7$

4. If $\sin\theta + \cos\theta = a$ and $\sec\theta + \mathrm{cosec}\,\theta = b$, then the value of b $(a^2 - 1)$ is equal to

(1) 2a (2) 3a
(3) 0 (4) 2ab **[Ans : (1)]**

Sol :

$$b(a^2-1) = (\sec\theta + \mathrm{cosec}\,\theta)[(\sin\theta+\cos\theta)^2 - 1]$$

$$= (\sec\theta + \mathrm{cosec}\,\theta)[\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1]$$

$$= \left(\frac{1}{\cos\theta} + \frac{1}{\sin\theta}\right)(1 + 2\sin\theta\cos\theta - 1)$$

$$= \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} \times 2\sin\theta\cos\theta = 2a$$

5. If $5x = \sec\theta$ and $\frac{5}{x} = \tan\theta$, then $x^2 - \frac{1}{x^2}$ is equal to

(1) 25 (2) $\frac{1}{25}$
(3) 5 (4) 1 **[Ans : (2)]**

Sol : $5x = \sec\theta \Rightarrow 25x^2 = \sec^2\theta$ ... (1)

$\frac{5}{x} = \tan\theta \Rightarrow \frac{25}{x^2} = \tan^2\theta$ ... (2)

Subtract (1) and (2)

$$\Rightarrow 25x^2 - \frac{25}{x^2} = \sec^2\theta - \tan^2\theta$$

$$25\left(x^2 - \frac{1}{x^2}\right) = 1$$

$$x^2 - \frac{1}{x^2} = \frac{1}{25}$$

6. If $\sin\theta = \cos\theta$, then $2\tan^2\theta + \sin^2\theta - 1$ is equal to

(1) $\frac{-3}{2}$ (2) $\frac{3}{2}$

(3) $\frac{2}{3}$ (4) $\frac{-2}{3}$ **[Ans : (2)]**

Sol : $\sin\theta = \cos\theta \Rightarrow \theta = 45^\circ$

$$= 2\tan^2 45^\circ + \sin^2 45^\circ - 1$$

$$= 2(1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 - 1$$

$$= 2 + \frac{1}{2} - 1$$

$$= 1 + \frac{1}{2}$$

$$= \frac{3}{2}$$

7. If $x = a\tan\theta$ and $y = b\sec\theta$ then

(1) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(3) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

[Ans : (1)]

Sol : $x = a\tan\theta$ $y = b\sec\theta$

$x^2 = a^2\tan^2\theta$ $y^2 = b^2\sec^2\theta$

$\frac{x^2}{a^2} = \tan^2\theta$ $\frac{y^2}{b^2} = \sec^2\theta$

$$\sec^2\theta - \tan^2\theta = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

8. $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \mathrm{cosec}\,\theta)$ is equal to

(1) 0 (2) 1

(3) 2 (4) -1 **[Ans : (3)]**

Sol :

$(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \mathrm{cosec}\,\theta)$

$$= \left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)$$

$$= \left(\frac{(\cos\theta + \sin\theta) + 1}{\cos\theta}\right)\left(\frac{(\sin\theta + \cos\theta) - 1}{\sin\theta}\right)$$

$$= \frac{(\sin\theta + \cos\theta)^2 - 1^2}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$$

$$= \frac{1 + 2\sin\theta\cos\theta - 1}{\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= 2$$

9. $a\cot\theta + b\,\mathrm{cosec}\,\theta = p$ and $b\cot\theta + a\,\mathrm{cosec}\,\theta = q$ then $p^2 - q^2$ is equal to

(1) $a^2 - b^2$ (2) $b^2 - a^2$

(3) $a^2 + b^2$ (4) $b - a$ **[Ans : (2)]**

Sol :

$p^2 - q^2 = (a\cot\theta + b\,\mathrm{cosec}\,\theta)^2 - (b\cot\theta + a\,\mathrm{cosec}\,\theta)^2$

$= a^2\cot^2\theta + b^2\mathrm{cosec}^2\theta + 2ab\cot\theta\,\mathrm{cosec}\,\theta$

$\quad - (b^2\cot^2\theta + a^2\mathrm{cosec}^2\theta + 2ab\cot\theta\,\mathrm{cosec}\,\theta)$

$= a^2\cot^2\theta + b^2\mathrm{cosec}^2\theta + 2ab\cot\theta\,\mathrm{cosec}\,\theta$

$\quad - b^2\cot^2\theta - a^2\mathrm{cosec}^2\theta - 2ab\cot\theta\,\mathrm{cosec}\,\theta$

$= a^2(\cot^2\theta - \mathrm{cosec}^2\theta) + b^2(\mathrm{cosec}^2\theta - \cot^2\theta)$

$= a^2(-1) + b^2(1)$

$= -a^2 + b^2$

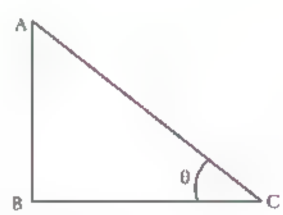
$= b^2 - a^2$

10. If the ratio of the height of a tower and the length of its shadow is $\sqrt{3}:1$, then the angle of elevation of the sun has measure

(1) 45° (2) 30°

(3) 90° (4) 60° **[Ans : (4)]**

Sol :



BC =Length of shadow

AB =Height of the tower

$$\tan\theta = \frac{AB}{BC} = \frac{\sqrt{3}}{1} \text{ (Given)}$$

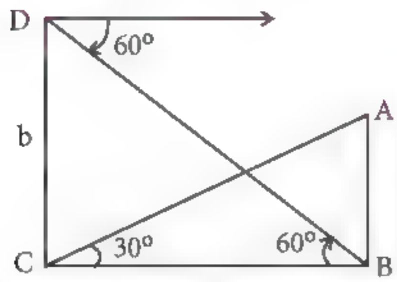
$$\tan 60^\circ = \sqrt{3}; \quad \theta = 60^\circ$$

11. The electric pole subtends an angle of 30° at a point on the same level as its foot. At a second point 'b' metres above the first, the depression of the foot of the tower is 60°. The height of the tower (in metres) is equal to

(1) $\sqrt{3}\,b$ (2) $\frac{b}{3}$

(3) $\frac{b}{2}$ (4) $\frac{b}{\sqrt{3}}$ **[Ans : (2)]**

Sol :



Let AB - tower, DC - electric pole

In $\Delta\, ABC$

$$\tan 30^\circ = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC}$$

$$BC = AB\sqrt{3} \quad \text{... (1)}$$

In $\Delta\, DCB$

$$\tan 60^\circ = \frac{DC}{BC}$$

$$\sqrt{3} = \frac{b}{BC}$$

$$BC = \frac{b}{\sqrt{3}} \quad \text{... (2)}$$

From (1) and (2)

$$AB\sqrt{3} = \frac{b}{\sqrt{3}}$$

$$AB = \frac{b}{\sqrt{3}\sqrt{3}}$$

$$= \frac{b}{3}$$

12. A tower is 60 m height. Its shadow is x metres shorter when the sun's altitude is 45° than when it has been 30°, then x is equal to

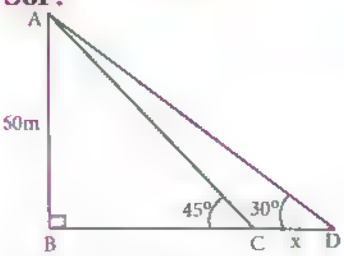
(1) 41.92 m

(2) 43.92 m

(3) 43 m

(4) 45.6 m **[Ans : (2)]**

Sol :



From the figure

AB - tower

BC - shadow when altitude is 45°

$$\tan 45^\circ = \frac{AB}{BC}$$

$$1 = \frac{60}{BC}$$

$$BC = 60 \text{ m} \quad \text{... (1)}$$

$$\tan 30^\circ = \frac{AB}{BD}$$

$$= \frac{60}{BC + CD}$$

$$\frac{1}{\sqrt{3}} = \frac{60}{60 + x} \quad [\therefore \text{ from (1)}]$$

$$60 + x = 60\sqrt{3}$$

$$x = 60\sqrt{3} - 60$$

$$= 60(\sqrt{3} - 1)$$

$$= 60\,(1.732 - 1)$$

$$= 60 \times 0.732$$

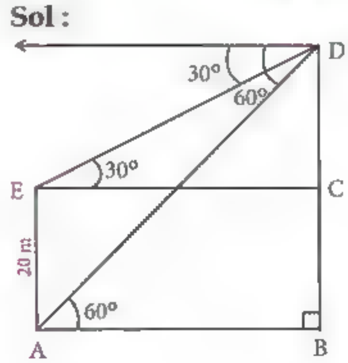
$$= 43.92 \text{ m}$$

13. The angle of depression of the top and bottom of 20 m tall building from the top of a multistoried building are 30° and 60° respectively. The height of the multistoried building and the distance between two buildings (in metres) is

(1) $20, 10\sqrt{3}$ (2) $30, 5\sqrt{3}$

(3) 20, 10 (4) $30, 10\sqrt{3}$ **[Ans : (4)]**

Sol :



From ΔABD

$$\tan 60^\circ = \frac{BD}{AB}$$

$$\sqrt{3} = \frac{BD}{AB}$$

$$AB = \frac{BD}{\sqrt{3}} \quad \dots (1)$$

From ΔDEC

$$\tan 30^\circ = \frac{DC}{EC}$$

$$\frac{1}{\sqrt{3}} = \frac{DC}{AB}$$

$$AB = DC\sqrt{3} \quad \dots (2)$$

(1) and (2) $\Rightarrow$

$$\frac{BD}{\sqrt{3}} = DC\sqrt{3}$$

$$\frac{BC+CD}{\sqrt{3}} = DC\sqrt{3}$$

$$BC + CD = DC \times 3$$

$$BC = 3\,CD - CD$$

$$BC = 2\,CD$$

$$AE = 2\,CD$$

$$20 = 2\,CD$$

$$CD = \frac{20}{2}$$

$$= 10 \text{ m}$$

Height of the building

$$BD = BC + CD = 20 + 10 = 30 \text{ m}$$

Distance between buildings

$$AB = DC\sqrt{3} \quad [\because \text{ from } (2)]$$

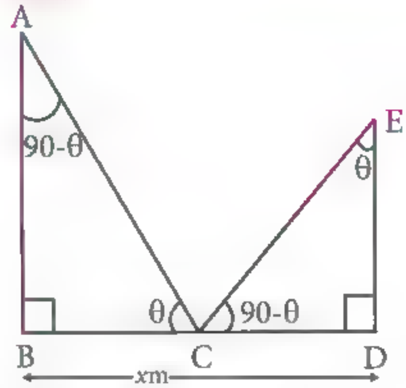
$$= 10\sqrt{3}\,m$$

14. Two persons are standing 'x' metres apart from each other and the height of the first person is double that of the other. If from the middle point of the line joining their feet an observer finds the angular elevations of their tops to be complementary, then the height of the shorter person (in metres) is

(1) $\sqrt{2}\,x$ (2) $\frac{x}{2\sqrt{2}}$

(3) $\frac{x}{\sqrt{2}}$ (4) $2x$ **[Ans : (2)]**

Sol :



AB and DE - persons

$$AB = 2\,DE$$

$$BD = x \Rightarrow BC = CD = \frac{x}{2}$$

From ΔABC

$$\tan\theta = \frac{AB}{BC}$$

$$\tan\theta = \frac{2\,DE}{BC} = \frac{2\,DE}{\frac{x}{2}} = \frac{4\,DE}{x} \quad \dots (1)$$

From ΔEDC

$$\tan\theta = \frac{CD}{ED} = \frac{\frac{x}{2}}{ED}$$

$$\tan\theta = \frac{x}{2\,ED} \quad \dots (2)$$

From (1) and (2)

$$\frac{4\,DE}{x} = \frac{x}{2\,ED}$$

$$8\,DE^2 = x^2$$

$$DE^2 = \frac{x^2}{8}$$

$$DE = \frac{x}{\sqrt{8}}$$

$$DE = \frac{x}{2\sqrt{2}}$$

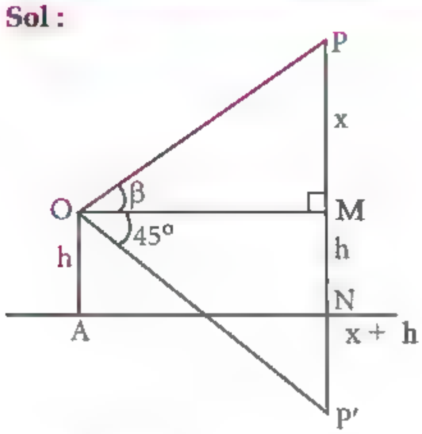
15. The angle of elevation of a cloud from a point h metres above a lake is β. The angle of depression of its reflection in the lake is 45°. The height of location of the cloud from the lake is

(1) $\frac{h(1+\tan\beta)}{1-\tan\beta}$ (2) $\frac{h(1-\tan\beta)}{1+\tan\beta}$

(3) $h\tan(45^\circ - \beta)$ (4) none of these

[Ans : (1)]

Sol :



From $\Delta OMP'$

$$\tan 45° = \frac{h+(x+h)}{OM}$$

$$1 = \frac{x+2h}{OM}$$

$$OM = x+2h \quad \dots (1)$$

From ΔOMP

$$\tan\beta = \frac{x}{OM}$$

$$OM = \frac{x}{\tan\beta} \quad \dots (2)$$

From (1) and (2)

$$x+2h = \frac{x}{\tan\beta}$$

$$\frac{x}{\tan\beta} - x = 2h$$

$$\frac{x - x\tan\beta}{\tan\beta} = 2h$$

$$x(1-\tan\beta) = 2h\tan\beta$$

$$x = \frac{2h\tan\beta}{1-\tan\beta}$$

$$PN = x+h = \frac{2h\tan\beta}{1-\tan\beta}+h$$

$$= \frac{2h\tan\beta + h - h\tan\beta}{1-\tan\beta}$$

$$= \frac{h\tan\beta + h}{1-\tan\beta}$$

$$= \frac{h(1+\tan\beta)}{1-\tan\beta}$$

$\therefore$ Height of the cloud from lake $= \dfrac{h(1+\tan\beta)}{1-\tan\beta}$

UNIT EXERCISE - 6

1. Prove that (i) $\cot^2 A\left(\frac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\frac{\sin A-1}{1+\sec A}\right)=0.$

(ii) $\frac{\tan^2\theta-1}{\tan^2\theta+1} = 1-2\cos^2\theta.$

Sol :

(i) LHS $= \cot^2 A\left(\frac{\sec A-1}{1+\sin A}\right)+\sec^2 A\left(\frac{\sin A-1}{1+\sec A}\right)$

$$= \frac{\cot^2 A(\sec A-1)(\sec A+1)+\sec^2 A(\sin A-1)(1+\sin A)}{(1+\sin A)(1+\sec A)}$$

$$= \frac{\cot^2 A(\sec^2 A-1)+\sec^2 A(\sin^2 A-1)}{(1+\sin A)(1+\sec A)}$$

$$= \frac{\cot^2 A\tan^2 A-\sec^2 A(1-\sin^2 A)}{(1+\sin A)(1+\sec A)}$$

$[\because \sec^2 A - \tan^2 A = 1]$

$$= \frac{\cot^2 A\tan^2 A-\sec^2 A\cos^2 A}{(1+\sin A)(1+\sec A)}$$

$$= \frac{\frac{1}{\tan^2 A}\tan^2 A - \frac{1}{\cos^2 A}\cos^2 A}{(1+\sin A)(1+\sec A)}$$

$$= \frac{1-1}{(1+\sin A)(1-\sec A)} = 0 = \text{RHS}.$$

(ii) $\frac{\tan^2\theta-1}{\tan^2\theta+1} = 1-2\cos^2\theta$

$$\text{LHS} = \frac{\tan^2\theta-1}{\tan^2\theta+1}$$

$$= \frac{\frac{\sin^2\theta}{\cos^2\theta}-1}{\frac{\sin^2\theta}{\cos^2\theta}+1}$$

$$= \frac{\dfrac{\sin^2\theta - \cos^2\theta}{\cos^2\theta}}{\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta} \times \frac{\cos^2\theta}{(\sin^2\theta + \cos^2\theta)}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{1} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= (1 - \cos^2\theta) - \cos^2\theta \quad [\therefore \sin^2\theta = 1 - \cos^2\theta]$$

$$= 1 - \cos^2\theta - \cos^2\theta$$

$$= 1 - 2\cos^2\theta = \text{RHS}$$

2. Prove that $\left(\dfrac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta}\right)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

Sol :

$$\text{LHS} = \left(\frac{1+\sin\theta - \cos\theta}{1+\sin\theta+\cos\theta}\right)^2$$

$$= \left(\frac{(1+\sin\theta)-\cos\theta}{(1+\sin\theta)+\cos\theta} \times \frac{(1+\sin\theta)-\cos\theta}{(1+\sin\theta)-\cos\theta}\right)^2$$

$$= \left[\frac{((1+\sin\theta)-\cos\theta)^2}{(1+\sin\theta)^2 - (\cos\theta)^2}\right]^2$$

$$= \left[\frac{1+\sin^2\theta + 2\sin\theta + \cos^2\theta - 2\cos\theta(1+\sin\theta)}{1+\sin^2\theta+2\sin\theta - \cos^2\theta}\right]^2$$

$$[\because (a-b)^2 = a^2 + b^2 - 2ab]$$

$$= \left[\frac{\sin^2\theta + 2\sin\theta + 1 + \cos^2\theta - 2\cos\theta - 2\sin\theta\cos\theta}{1 + (1-\cos^2\theta) + 2\sin\theta - \cos^2\theta}\right]^2$$

$$= \left[\frac{(\cos^2\theta + \sin^2\theta) + 2\sin\theta + 1 - 2\cos\theta(1+\sin\theta)}{1+1-2\cos^2\theta + 2\sin\theta}\right]^2$$

$$= \left[\frac{1+1+2\sin\theta - 2\cos\theta(1+\sin\theta)}{2-2\cos^2\theta+2\sin\theta}\right]^2$$

$$= \left[\frac{2+2\sin\theta - 2\cos\theta(1+\sin\theta)}{2-2\cos^2\theta + 2\sin\theta}\right]^2$$

$$= \left(\frac{2(1+\sin\theta) - 2\cos\theta(1+\sin\theta)}{2(1-\cos^2\theta) + 2\sin\theta}\right)^2$$

$$= \left(\frac{(1+\sin\theta)\,2(1-\cos\theta)}{2\sin^2\theta + 2\sin\theta}\right)^2$$

$$= \left(\frac{2(1+\sin\theta)(1-\cos\theta)}{2\sin\theta(1+\sin\theta)}\right)^2$$

$$= \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{(1-\cos\theta)^2}{\sin^2\theta}$$

$$= \frac{(1-\cos\theta)(1-\cos\theta)}{1-\cos^2\theta}$$

$$= \frac{(1-\cos\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$= \frac{1-\cos\theta}{1+\cos\theta}$$

$$= \text{RHS}$$

3. If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$ **and** $x\sin\theta = y\cos\theta$ **then prove that** $x^2 + y^2 = 1$.

Sol :

We have $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$

$\Rightarrow \quad (x\sin\theta)(\sin^2\theta) + (y\cos\theta)\cos^2\theta = \sin\theta\cos\theta$

$\Rightarrow \quad x\sin\theta(\sin^2\theta) + (x\sin\theta)\cos^2\theta = \sin\theta\cos\theta \quad [\because x\sin\theta = y\cos\theta]$S

$\Rightarrow \quad x\sin\theta(\sin^2\theta + \cos^2\theta) = \sin\theta\cos\theta$

$\Rightarrow \quad x\sin\theta = \sin\theta\cos\theta$

$x = \cos\theta \quad \dots(1)$

Now, $\quad x\sin\theta = y\cos\theta$

$\Rightarrow \quad \cos\theta\sin\theta = y\cos\theta \quad [\because x = \cos\theta \text{ from (1)}]$

$\Rightarrow \quad y = \sin\theta \quad \dots(2)$

From (1) and (2)

$x^2 + y^2 = \cos^2\theta + \sin^2\theta$

$= 1$

$\therefore \quad x^2 + y^2 = 1$

4. If $a\cos\theta - b\sin\theta = c$**, then prove that** $(a\sin\theta + b\cos\theta) = \pm\sqrt{a^2+b^2-c^2}$.

Sol :

We have $a\cos\theta - b\sin\theta = c$

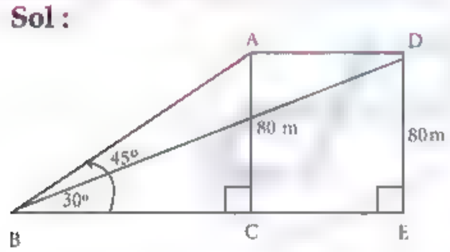
Squaring on both the sides.

$(a\cos\theta - b\sin\theta)^2 = c^2$

$$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\cos\theta\sin\theta = c^2$$
$$a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) - 2ab\sin\theta\cos\theta = c^2$$
$$a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2$$
$$a^2 + b^2 - c^2 = a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta$$
$$a^2 + b^2 - c^2 = (a\sin\theta + b\cos\theta)^2$$
$$\pm\sqrt{a^2+b^2-c^2} = a\sin\theta + b\cos\theta$$
$$\therefore\ a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$$

5. A bird is sitting on the top of a 80 m high tree. From a point on the ground, the angle of elevation of the bird is 45°. The bird flies away horizontally in such away that it remained at a constant height from the ground. After 2 seconds, the angle of elevation of the bird from the same point is 30°. Determine the speed at which the bird flies.($\sqrt{3}$ = 1.732)

Sol :



Let the initial position of the bird be A and after two seconds its position is at D.

$$AC = DE = 80 \text{ m}$$
$$\angle ABC = 45°$$
$$\angle DBC = 30°$$

In right ΔABC

$$\tan 45° = \frac{AC}{BC}$$
$$1 = \frac{80}{BC}$$
$$BC = 80 \text{ m}$$

In right triangle ΔDBE

$$\tan 30° = \frac{DE}{BE}$$
$$\frac{1}{\sqrt{3}} = \frac{80}{BC+CE}$$
$$\frac{1}{\sqrt{3}} = \frac{80}{80+CE}$$
$$80 + CE = 80\sqrt{3}$$
$$CE = (80\sqrt{3} - 80)$$
$$= 80(\sqrt{3}-1)$$
$$= 80\,(1.732 - 1)$$
$$= 80 \times 0.732$$
$$CE = 58.56 \text{ m}$$

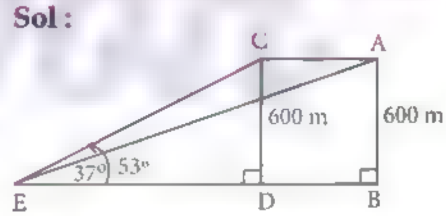
$\therefore$ The bird travelled 58.56 m in 2 seconds.

$$\text{Speed of the bird} = \frac{\textit{Distance travelled}}{\textit{Time taken}}$$
$$= \frac{58.56}{2}$$
$$= 29.28 \text{ m/s}$$

$\therefore$ Speed of flying bird = 29.28 m/s.

6. An aeroplane is flying parallel to the Earth's surface at a speed of 175 m/sec and at a height of 600 m. The angle of elevation of the aeroplane from a point on the Earth's surface is 37° at a given point. After what period of time does the angle of elevation increase to 53°? (tan 53° = 1.3270, tan 37° = 0.7536)

Sol :



Let A be the initial position of the aeroplane and C be the position of the aeroplane at an angle of elevation 53°

$$\angle AEB = 37°$$
$$\angle CED = 53°$$

In the right triangle ΔAEB

$$\tan 37° = \frac{AB}{BE}$$
$$0.7536 = \frac{600}{BD+DE}$$
$$BD + DE = \frac{600}{0.7536}$$
$$= \frac{60{,}00{,}000}{7536}$$
$$BD + DE = 796.18 \text{ m} \qquad \text{... (1)}$$

In right ΔCED

$$\tan 53° = \frac{CD}{DE}$$

$$1.3270 = \frac{600}{DE}$$

$$DE = \frac{600}{1.3270}$$

$$= \frac{6,00,000}{1327}$$

$$DE = 452.15 \text{ m} \quad \text{... (2)}$$

$(1) - (2) \Rightarrow$ BD + DE – DE = 796.18 – 452.15

$$BD = 344.03 \text{ m}$$

$\therefore$ CA = 344.03 m [$\because$ BD = CA]

$\therefore$ Distance travelled by the aeroplane = 344.03 m

Speed of the aeroplane = 175 m/s

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{344.03}{175}$$

$$= 1.965 \text{ seconds.}$$

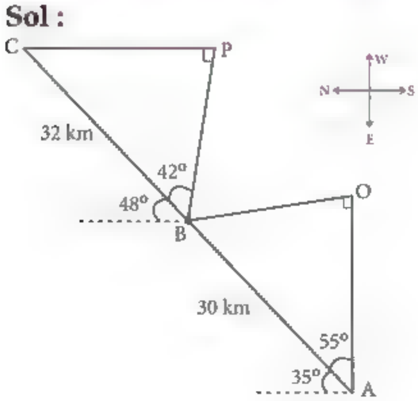
$$= 1.97 \text{ seconds.}$$

$\therefore$ After 1.97 seconds the angle of elevation is 53°.

7. A bird is flying from A towards B at an angle of 35°, a point 30 km away from A. At B it changes its course of flight and heads towards C on a bearing of 48° and distance 32 km away.
(i) How far is B to the North of A?
(ii) How far is B to the West of A?
(iii) How far is C to the North of B?
(iv) How far is C to the East of B?

$$\begin{pmatrix} \sin 55^\circ = 0.8192, \cos 55^\circ = 0.5736 \\ \sin 42^\circ = 0.6691, \cos 42^\circ = 0.7431 \end{pmatrix}$$

Sol :



Let A be the initial position of the bird.

B be the position after travelling 30 km at an angle of 35° from A.

Let C be the position after travelling 32 km at an angle of 48° from B.

$\therefore$ $\angle OAP = 55^\circ$

$\angle PBC = 42^\circ$

[$\because$ complementary angle]

(i) In Δ AOB

$$\sin 55^\circ = \frac{OB}{AB}$$

$$0.8192 = \frac{OB}{30}$$

$$OB = 30 \times 0.8192 = 24.58 \text{ km}$$

(ii) From the right triangle Δ AOB

$$\cos 55^\circ = \frac{OA}{AB}$$

$$0.5736 = \frac{OA}{30}$$

$$OA = 30 \times 0.5736 = 17.21 \text{ km}$$

B is 17.21 km to the West of A.

(iii)

$$\sin 42^\circ = \frac{PC}{BC}$$

$$0.6691 = \frac{PC}{32}$$

$$PC = 32 \times 0.6691 = 21.41 \text{ km}$$

C is 21.41 km to the North of B.

(iv) In the right Δ BPC

$$\cos 42^\circ = \frac{BP}{BC}$$

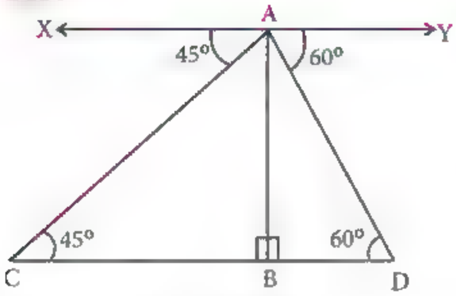
$$0.7431 = \frac{BP}{32}$$

$$BP = 23.78 \text{ km}$$

C is 23.78 km to the East of B.

8. Two ships are sailing in the sea on either side of the lighthouse. The angles of depression of two ships as observed from the top of the lighthouse are 60° and 45° respectively. If the distance between the ships is $200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$ metres, find the height of the lighthouse.

Sol :



Let C and D are two ships.

Let AB be the height of the light house.

$$\angle XAC = \angle ACB = 45°$$

$$\angle YAD = \angle ADB = 60°$$

$$CD = 200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right) m$$

In the right $\Delta\ ABD$

$$\tan 60° = \frac{AB}{BD}$$

$$\sqrt{3} = \frac{AB}{BD}$$

$$BD = \frac{AB}{\sqrt{3}} \quad \text{... (1)}$$

In the right triangle ABC

$$\tan 45° = \frac{AB}{BC}$$

$$1 = \frac{AB}{BC}$$

$$BC = AB \quad \text{... (2)}$$

$$(1) + (2) \Rightarrow BD + BC = \frac{AB}{\sqrt{3}} + AB$$

$$CD = AB\left(\frac{1}{\sqrt{3}}+1\right) \quad [\because CB + BD = CD]$$

$$\frac{CD}{\left(\frac{1}{\sqrt{3}}+1\right)} = AB$$

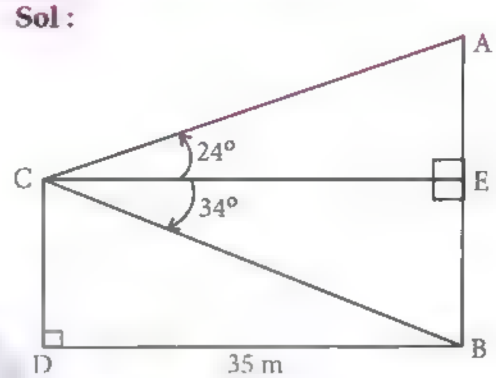
$$AB = \frac{200\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)}{\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)}$$

$$AB = 200 \text{ m}$$

$\therefore$ Height of the light house is 200 m.

9. A building and a statue are in opposite side of a street from each other 35 m apart. From a point on the roof of building the angle of elevation of the top of statue is 24° and the angle of depression of base of the statue is 34°. Find the height of the statue. ($\tan 24° = 0.4452$, $\tan 34° = 0.6745$)

Sol :



Let AB be the statue CD be the building Given BD = 35 m.

$$\angle ACE = 24°$$

$$\angle ECB = 34°$$

$$DB = CE = 35 \text{ m}$$

In right triangle $\Delta\ ECB$

$$\tan 34° = \frac{EB}{CE}$$

$$EB = CE \times \tan 34°$$

$$= 35 \times 0.6745 \text{ m}$$

$$EB = 23.61 \text{ m} \quad \text{... (1)}$$

In right $\Delta\ AEC$

$$\tan 24° = \frac{AE}{EC}$$

$$AE = \tan 24° \times EC$$

$$= 0.4452 \times 35$$

$$AE = 15.58 \text{ m} \quad \text{... (2)}$$

$$(1) + (2) \Rightarrow AE + EB = 15.58 + 23.61$$

$$AB = 39.19 \text{ m}$$

$\therefore$ Height of the statue = 39.19 m.

CREATIVE QUESTIONS

I. Multiple Choice Questions

Trigonometric Ratios

1. If $\tan\theta = \frac{a}{b}$, then $\frac{a\sin\theta + b\cos\theta}{a\sin\theta - b\cos\theta}$ is

(1) $\frac{a^2+b^2}{a^2-b^2}$ (2) $\frac{a^2-b^2}{a^2+b^2}$

(3) $\frac{a+b}{a-b}$ (4) $\frac{a-b}{a+b}$ **[Ans : (1)]**

Sol :

$$\tan\theta = \frac{a}{b}$$

$$\frac{a\sin\theta + b\cos\theta}{a\sin\theta - b\cos\theta} = \frac{a\frac{\sin\theta}{\cos\theta} + b\frac{\cos\theta}{\cos\theta}}{a\frac{\sin\theta}{\cos\theta} - b\frac{\cos\theta}{\cos\theta}}$$

[Dividing by cos θ]

$$= \frac{a\tan\theta + b}{a\tan\theta - b}$$

$$= \frac{a\times\frac{a}{b}+b}{a\times\frac{a}{b}-b}$$

$$= \frac{\frac{a^2}{b}+b}{\frac{a^2}{b}-b}$$

$$= \frac{\frac{a^2+b^2}{b}}{\frac{a^2-b^2}{b}}$$

$$= \frac{a^2+b^2}{b}\times\frac{b}{a^2-b^2} = \frac{a^2+b^2}{a^2-b^2}$$

2. If A and B are complementary angles then

(1) sin A = sin B (2) cos A = cos B

(3) tan A = tan B (4) sec A = cosec B

[Ans : (4)]

Sol :

A and B are complementary angles

$$A + B = 90^\circ$$
$$A = 90^\circ - B$$
$$\text{Sec } A = \text{Sec }(90 - B) = \text{Cosec } B$$

3. If $x\sin(90^\circ-\theta)\cot(90^\circ-\theta) = \cos(90^\circ-\theta)$ then x =

(1) 0 (2) 1

(3) – 1 (4) 2 **[Ans : (2)]**

Sol :

$$x\sin(90-\theta)\cot(90^\circ-\theta) = \cos(90^\circ-\theta)$$

$$\Rightarrow x\cos\theta\tan\theta = \sin\theta$$

$$\Rightarrow x = \frac{\sin\theta}{\cos\theta\tan\theta}$$

$$= \frac{\sin\theta}{\cos\theta}\times\cot\theta$$

$$= \tan\theta\cot\theta$$

$$= \tan\theta\times\frac{1}{\tan\theta} = 1$$

4. If x tan 45° cos 60° = sin 60° cot 60°, then x is

(1) 1 (2) $\sqrt{3}$

(3) $\frac{1}{2}$ (4) $\frac{1}{\sqrt{2}}$ **[Ans : (1)]**

Sol :

$$x\tan 45^\circ\cos 60^\circ = \sin 60^\circ\cot 60^\circ$$

$$x\times 1\times\frac{1}{2} = \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{3}}$$

$$x\times\frac{1}{2} = \frac{1}{2}$$

$$x = \frac{2}{2} = 1$$

5. $\frac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ}$ =

(1) tan 90° (2) 1

(3) sin 45° (4) sin 0° **[Ans : (4)]**

Sol :

$$\frac{1-\tan^2 45^\circ}{1+\tan^2 45^\circ} = \frac{1-1}{1+1} = \frac{0}{2} = 0$$

Trigonometric Identities:

6. If $\sec\theta + \tan\theta = x$, **then** $\sec\theta =$

(1) $\frac{x^2+1}{x}$ (2) $\frac{x^2+1}{2x}$

(3) $\frac{x^2-1}{2x}$ (4) $\frac{x^2-1}{x}$ **[Ans : (2)]**

Sol :

$$\sec\theta + \tan\theta = x$$
$$(\sec\theta + \tan\theta)^2 = x^2$$
$$\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta = x^2$$
$$\sec^2\theta + \sec^2\theta - 1 + 2\sec\theta\tan\theta = x^2$$
$$2\sec^2\theta + 2\sec\theta\tan\theta = x^2 + 1$$
$$2\sec\theta(\sec\theta + \tan\theta) = x^2 + 1$$
$$\sec\theta\ (x) = \frac{x^2+1}{2}$$
$$\sec\theta = \frac{x^2+1}{2x}$$

7. $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} =$

(1) $\sec\theta + \tan\theta$ (2) $\sec\theta - \tan\theta$

(3) $\sec^2\theta + \tan^2\theta$ (4) $\sec^2\theta - \tan^2\theta$

[Ans : (1)]

Sol :

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}}$$
$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$$
$$= \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$
$$= \frac{1+\sin\theta}{\cos\theta}$$
$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$
$$= \sec\theta + \tan\theta$$

8. $\cos^4 A - \sin^4 A =$

(1) $2\cos^2 A + 1$ (2) $2\cos^2 A - 1$

(3) $2\sin^2 A - 1$ (4) $2\sin^2 A + 1$ **[Ans : (2)]**

Sol :

$$\cos^4 A - \sin^4 A = (\cos^2 A)^2 - (\sin^2 A)^2$$
$$= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= (1)(\cos^2 A - (1 - \cos^2 A))$$
$$= \cos^2 A - 1 + \cos^2 A$$
$$= 2\cos^2 A - 1$$

9. $\frac{\sin\theta}{1+\cos\theta} =$

(1) $\frac{1+\cos\theta}{\sin\theta}$ (2) $\frac{1-\cos\theta}{\cos\theta}$

(3) $\frac{1-\cos\theta}{\sin\theta}$ (4) $\frac{1-\sin\theta}{\cos\theta}$ **[Ans : (3)]**

Sol :

$$\frac{\sin\theta}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta}$$
$$= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$$
$$= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$

10. If $\sin\theta + \sin^2\theta = 1$ **then** $\cos^2\theta + \cos^4\theta =$

(1) -1 (2) 1

(3) 0 (4) None of these

[Ans : (2)]

Sol :

$$\sin^2\theta + \cos^2\theta = 1$$
$$\text{Taking } \sin\theta + \sin^2\theta = 1$$
$$\sin\theta + (1 - \cos^2\theta) = 1$$
$$\sin\theta + 1 - \cos^2\theta = 1$$
$$\sin\theta = 1 - 1 + \cos^2\theta$$
$$\sin\theta = \cos^2\theta$$
$$\sin^2\theta = \cos^4\theta \quad \text{... (1)}$$
$$\text{Now } \sin^2\theta + \cos^2\theta = 1$$

$$1-\cos^2\theta = \sin^2\theta$$

$$1-\cos^2\theta = \cos^4\theta$$

using (1)

$$\cos^4\theta+\cos^2\theta = 1$$

$$\cos^2\theta+\cos^4\theta = 1$$

Heights and Distances – Angle of Elevation:

11. From a given point when height of an object increases the angle of elevation _____

(1) increases
(2) decreases
(3) neither increases nor decreases
(4) equal. **[Ans : (1)]**

12. The ratio of the length of a rod and its shadow is $1:\sqrt{3}$. The angle of elevation of the sum is

(1) 30° (2) 45°
(3) 60° (4) 90° **[Ans : (1)]**

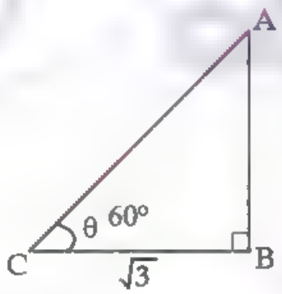
Sol :

$$\tan\theta = \frac{AB}{BC}$$

$$= \frac{1}{\sqrt{3}}$$

$$= \tan 30^\circ$$

$$\theta = 30^\circ$$

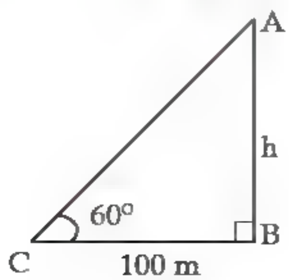


Angle of elevation is 30°

13. If the angle of elevation of a tower from a distance of 100 m from its foot is 60°, then the height of the tower is

(1) $100\sqrt{3}\,m$ (2) $\frac{100}{\sqrt{3}}m$
(3) $50\sqrt{3}\,m$ (4) $\frac{200}{\sqrt{3}}m$ **[Ans : (2)]**

Sol :



AB – tower

From C distance = 100 m

Angle of elevation θ = 60°

From right ΔABC

$$\tan\theta = \frac{AB}{BC}$$

$$\tan 60^\circ = \frac{h}{100}$$

$$\sqrt{3} = \frac{h}{100}$$

$$h = 100\sqrt{3}$$

Height of the tower = $100\sqrt{3}$ m.

14. If the altitude of the sun is at 60°, then the height of the vertical tower that will cast a shadow of length 30 m is

(1) $30\sqrt{3}\,m$ (2) 15 m
(3) $\frac{30}{\sqrt{3}}m$ (4) $15\sqrt{2}\,m$ **[Ans : (1)]**

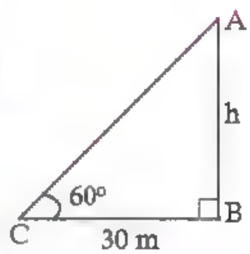
Sol :

From the right ΔABC

$$\tan 60^\circ = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{h}{30}$$

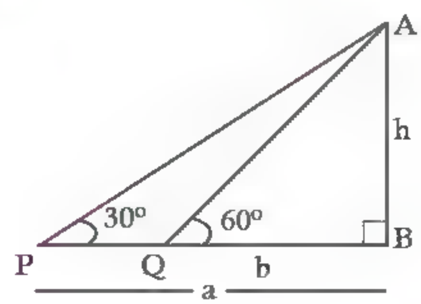
$$h = 30\sqrt{3}\text{ m}$$



15. The angles of elevation of a tower from two points distant a and b (a>b) from its foot and in the same straight line from if are 30° and 60°, then the height of the tower is

(1) $\sqrt{a+b}$ (2) $\sqrt{ab}$
(3) $\sqrt{a-b}$ (4) $\sqrt{\frac{a}{b}}$ **[Ans : (2)]**

Sol :



$$\tan 30° = \frac{h}{a} = \frac{1}{\sqrt{3}}$$

$$\frac{h}{a} = \frac{1}{\sqrt{3}} \quad ...(1)$$

$$\tan 60° = \frac{AB}{QB}$$

$$\sqrt{3} = \frac{h}{b} \quad ...(2)$$

$$(1)\times(2) \Rightarrow \frac{h}{a}\times\frac{h}{b} = \frac{1}{\sqrt{3}}\times\sqrt{3}$$

$$\frac{h^2}{ab} = 1$$

$$h^2 = ab$$

Height of the tower, $h = \sqrt{ab}$

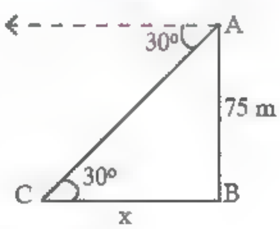
Angle of Depression

16. The angle of depression of a car, standing on the ground from the top of a 75 m tower is 30°. The distance of the car from the base of the tower in metres is

(1) $25\sqrt{3}$ (2) $50\sqrt{3}$

(3) $75\sqrt{3}$ (4) 150 **[Ans : (3)]**

Sol :



From the right ΔABC

$$\tan 30° = \frac{75}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{75}{x}$$

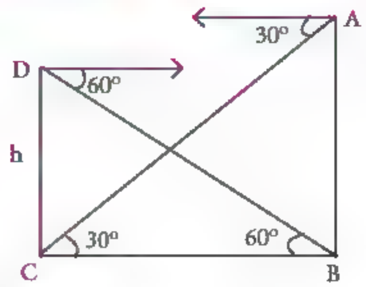
$$x = 75\sqrt{3} \text{ m}$$

17. A tower subtends an angle 30° at a point on the same level as its foot. At a second point h metres above the first the depression of the foot of the tower is 60°. The height of the tower is

(1) $\frac{h}{2}m$ (2) $\sqrt{3}\,h\,m$

(3) $\frac{h}{3}m$ (4) $\frac{h}{\sqrt{3}}m$ **[Ans : (3)]**

Sol :



Let AB be the tower

From the right Δ DCB

$$\tan 60° = \frac{h}{BC}$$

$$\sqrt{3} = \frac{h}{BC}$$

$$BC = \frac{h}{\sqrt{3}} \quad ...(1)$$

From the right ΔCBA

$$\tan 30° = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC}$$

$$BC = AB\sqrt{3} \quad ...(2)$$

From (1) and (2)

$$AB\sqrt{3} = \frac{h}{\sqrt{3}}$$

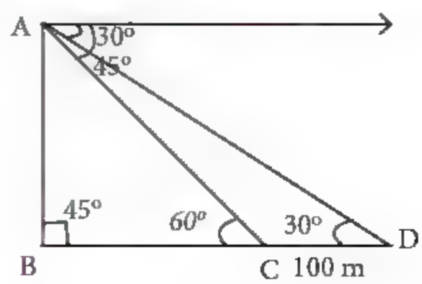
$$AB = \frac{h}{\sqrt{3}\times\sqrt{3}} = \frac{h}{3} \text{ m}$$

18. The angles of depression of two ships from the top of a light house are 45° and 30° towards east. If the ships are 100 m apart, the height of the light house is

(1) $\frac{50}{\sqrt{3}+1}m$ (2) $\frac{50}{\sqrt{3}-1}m$

(3) $50(\sqrt{3}-1)\,m$ (4) $50(\sqrt{3}+1)\,m$ **[Ans : (4)]**

Sol :



Let AB be the light house C and D are two ships.
From the right Δ ABC

$$\tan 45^\circ = \frac{AB}{BC}$$

$$1 = \frac{AB}{BC}$$

$$BC = AB \quad \text{... (1)}$$

From the right $\Delta\, ABD$

$$\tan 30^\circ = \frac{AB}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{BC+CD}$$

$$BC + 100 = \sqrt{3}\, AB \quad \text{... (2)}$$

From (1) and (2)

$$\sqrt{3}\, AB = AB + 100$$

$$\sqrt{3}\, AB - AB = 100$$

$$(\sqrt{3}-1)\, AB = 100$$

$$AB = \frac{100}{(\sqrt{3}-1)}$$

$$= \frac{100 \times (\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{100\ (\sqrt{3}+1)}{(\sqrt{3})^2 - 1^2}$$

$$= \frac{100\ (\sqrt{3}+1)}{3-1}$$

$$= \frac{100}{2}(\sqrt{3}+1)$$

$$AB = 50(\sqrt{3}+1)\,\text{m}$$

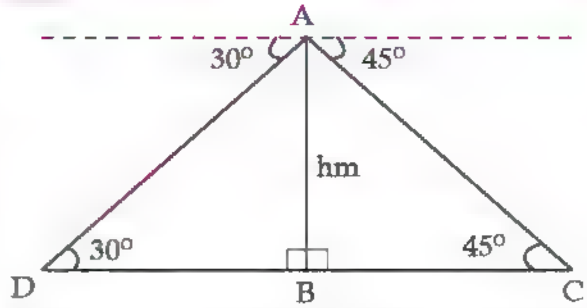
19. If the altitude of the light house is h metres and from it the angle of depression of two ships on opposite sides of the light house are observed to be 30° and 45°, then the distance between the ships are

(1) $(\sqrt{3}+1)h$ metres (2) $(\sqrt{3}-1)h$ metres

(3) $\sqrt{3}\,h$ metres (4) $1+\left(1+\frac{1}{\sqrt{3}}\right)h$ metres

[Ans : (1)]

Sol :



Let AB be the light house C and D be the two ships.
CD is the distance between two ships
From the right $\Delta\, ABD$

$$\tan 30^\circ = \frac{AB}{BD}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{DB}$$

$$DB = h\sqrt{3} \quad \text{... (1)}$$

From the right $\Delta\, ABC$

$$\tan 45^\circ = \frac{AB}{BC}$$

$$1 = \frac{h}{BC}$$

$$BC = h \quad \text{... (2)}$$

$$\text{From (1) and (2) } BC + DB = h + h\sqrt{3}$$

$$DC = (1+\sqrt{3})\,h$$

$$\text{Distance between ships} = (\sqrt{3}+1)\ h\ \text{m}$$

II. Very Short Answer Questions

1. If $\sin\theta + \sin^2\theta = 1$ **prove that** $\cos^2\theta + \cos^4\theta = 1$.

Sol :

We have $\quad \sin\theta + \sin^2\theta = 1$

$$\Rightarrow \sin\theta = 1 - \sin^2\theta$$

$$\Rightarrow \sin\theta = \cos^2\theta \quad \text{... (1)}$$

Now $\quad \cos^2\theta + \cos^4\theta = \cos^2\theta + (\cos^2\theta)^2$

$$\Rightarrow \cos^2\theta + \cos^4\theta = \cos^2\theta + \sin^2\theta$$

[∵ *From* (1)]

$$\Rightarrow \cos^2\theta + \cos^4\theta = 1$$

[∵ $\sin^2\theta + \cos^2\theta = 1$]

2. Prove the trigonometrical identity $(1-\sin^2\theta)\sec^2\theta$ **= 1.**

Sol :

$$\text{LHS} = (1-\sin^2\theta)\sec^2\theta = \cos^2\theta\ \sec^2\theta$$

$$= \cos^2\theta\left(\frac{1}{\cos^2\theta}\right) = 1 = \text{RHS}$$

3. Prove that trigonometrical identity

$\cos^2\theta(1+\tan^2\theta) = 1.$

Sol :

$$\text{LHS} = \cos^2\theta(1+\tan^2\theta)$$

$$= \cos^2\theta\sec^2\theta \qquad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \cos^2\theta\times\frac{1}{\cos^2\theta} = 1 = \text{RHS}$$

4. Prove the trigonometrical identity

$\cos^2\theta + \dfrac{1}{1+\cot^2\theta} = 1.$

Sol :

$$\text{LHS} = \cos^2\theta + \frac{1}{1+\cot^2\theta}$$

$$= \cos^2\theta + \frac{1}{\text{cosec}^2\theta} \qquad [\because 1+\cot^2\theta = \text{cosec}^2\theta]$$

$$= \cos^2\theta + \sin^2\theta \qquad \left[\because \frac{1}{\text{cosec}\,\theta}\right] = \sin\theta$$

$$= 1 = \text{RHS}$$

5. Prove the trigonometrical identity

$\dfrac{1}{1+\sin\theta} + \dfrac{1}{1-\sin\theta} = 2\sec^2\theta.$

Sol :

$$\text{LHS} = \frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}$$

$$= \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{2}{1-\sin^2\theta} = \frac{2}{\cos^2\theta}$$

$$= 2\sec^2\theta \qquad \left[\because \frac{1}{\cos\theta} = \sec\theta\right]$$

$$= \text{RHS}$$

6. Prove the trigonometrical identity

$\text{cosec}^2\theta + \sec^2\theta = \text{cosec}^2\theta\sec^2\theta.$

Sol :

$$\text{LHS} = \text{cosec}^2\theta + \sec^2\theta$$

$$= \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$$

$$= \frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta} \qquad \left[\because \frac{1}{\sin\theta} = \text{cosec}\,\theta,\ \frac{1}{\cos\theta} = \sec\theta\right]$$

$$= \frac{1}{\sin^2\theta}\ \frac{1}{\cos^2\theta}$$

$$= \text{cosec}^2\theta\sec^2\theta = \text{RHS}$$

7. Prove the trigonometric identity

$\cot^2\theta - \dfrac{1}{\sin^2\theta} = -1.$

Sol :

$$\text{LHS} = \cot^2\theta - \frac{1}{\sin^2\theta}$$

$$= \cot^2\theta - \text{cosec}^2\theta$$

$$= -(\text{cosec}^2\theta - \cot^2\theta) \qquad [\because \text{cosec}^2\theta - \cot^2\theta = 1]$$

$$= -1 = \text{RHS}$$

8. Prove the trigonometric identity

$(1+\tan^2\theta)(1+\sin\theta)(1-\sin\theta) = 1.$

Sol :

$$\text{LHS} = (1+\tan^2\theta)(1+\sin\theta)(1-\sin\theta)$$

$$= (1+\tan^2\theta)\{(1+\sin\theta)(1-\sin\theta)\}$$

$$= (1+\tan^2\theta)(1-\sin^2\theta)$$

$$[\because (a+b)(a-b) = a^2-b^2]$$

$$= (1+\tan^2\theta)(\cos^2\theta)$$

$$= \sec^2\theta\cos^2\theta \qquad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \frac{1}{\cos^2\theta}\times\cos^2\theta = 1 = \text{RHS}$$

9. Prove the trigonometric identity

$(1+\cot^2\theta)(1-\cos\theta)(1+\cos\theta) = 1$

Sol :

$$\text{LHS} = (1+\cot^2\theta)(1-\cos\theta)(1+\cos\theta)$$

$$= (1+\cot^2\theta)(1-\cos^2\theta)$$

$$= \text{cosec}^2\theta\sin^2\theta \qquad \left[\because 1-\cos^2\theta = \sin^2\theta \text{ and } 1+\cot^2\theta = \text{cosec}^2\theta\right]$$

$$= \frac{1}{\sin^2\theta}\sin^2\theta = 1 = \text{RHS}$$

10. Prove the trigonometric identity

$$\tan^2\theta - \frac{1}{\cos^2\theta} = -1$$

Sol :

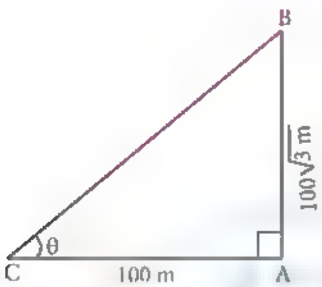
$$\text{LHS} = \tan^2\theta - \frac{1}{\cos^2\theta}$$

$$= \tan^2\theta - \sec^2\theta = -(\sec^2\theta - \tan^2\theta)$$

$$= -1 = \text{RHS} \qquad [\because \sec^2\theta - \tan^2\theta = 1]$$

11. A tower is $100\sqrt{3}$ metres high. Find the angle of elevation of its top from a point 100 metres away from its foot.

Sol :



Let AB be the tower AC be the distance from the point to the foot of the tower.

From the right triangle ΔCAB

$$\tan\theta = \frac{BA}{AC}$$

$$= \frac{100\sqrt{3}}{100}$$

$$= \sqrt{3}$$

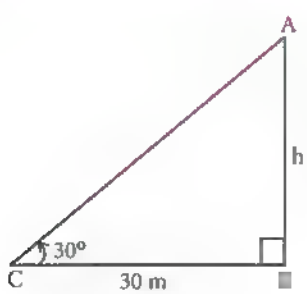
$$\theta = \tan^{-1}\left(\sqrt{3}\right)$$

$$\theta = 60^\circ$$

Angle of elevation is 60°

12. The angle of elevation of the top of a tower from a point on the ground, which is 30 m away from the foot of the tower is 30°. Find the height of the tower.

Sol :



Let AB be the tower.

BC is the distance between the point and the foot of the tower.

From the right triangle ΔABC

$$\tan 30^\circ = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{30}$$

$$AB = \frac{30}{\sqrt{3}}$$

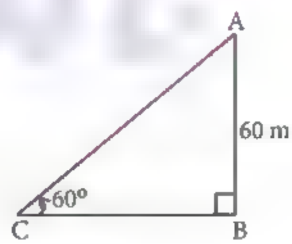
$$= \frac{3\times 10}{\sqrt{3}}$$

$$= \frac{\sqrt{3}\times\sqrt{3}\times 10}{\sqrt{3}} = 10\sqrt{3}\,m$$

$\therefore$ Height of the tower is $10\sqrt{3}\,m$

13. A kite is flying at a height of 60 m above the ground. The inclination of the string with the ground where its string is tied is 60°. Find the length of the string.

Sol :



Let AB be the height of the kite from the ground.

AC is the length of the string.

From the right triangle ΔABC

$$\sin 60^\circ = \frac{AB}{AC}$$

$$\frac{\sqrt{3}}{2} = \frac{60}{AC}$$

$$AC = \frac{2\times 60}{\sqrt{3}}$$

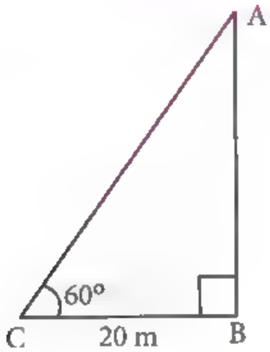
$$= \frac{2\times 20\times\sqrt{3}\times\sqrt{3}}{\sqrt{3}}$$

$$= 40\sqrt{3}\,m$$

Length of the string $= 40\sqrt{3}\,m$

14. A tower stands vertically on the ground. From a point on the ground which is 20 m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.

Sol :



Let AB is the tower.

In the right triangle ΔABC

$$\tan 60° = \frac{AB}{BC}$$

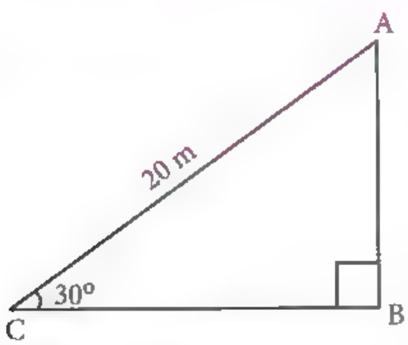
$$\sqrt{3} = \frac{AB}{20}$$

$$AB = 20\sqrt{3}\,m$$

$\therefore$ Height of the tower is $20\sqrt{3}\,m$.

15. A circus artist is climbing a 20 m long rope which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole if the angle made by the rope with the ground level is 30°.

Sol :



Let AB be the vertical height of the pole.

AC be the length of the rope.

From the right triangle ΔABC

$$\sin 30° = \frac{AB}{AC}$$

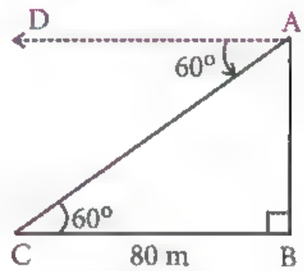
$$\frac{1}{2} = \frac{AB}{20}$$

$$AB = \frac{20}{2} = 10 \text{ m}$$

$\therefore$ Height of the pole = 10 m.

16. The angle between the top of a building and a point 80 m away from the base on level ground is 60°. How tall is the building?

Sol :



Let AB is the building

$$\angle DAC = \angle ACB = 60°$$

In right triangle ΔABC

$$\tan 60° = \frac{AB}{BC}$$

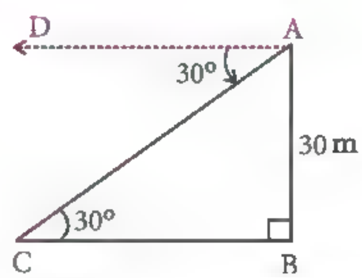
$$\sqrt{3} = \frac{AB}{80}$$

$$AB = 80\sqrt{3}\,m$$

Height of the building is $80\sqrt{3}\,m$.

17. From the top of the tower 30 m height a man is observing the base of a tree at an angle of depression measuring 30°. Find the distance between the tree and the tower.

Sol :



Let AB = 30 m is the height of the tower.

$$\angle DAC = \angle ACB = 30°$$

CB is the distance between the tree and the tower.

From right triangle ΔABC

$$\tan 30° = \frac{AB}{BC}$$

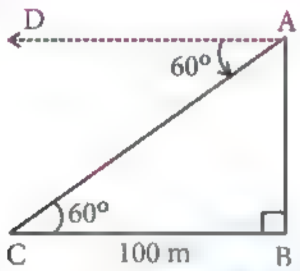
$$\frac{1}{\sqrt{3}} = \frac{30}{BC}$$

$$BC = 30\sqrt{3}\ m$$

$\therefore$ Distance between the tree and tower $= 30\sqrt{3}\ m$.

18. The angle of depression of a vehicle on the ground from the top of a tower is 60°. If the vehicle is at a distance of 100 m away from the building, find the height of the tower.

Sol :



Let AB is the tower.

$$\angle DAC = \angle ACB = 60°$$

In right triangle ΔABC

$$\tan 60° = \frac{AB}{BC}$$

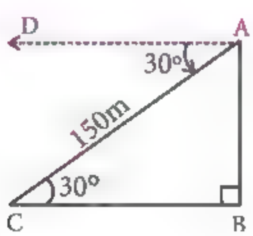
$$\sqrt{3} = \frac{AB}{100}$$

$$AB = 100\sqrt{3}\ m$$

$\therefore$ Height of the tower $= 100\sqrt{3}\ m$.

19. Anu was flying a kite on a hill, but he dumped his kite into the pond below. If the length of the string of his kite is 150 m and the angle of depression from his position to the kite is 30° then how high is the hill where he is standing?

Sol :



Let AB be the hill.

C be the pond.

$$\angle DAC = \angle ACB = 30°$$

Distance between the pond and top of the hill is AC = 150 m.

In right ΔABC

$$\sin 30° = \frac{AB}{AC}$$

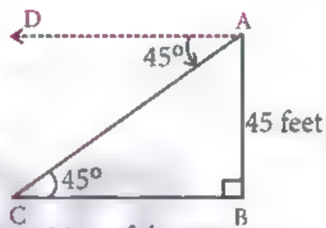
$$\frac{1}{2} = \frac{AB}{150}$$

$$AB = \frac{150}{2} = 75\text{ m}$$

Height of the hill = 75 m.

20. From the top of a fire tower, a forest ranger sees his partner on the ground at an angle of depression of 45°. If the tower is 45 feet in height, how far is the partner from the base of the tower?

Sol :



Let C be the position of the partner.

AB be the tower.

$$\angle DAC = \angle ACB = 45°$$

In right triangle ΔACB

$$\tan 45° = \frac{AB}{BC}$$

$$1 = \frac{45}{BC}$$

$$BC = 45\text{ ft}$$

$\therefore$ The partner is 45 ft far from the base of the tower.

III. Short Answer Questions:

1. Prove the trigonometric identity $\frac{\sin\theta}{1-\cos\theta} = \text{cosec }\theta + \cot\theta$

Sol :

$$\text{LHS} = \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{\sin\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$$

$$= \frac{\sin\theta\,(1+\cos\theta)}{1-\cos^2\theta} \qquad [\because 1-\cos^2\theta = \sin^2\theta]$$

$$= \frac{1}{\sin\theta}(1+\cos\theta) = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \operatorname{cosec}\theta + \cot\theta = \text{RHS}$$

2. Prove the trigonometric identity

$$\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\sec\theta+1}{\sec\theta-1}$$

Sol :

$$\text{LHS} = \frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta}$$

$$= \frac{\dfrac{\sin\theta}{\cos\theta}+\sin\theta}{\dfrac{\sin\theta}{\cos\theta}-\sin\theta}$$

$$= \frac{\sin\theta\left(\dfrac{1}{\cos\theta}+1\right)}{\sin\theta\left(\dfrac{1}{\cos\theta}-1\right)} = \frac{\sec\theta+1}{\sec\theta-1} = \text{RHS}$$

3. Prove the trigonometric identity

$$\cot\theta-\tan\theta = \frac{2\cos^2\theta-1}{\sin\theta\cos\theta}$$

Sol :

$$\text{LHS} = \cot\theta-\tan\theta$$

$$= \frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta-\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta-(1-\cos^2\theta)}{\sin\theta\cos\theta} \quad [\because \sin^2\theta = 1-\cos^2\theta]$$

$$= \frac{\cos^2\theta-1+\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos^2\theta-1}{\sin\theta\cos\theta} = \text{RHS}$$

4. Prove the trigonometric identity

$$\tan\theta-\cot\theta = \frac{2\sin^2\theta-1}{\sin\theta\cos\theta}$$

Sol :

$$\text{LHS} = \tan\theta-\cot\theta$$

$$= \frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta-\cos^2\theta}{\sin\theta\ \cos\theta}$$

$$= \frac{\sin^2\theta-(1-\sin^2\theta)}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta-1+\sin^2\theta}{\sin\theta\cos\theta} = \frac{2\sin^2\theta-1}{\sin\theta\cos\theta} = \text{RHS}$$

5. Prove the trigonometric identity

$$\sec^4\theta-\sec^2\theta = \tan^2\theta+\tan^4\theta$$

Sol :

$$\text{LHS} = \sec^4\theta-\sec^2\theta$$

$$= \sec^2\theta(\sec^2\theta-1)$$

$$= (1+\tan^2\theta)(1+\tan^2\theta-1)$$

$$[\because \sec^2\theta = 1+\tan^2\theta]$$

$$= (1+\tan^2\theta)\tan^2\theta$$

$$= \tan^2\theta+\tan^4\theta = \text{RHS}$$

6. Prove the trigonometric identity

$$(\operatorname{cosec}\theta-\cot\theta)^2 = \frac{1-\cos\theta}{1+\cos\theta}$$

Solution:

$$\text{LHS} = (\operatorname{cosec}\theta-\cot\theta)^2$$

$$= \left(\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1-\cos\theta}{\sin\theta}\right)^2$$

$$= \frac{(1-\cos\theta)^2}{\sin^2\theta} = \frac{(1-\cos\theta)^2}{1-\cos^2\theta}$$

$$= \frac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{1-\cos\theta}{1+\cos\theta} = \text{RHS}$$

7. Prove the trigonometric identity

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta-\tan\theta$$

Sol :

$$\text{LHS} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)}{1+\sin\theta} \times \frac{(1-\sin\theta)}{(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \sqrt{\left(\frac{1-\sin\theta}{\cos\theta}\right)^2} = \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{RHS}$$

8. Prove the trigonometric identity

$$\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$$

Sol :

$$\text{LHS} = \frac{1-\sin\theta}{1+\sin\theta} = \frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}$$

$$= \frac{(1-\sin\theta)^2}{1-\sin^2\theta} = \frac{(1-\sin\theta)^2}{\cos^2\theta}$$

$$= \left(\frac{1-\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2$$

$$= (\sec\theta - \tan\theta)^2 = \text{RHS}$$

9. Prove the trigonometric identity

$$\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = (\text{cosec }\theta - \cot\theta)$$

Sol :

$$\text{LHS} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta} \times \frac{1-\cos\theta}{1-\cos\theta}}$$

$$= \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} = \sqrt{\left(\frac{1-\cos\theta}{\sin\theta}\right)^2}$$

$$= \sqrt{\left(\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\right)^2}$$

$$= \sqrt{(\text{cosec }\theta - \cot\theta)^2}$$

$$= \text{cosec }\theta - \cot\theta = \text{RHS}$$

10. Prove the trigonometric identity

$$\frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta} = 2\sec\theta$$

Sol :

$$\text{LHS} = \frac{\cos\theta}{1-\sin\theta} + \frac{\cos\theta}{1+\sin\theta}$$

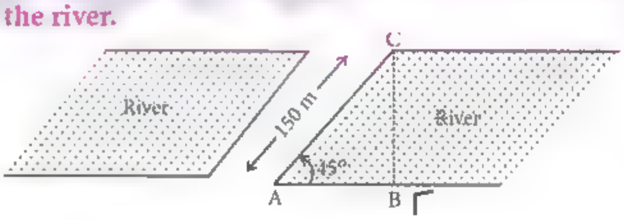
$$= \frac{\cos\theta(1+\sin\theta) + \cos\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{\cos\theta + \cos\theta\sin\theta + \cos\theta - \cos\theta\sin\theta}{1-\sin^2\theta}$$

$$= \frac{2\cos\theta}{\cos^2\theta}$$

$$= \frac{2}{\cos\theta} = 2\sec\theta = \text{RHS}$$

11. A bridge across a river makes an angle of 45° with the river bank. If the length of the bridge across the river is 150 m, what is the width of the river.



Sol :

Let BC be the width of the river.

AC is the length of the bridge.

From the right triangle CBA

$$\sin 45^\circ = \frac{BC}{AC}$$

$$\frac{1}{\sqrt{2}} = \frac{BC}{150}$$

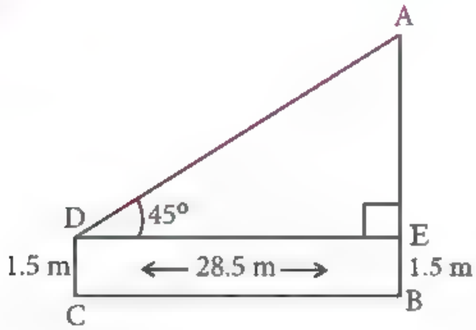
$$BC = \frac{150}{\sqrt{2}}$$

$$= \frac{75\sqrt{2}\sqrt{2}}{\sqrt{2}} = 75\sqrt{2}\,m$$

$\therefore$ Width of the river is $75\sqrt{2}\,m$.

12. An observer 1.5 m tall is 28.5 away from a tower. The angle of elevation of the top of the tower from her eyes is 45° what is the height of the tower?

Sol :



Let AB is the tower.

CD is the observer of height 1.5m.

CB is the Distance between the observer and tower.

From the right triangle ΔAED

$$\tan 45^\circ = \frac{AE}{DE}$$

$$1 = \frac{AE}{28.5}$$

$$AE = 28.5 \text{ m}$$

$$AB = AE + BE$$

$$= AE + DC$$

$$= (28.5 + 1.5) \text{ m} = 30 \text{ m}$$

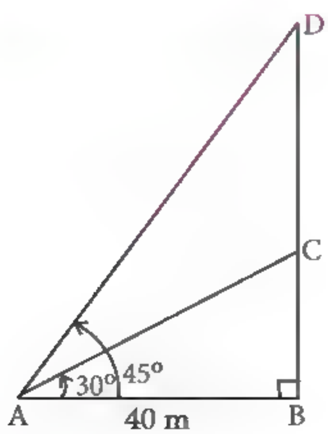
$\therefore$ Height of the tower is 30 m.

13. From a point on the ground 40 m away from the foot of a tower, the angle of elevation of the top of the tower is 30°. The angle of elevation of the top of the water tank on the tower is 45°. Find

(i) The height of the tower and

(ii) The depth of the tank.

Sol :



Let BC is the tower.

CD is the water tank.

In right triangle ΔABD

$$\tan 45^\circ = \frac{BD}{AB}$$

$$1 = \frac{BC+CD}{40}$$

$$BC + CD = 40 \text{ m} \qquad ...(1)$$

In the right triangle ΔABC

$$\tan 30^\circ = \frac{BC}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{BC}{40}$$

$$BC = \frac{40}{\sqrt{3}} m$$

$$BC = 23.1 \text{ m.}$$

Substituting BC = 23.1 m in (1)

$$23.1 + CD = 40$$

$$CD = 40 - 23.1$$

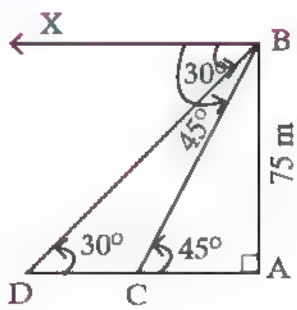
$$= 16.9 \text{ m.}$$

$\therefore$ Height of the tower = 23.1 m.

Depth of the tank = 16.9 m.

14. As observed from the top of a 75 m high lighthouse from the sea level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the light house find the distance between the two ships.

Sol :



Let C and D be the ships.

In right triangle ΔABC

$$\frac{AB}{AC} = \tan 45^\circ$$

$$\frac{75}{AC} = 1$$

$$AC = 75 \text{ m}$$

In right triangle ΔABD

$$\tan 30^\circ = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{75}{AD}$$

$$AD = 75\sqrt{3}$$

$$\therefore\ CD = AD - AC$$

$$= 75\sqrt{3} - 75 = 75(\sqrt{3}-1)$$

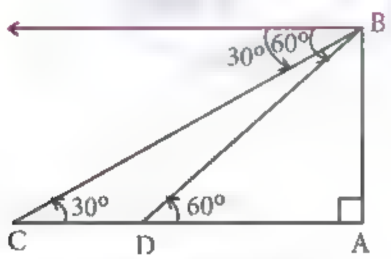
$$= 75\,[1.732 - 1]$$

$$= 75 \times 0.732 = 54.9 \text{ m}$$

$\therefore$ Distance between ships is 54.9 m.

15. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

Sol :



Let AB be the height of the tower C and D be the two position of the car.

In right ΔABD, we have,

$$\frac{AB}{AD} = \tan 60^\circ$$

$$\frac{AB}{AD} = \sqrt{3}$$

$$AB = \sqrt{3}\ AD \quad \text{... (1)}$$

In right triangle ΔABC

$$\frac{AB}{AC} = \tan 30^\circ$$

$$\frac{AB}{AC} = \frac{1}{\sqrt{3}}$$

$$AB = \frac{AC}{\sqrt{3}} \quad \text{... (2)}$$

From (1) and (2)

$$\sqrt{3}\ AD = \frac{AC}{\sqrt{3}}$$

$$AC = \sqrt{3}\times\sqrt{3}\times AD = 3\ AD$$

$$CD = AC - AD$$

$$= 3\ AD - AD = 2\ AD$$

Since the distance 2AD is covered in 6 second, the distance AD will be covered in 6/2 = 3 seconds.

IV. Long Answer Questions

1. If $\tan^2\theta = 1 - a^2$ prove that

$$\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-a^2)^{\frac{3}{2}}$$

Sol :

$$\text{LHS} = \sec\theta + \tan^3\theta\operatorname{cosec}\theta$$

$$= \sec\theta\left\{\frac{\sec\theta + \tan^3\theta\operatorname{cosec}\theta}{\sec\theta}\right\}$$

[$\because$ Multiplying and dividing by $\sec\theta$]

$$= \sec\theta\left\{\frac{\frac{1}{\cos\theta} + \tan^3\theta\operatorname{cosec}\theta}{\frac{1}{\cos\theta}}\right\}$$

$$= \sec\theta\left\{\frac{\frac{1+\tan^3\theta\cos\theta.\operatorname{cosec}\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\right\}$$

$$= \sec\theta\ \frac{(1+\tan^3\theta\operatorname{cosec}\theta\sec\theta}{\cos\theta} \times \frac{\cos\theta}{1}$$

$$= \sec\theta\left[1+\tan^3\theta\frac{\cos\theta}{\sin\theta}\right]$$

$$= \sec\theta\,(1+\tan^3\theta\cot\theta)$$

$$= \sqrt{1+\tan^2\theta}\left\{1+\tan^3\theta\times\frac{1}{\tan\theta}\right\}$$

$$= \sqrt{1+\tan^2\theta}\,(1+\tan^2\theta)$$

$$= (1+\tan^2\theta)^{\frac{1}{2}}\,(1+\tan^2\theta)$$

$$= (1+\tan^2\theta)^{\frac{3}{2}} = \left(1+(1-a^2)\right)^{\frac{3}{2}}$$

$$= \left(1+1-a^2\right)^{\frac{3}{2}} = (2-a^2)^{\frac{3}{2}} = \text{RHS}$$

2. If $a\cos\theta + b\sin\theta = m$ and $a\sin\theta - b\cos\theta = n$. prove that $a^2 + b^2 = m^2 + n^2$.

Sol :

Given
$$a\cos\theta + b\sin\theta = m$$
$$a\sin\theta - b\cos\theta = n$$

$$\text{RHS} = m^2 + n^2$$
$$= (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$
$$= a^2\cos^2\theta + b^2\sin^2\theta + ab\sin\theta\cos\theta + a^2\sin^2\theta + b^2\cos^2\theta - ab\sin\theta\cos\theta$$
$$= a^2(\sin^2\theta + \cos^2\theta) + b^2(\sin^2\theta + \cos^2\theta)$$
$$= a^2 + b^2 = \text{LHS}$$

3. If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, prove that $(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = 1$.

Sol :

Given $\operatorname{cosec}\theta - \sin\theta = m$

$$\Rightarrow \frac{1}{\sin\theta} - \sin\theta = m$$
$$\frac{1-\sin^2\theta}{\sin\theta} = m$$
$$\frac{\cos^2\theta}{\sin\theta} = m$$

Also $\sec\theta - \cos\theta = n$

$$\frac{1}{\cos\theta} - \cos\theta = n$$
$$\Rightarrow \frac{1-\cos^2\theta}{\cos\theta} = n$$
$$\Rightarrow \frac{\sin^2\theta}{\cos\theta} = n$$

$$\text{LHS} = (m^2n)^{2/3} + (mn^2)^{2/3}$$
$$= \left(\frac{\cos^4\theta}{\sin^2\theta} \times \frac{\sin^2\theta}{\cos\theta}\right)^{\frac{2}{3}} + \left(\frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right)^{\frac{2}{3}}$$
$$= (\cos^3\theta)^{\frac{2}{3}} + (\sin^3\theta)^{\frac{2}{3}}$$
$$= \cos^2\theta + \sin^2\theta$$
$$= 1 = \text{RHS}$$

4. If $\tan A = n\tan B$ and $\sin A = m\sin B$, Prove that $\cos^2 A = \frac{m^2-1}{n^2-1}$.

Sol : Given $\tan A = n\tan B$

$$\Rightarrow \tan B = \frac{1}{n}\tan A$$
$$\Rightarrow \frac{1}{\tan B} = \frac{n}{\tan A}$$
$$\Rightarrow \cot B = \frac{n}{\tan A} \quad \text{...(1)}$$

Also $\sin A = m\sin B$

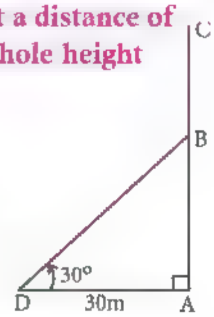
$$\Rightarrow \sin B = \frac{1}{m}\sin A$$
$$\Rightarrow \frac{1}{\sin B} = \frac{m}{\sin A}$$
$$\Rightarrow \operatorname{cosec} B = \frac{m}{\sin A} \quad \text{...(2)}$$

We know that $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Now $\operatorname{cosec}^2 B - \cot^2 B = 1$

$$\Rightarrow \frac{m^2}{\sin^2 A} - \frac{n^2}{\tan^2 A} = 1$$
$$\Rightarrow \frac{m^2}{\sin^2 A} - n^2\frac{\cos^2 A}{\sin^2 A} = 1$$
$$\frac{m^2 - n^2\cos^2 A}{\sin^2 A} = 1$$
$$m^2 - n^2\cos^2 A = \sin^2 A$$
$$m^2 = \sin^2 A + n^2\cos^2 A$$
$$m^2 = 1 - \cos^2 A + n^2\cos^2 A$$
$$\Rightarrow m^2 - 1 = n^2\cos^2 A - \cos^2 A$$
$$\Rightarrow m^2 - 1 = (n^2 - 1)\cos^2 A$$
$$\Rightarrow \frac{m^2-1}{n^2-1} = \cos^2 A$$
$$\cos^2 A = \frac{m^2-1}{n^2-1}$$

5. A tree is broken by the wind. the top struck the ground at an angle of 30° and at a distance of 30 m from the root. Find the whole height of the tree.



Sol :

Let AC be the tree.

BD be the broken part of the tree.

BD = BC

In the right triangle ΔABD

$$\tan 30° = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{30}$$

$$AB = \frac{30}{\sqrt{3}} m \quad ...(1)$$

Also In $\Delta\, ABD$

$$\cos 30° = \frac{AD}{BD}$$

$$\frac{\sqrt{3}}{2} = \frac{30}{BD}$$

$$BD = \frac{2 \times 30}{\sqrt{3}} m = \frac{60}{\sqrt{3}} \text{m} \quad ...(2)$$

$\therefore$ Height of the tree $= AB + BC$

$= AB + BD$

$= \frac{30}{\sqrt{3}} + \frac{60}{\sqrt{3}}$

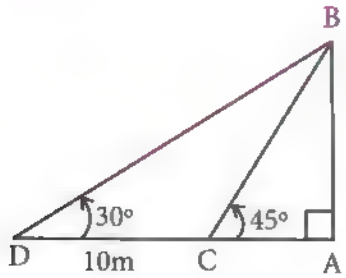
$= \frac{30+60}{\sqrt{3}} = \frac{90}{\sqrt{3}}$

$= \frac{30 \times \sqrt{3} \times \sqrt{3}}{\sqrt{3}} = 30\sqrt{3}$ m

$\therefore$ Height of the tree $= 30\sqrt{3}\, m$.

6. The shadow of a vertical tower on level ground increases by 10 m, when the altitude of the sun changes from angle of elevation 45° to 30°. Find the height of the tower, correct to one place of decimal $\left(\sqrt{3} = 1.732\right)$

Sol :



Let AB is the tower.

AC and AD are shadows when the angle of elevation of the sun are 45° and 30° respectively.

$$CD = 10 \text{ m}$$

In $\Delta\, CAB$

$$\tan 45° = \frac{AB}{AC}$$

$$1 = \frac{AB}{AC}$$

$$AC = AB \quad ...(1)$$

In the right triangle ΔDAB

$$\tan 30° = \frac{AB}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{AB}{AC+CD} = \frac{AB}{AC+10}$$

$$AC + 10 = AB\sqrt{3} \quad ...(2)$$

Using (1) $AC + 10 = \sqrt{3}\, AC$

$$\sqrt{3}\, AC - AC = 10$$

$$\left(\sqrt{3}-1\right) AC = 10$$

$$AC = \frac{10}{\sqrt{3}-1}$$

$$AC = \frac{10}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{10\left(\sqrt{3}+1\right)}{3-1}$$

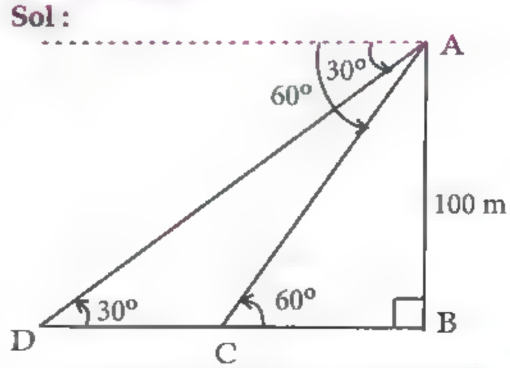
$$= \frac{10\left(\sqrt{3}+1\right)}{2} = 5\left(\sqrt{3}+1\right)$$

$$= 5\,(1.732 + 1) = 13.65 \text{ m}$$

$\therefore$ Height of the tower $= 13.65$ m.

7. As observed from the top of a light house 100 m high above sea level, the angle of depression of a ship sailing directly towards it, changes from 30° to 60°. Determine the distances travelled by the ship during the period of observation [$\sqrt{3}$ = 1.732]

Sol :



Let A represents the position of the observer

$$AB = 100$$

In right triangle ΔABC

$$\tan 60° = \frac{AB}{BC}$$

$$\sqrt{3} = \frac{100}{BC}$$

$$BC = \frac{100}{\sqrt{3}}$$

$$= \frac{100\sqrt{3}}{3}$$

$$= \frac{100 \times 1.732}{3} = 57.73 \text{ m}$$

In right triangle ΔABD

$$\frac{AB}{BD} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} = \frac{100}{BD}$$

$$BD = \sqrt{3} \times 100$$

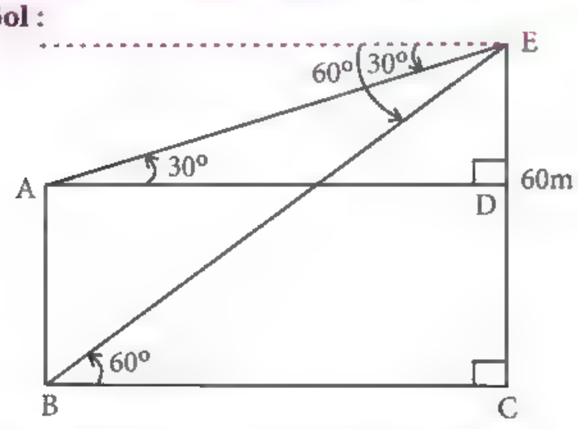
$$= 1.732 \times 100 = 173.2$$

$\therefore$ Distance travelled CD = BD – BC

$$= 173.2 - 57.73 = 115.47 \text{ m.}$$

8. From the top of a building 60 m high, the angles of depression of the top and bottom of a vertical lamp post are observed to be 30° and 60° respectively. Find (i) The horizontal distance between the building and the lamp post. (ii) The height of the lamp post ($\sqrt{3}$ = 1.732)

Sol :



Let CE be the building and AB be the lamp post

$$CE = 60 \text{ m}$$

In right ΔBCE

$$\frac{CE}{BC} = \tan 60°$$

$$\sqrt{3} = \frac{60}{BC}$$

$$BC = \frac{60}{\sqrt{3}}$$

$$= \frac{60\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{60\sqrt{3}}{3}$$

$$BC = 20\sqrt{3}\ m \qquad \text{... (1)}$$

In right triangle ΔADE

$$\tan 30° = \frac{DE}{AD}$$

$$\frac{1}{\sqrt{3}} = \frac{DE}{20\sqrt{3}}$$

[From (1) and BC = DE]

$$DE = \frac{20\sqrt{3}}{\sqrt{3}} = 20 \text{ m}$$

Height of the lamp post = AB = CD

= CE – DE

= 60 m – 20 m = 40 m

Distance between the lamp post and building

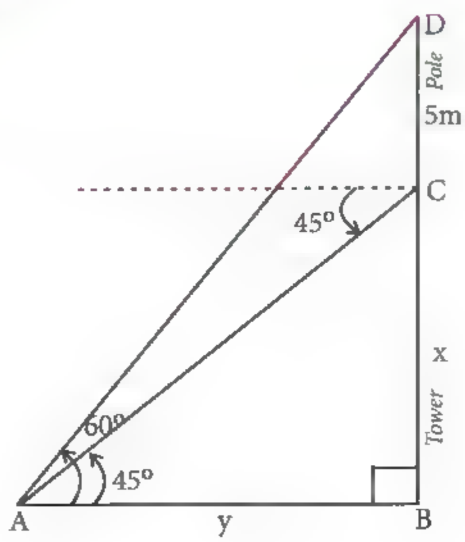
= $20\sqrt{3}\ m$

= 20 × 1.732 m

= 34.64 m.

9. A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point A on the ground is 60° and the angle of depression of the point A from the top of the tower is 45°. Find the height of the tower.

Sol :



In the figure, let BC be the tower and CD be the pole.

Let BC = 'x' m and AB = 'y' m

In right ΔABC

$$\frac{BC}{AB} = \tan 45^\circ = 1$$
$$BC = AB$$
$$y = x \quad \text{... (1)}$$

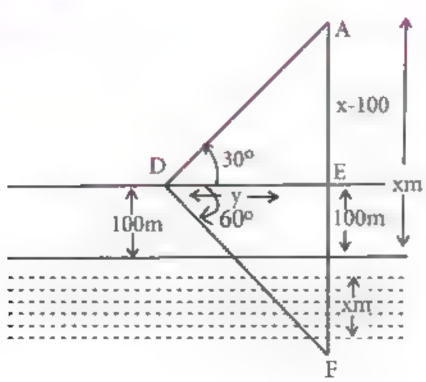
In right ΔABD

$$\frac{BD}{AB} = \tan 60^\circ = \sqrt{3}$$
$$\frac{x+5}{y} = \sqrt{3}$$
$$y\sqrt{3} = x+5$$
$$x\sqrt{3} = x+5 \quad [\because x = y \text{ from (1)}]$$
$$\sqrt{3}x - x = 5$$
$$(\sqrt{3}-1)x = 5$$
$$x = \frac{5}{\sqrt{3}-1}$$
$$= \frac{5}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$
$$= \frac{5(\sqrt{3}+1)}{3-1}$$
$$= \frac{5(1.732+1)}{2}$$
$$= \frac{5}{2} \times 2.732 = 6.83 \text{ m}$$

$\therefore$ Height of the tower is 6.83 m.

10. From a point 100 m above a lake the angle of elevation of a stationary helicopter is 30° and the angle of depression of reflection of the helicopter in the lake is 60°. Find the height of the helicopter.

Sol :



In the figure A is the stationary helicopter F is its reflection in the lake.

In right ΔAED

$$\tan 30^\circ = \frac{AE}{DE}$$
$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{AE}{DE}$$
$$\frac{1}{\sqrt{3}} = \frac{x-100}{y}$$
$$y = (x-100)\sqrt{3} \quad \text{... (1)}$$

In right ΔDEF

$$\tan 60^\circ = \frac{EF}{DE}$$
$$\frac{x+100}{y} = \sqrt{3}$$
$$\sqrt{3}\,y = x+100$$
$$y = \frac{(x+100)}{\sqrt{3}} \quad \text{... (2)}$$

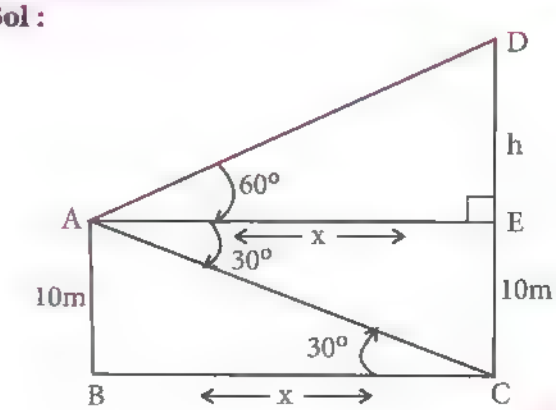
From (1) and (2) we have

$$\frac{x+100}{\sqrt{3}} = \sqrt{3}(x-100)$$
$$\sqrt{3} \times \sqrt{3}(x-100) = x+100$$
$$3(x-100) = x+100$$
$$3x - 300 - x - 100 = 0$$
$$2x = 400$$
$$x = 200$$

$\therefore$ Height of the helicopter = 200 m.

11. A man standing on the deck of a ship, which is 10 m above water level. He observes the angle of elevation of the top of a hill as 60° and the angle of depression of the base of the hill as 30°. Calculate the distance of the hill from the ship and the height of the hill.

Sol :



Let CD be the hill and the man is in A.

$$\angle EAD = 60^\circ ; \angle BCA = 30^\circ$$

In ΔAED

$$\tan 60^\circ = \frac{DE}{EA}$$

$$\sqrt{3} = \frac{h}{x}$$

$$h = \sqrt{3}\,x \qquad \ldots (1)$$

In ΔABC

$$\tan 30^\circ = \frac{AB}{BC}$$

$$\frac{1}{\sqrt{3}} = \frac{10}{x}$$

$$x = 10\sqrt{3} \qquad \ldots (2)$$

From (1) and (2) $h = \sqrt{3}\,(10\sqrt{3})$

$[\because x = 10\sqrt{3} \text{ in } (1)]$

$$= 10 \times 3 = 30$$

$$CD = CE + ED$$

$$= 10 + 30$$

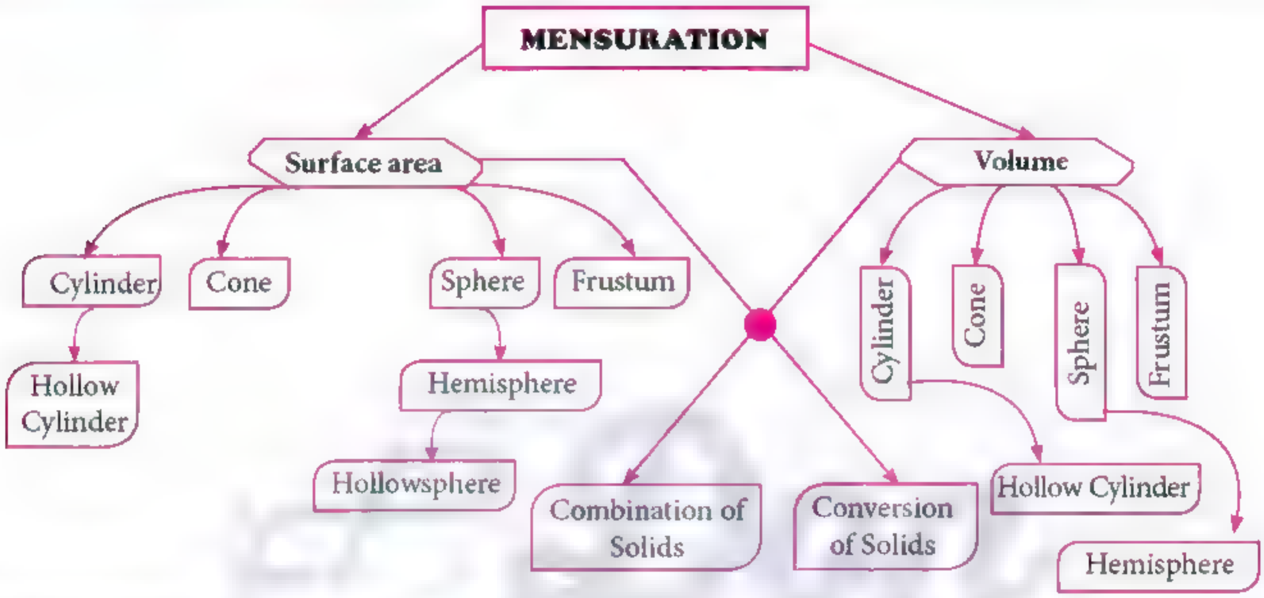
$$= 40 \text{ m}$$

Distance of the hill from the ship is $10\sqrt{3}\,m$

Height of the hill $= 40$ m

UNIT 7 MENSURATION

MIND MAP



SURFACE AREA

Key Points

- Surface area refers the term 'Total surface area'
- Use $\pi = \dfrac{22}{7}$ unless stated
- C.S.A of a right circular cylinder = $2\pi rh$ sq. units.
- T.S.A of a right circular cylinder = $2\pi r(h + r)$ sq. units
- C.S.A of a hollow cylinder = $2\pi(R + r)h$ sq. units
- T.S.A of a hollow cylinder = $2\pi(R + r)(R - r + h)$ sq. units
- C.S.A of a right circular cone = πrl sq. units
- T.S.A of a cone = $\pi r(l + r)$ sq. units
- C.S.A of a hemisphere = $2\pi r^2$ sq. units
- T.SA of a hemisphere = $3\pi r^2$ sq. units
- C.S.A of a hollow hemisphere = $2\pi(R^2 + r^2)$ sq. units
- T.S.A of a hollow hemisphere = $\pi(3R^2 + r^2)$ sq. units

Don

Worked Examples

7.1 **A cylindrical drum has a height of 20 cm and base radius of 14 cm. Find its curved surface area and the total surface area.**

Sol : Given that,

height of the cylinder h = 20 cm;

radius r = 14 cm

Now,

C.S.A of the cylinder = $2\pi rh$ square units

C.S.A of the cylinder = $2 \times \frac{22}{7} \times 14 \times 20$

$= 2 \times 22 \times 2 \times 20$

$= 1760 \text{ cm}^2$

T.S.A. of the cylinder = $2\pi r(h + r)$ sq. units

$= 2 \times \frac{22}{7} \times 14 \times (20 + 14)$

$= 2 \times \frac{22}{7} \times 14 \times 34$

$= 2992 \text{ cm}^2$

Therefore, C.S.A. = 1760 cm^2

and T.S.A. = 2992 cm^2

7.2 **The curved surface area of a right circular cylinder of height 14 cm is 88 cm². Find the diameter of the cylinder.**

Sol :

Given that, C.S.A of the cylinder = 88 sq. cm

$\Rightarrow \quad 2\pi rh = 88$

$\Rightarrow 2 \times \frac{22}{7} \times r \times 14 = 88$ (given h = 14 cm)

$2r = \frac{88 \times 7}{22 \times 14} = 2$

$= 2r = 2 \text{ cm}$

Therefore, diameter = 2 cm.

7.3 **A garden roller whose length is 3 m long and whose diameter is 2.8 m is rolled to level a garden. How much area will it cover in 8 revolutions?**

Sol :

Given that, diameter = 2r = 2.8 m and

height = 3m

radius r = 1.4 m

Area covered in one revolution = curved surface area of the cylinder

$= 2\pi rh$ square units

$= 2 \times \frac{22}{7} \times 1.4 \times 3 = 26.4$

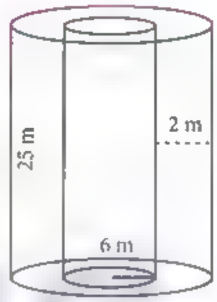
Area covered in 1 revolution = 26.4 m^2

Therefore, Area covered in 8 revolutions

$= 8 \times 26.4 = 211.2 \text{ m}^2$

7.4 **If one litre of paint covers 10 m², how many litres of paint is required to paint the internal and external surface areas of a cylindrical tunnel whose thickness is 2 m, internal radius is 6 m and height is 25 m?**

Sol :



Given that, height (h) = 25 m; thickness = 2 m.

internal radius (r) = 6 m

Now, external radius (R) = 6 + 2 = 8 m

C.S.A of cylindrical tunnel

= C.S.A of hollow cylinder

C.S.A of the hollow cylinder

$= 2\pi (R + r) h$ square units

$= 2 \times \frac{22}{7} (8 + 6) \times 25$

Hence, C.S.A. of the cylinder tunnel = 2200 m^2

Area covered by one litre of paint = 10 m^2

Therefore, No. of litres required to paint the

tunnel = $\frac{2200}{10} = 220$ litres.

$\therefore$ 220 litres of paint is needed to paint the tunnel.

7.5 **The radius of a conical tent is 7 m and the height is 24 m. Calculate the length of the canvas used to make the tent if the width of the rectangular canvas is 4 m?**

Sol:

Let r and h be the radius and height of the cone respectively.

Given that,

radius (r) = 7 m, height (h) = 24 m

Hence, $l = \sqrt{r^2 + h^2}$

$$= \sqrt{49 + 576}$$

$$l = \sqrt{625} = 25 \text{ m}$$

C.S.A of the conical tent $= \pi rl$ sq. units

Area of the canvas $= \frac{22}{7} \times 7 \times 25 = 550 \text{ m}^2$

Now, length of the canvas $= \frac{Area\ of\ canvas}{Width}$

$$= \frac{550}{4} = 137.5 \text{ m}$$

Therefore, the length of the canvas $=$ 137.5 m.

7.6 **If the total surface area of a cone of radius 7 cm is 704 cm² then find its slant height.**

Sol :

Given that, Radius r = 7 cm

Now, total surface area of the cone $= \pi r\ (l + r)$ square units

$$\text{T.S.A} = 704 \text{ cm}^2$$

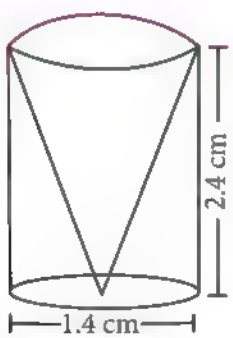
$$\Rightarrow \quad 704 = \frac{22}{7} \times 7\ (l + 7)$$

$$\Rightarrow \quad 32 = l + 7 \Rightarrow l = 25 \text{ cm}$$

Therefore, slant height of the cone $=$ 25 cm.

7.7 **From a solid cylinder whose height is 2.4 cm and diameter 1.4 cm, a conical cavity of the same height and base is hollowed out (Fig). Find the total surface area of the remaining solid.**

Sol :



Let h and r be the height and radius of the cone and cylinder respectively.

Let l be the slant height of the cone.

Given that, h = 2.4 cm and d = 1.4 cm

$$\Rightarrow \quad r = 0.7 \text{ cm}$$

Here, T.S.A of the remaining solid

= C.S.A of the cylinder + C.S.A of the cone + area of the bottom

$$= 2\pi rh + \pi rl + \pi r^2 \text{ sq. units}$$

Now, $l = \sqrt{r^2 + h^2} = \sqrt{0.49 + 5.76}$

$$= \sqrt{6.25} = 2.5 \text{ cm}$$

$$l = 2.5 \text{ cm}$$

Area of the remaining solid

$$= 2\pi rh + \pi rl + \pi r^2 \text{ sq. units}$$

$$= \pi r\ (2h + l + r)$$

$$= \frac{22}{7} \times 0.7 \times \left[(2 \times 2.4) + 2.5 + 0.7\right]$$

$$= 17.6 \text{ m}^2$$

Therefore, T.S.A of the remaining solid is 17.6 m².

7.8 **Find the diameter of a sphere whose surface area is 154 m².**

Sol :

Let r be the radius of the sphere.

Given that, surface area of sphere = 154 m²

$$4\pi r^2 = 154$$

$$\Rightarrow \quad 4 \times \frac{22}{7} \times r^2 = 154$$

$$\Rightarrow \quad r^2 = 154 \times \frac{1}{4} \times \frac{7}{22}$$

$$\Rightarrow \quad r^2 = \frac{49}{4} \Rightarrow r = \frac{7}{2}$$

Therefore, diameter $=$ 2r = 7 m.

7.9 **The radius of a spherical balloon increases from 12 cm to 16 cm as air being pumped into it. Find the ratio of the surface area of the balloons in the two cases.**

Sol :

Let r_1 and r_2 be the radii of the balloons.

Given that, $\frac{r_1}{r_2} = \frac{12}{16} = \frac{3}{4}$

Now, ratio of C.S.A of balloons

$$= \frac{4\pi r_1^2}{4\pi r_2^2} = \frac{r_1^2}{r_2^2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

Therefore, ratio of C.S.A of balloons is 9 : 16.

7.10 **If the base area of a hemispherical solid is 1386 sq. metres then find its total surface area.**

Sol :

Let r be the radius of the hemisphere.

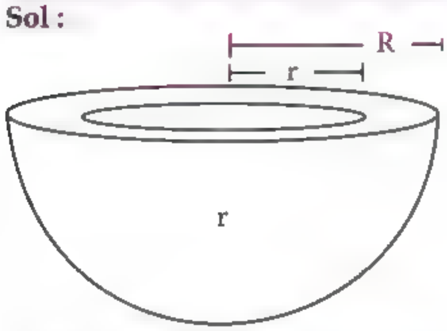
Given that, base area = πr^2 = 1386 sq. m

$$\text{T.S.A} = 3\pi r^2 \text{ sq. m}$$
$$= 3 \times 1386 = 4158 \text{ m}^2$$

Therefore, T.S.A of the hemispherical solid is 4158 m².

7.11 **The internal and external radii of a hollow hemispherical shell are 3 m and 5 m respectively. Find the T.S.A and C.S.A of the shell.**

Sol :



Let the internal and external radii of the hemispherical shell be r and R respectively.

Given that, R = 5 m, r = 3 m

C.S.A of the shell = $2\pi(R^2 + r^2)$ sq. units

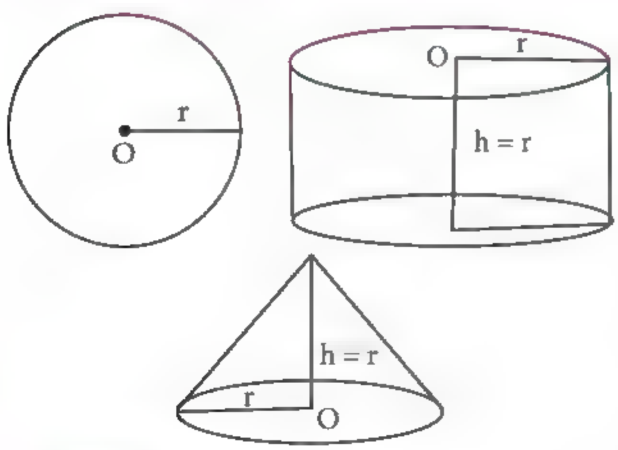
$= 2 \times \frac{22}{7} \times (25 + 9) = 213.71 \text{ m}^2$

T.S.A of the shell= $\pi(3R^2 + r^2)$ sq. units

$= \frac{22}{7}(75 + 9) = 264 \text{ m}^2$

Therefore, C.S.A= 213.71 m² and T.S.A = 264 m².

7.12 **A sphere, a cylinder and a cone (Figure) are of the same radius, where as cone and cylinder are of same height. Find the ratio of their curved surface areas.**



Sol :

Required Ratio = C.S.A of the sphere : C.S.A of the cylinder : C.S.A of the cone

$= 4\pi r^2 : 2\pi rh : \pi rl,$

$(l = \sqrt{r^2 + h^2} = \sqrt{2r^2} = \sqrt{2}r$ units)

$= 4 : 2 : \sqrt{2} = 2\sqrt{2} : \sqrt{2} : 1$

7.13 **The slant height of a frustum of a cone is 5 cm and the radii of its ends are 4 cm and 1 cm. Find its curved surface area.**

Sol :

Let l, R and r be the slant height, top radius and bottom radius of the frustum.

Given that, l = 5 cm, R = 4 cm, r = 1 cm

Now, C.S,A of the frustum

$$= \pi(R + r)l \text{ sq. units}$$
$$= \frac{22}{7} \times (4 + 1) \times 5$$
$$= \frac{550}{7}$$

Therefore, C.S.A = 78.57 cm².

7.14 **An industrial metallic bucket is in the shape of the frustum of a right circular cone whose top and bottom diameters are 10 m and 4 m and whose height is 4 m. Find the curved and total surface area of the bucket.**

Sol :

Let h, l, R and r be the height, slant height, outer radius and inner radius of the frustum.

Given that, diameter of the top = 10 m;

Radius of the top R = 5 m.

Diameter of the bottom = 4 m;

Radius of the bottom r = 2 m, Height h = 4 m

Now, $l = \sqrt{h^2 + (R - r)^2}$

$= \sqrt{4^2 + (5 - 2)^2}$

$l = \sqrt{16 + 9} = \sqrt{25} = 5$ m

Here, C.S.A = $\pi(R + r)l$ sq. units

$$= \frac{22}{7}(5+2)\times 5 = 110\ m^2$$

$$T.S.A = \pi(R+r)l + \pi R^2 + \pi r^2 \text{ sq. units}$$

$$= \frac{22}{7}\left[(5+2)5 + 25 + 4\right]$$

$$= \frac{1408}{7} = 201.14\ m^2$$

Therefore, C.S.A= 110 m^2 and T.S.A = 201.14 m^2.

Progress Check

1. Right circular cylinder is a solid obtained by revolving _____ about _____.

Ans : Rectangle, one of its sides as axis.

2. In a right circular cylinder the axis is _____ to the diameter.

Ans : Perpendicular.

3. The difference between the C.S.A and T.S.A of a right circular cylinder is _____.

Ans : $2\pi r^2$ i.e., area of two circles.

4. The C.S.A of a right circular cylinder of equal radius and height is _____ the area of its base.

Ans : 2 times.

5. Right circular cone is a solid obtained by revolving _____ about _____.

Ans : right angled triangle, one of its sides containing right angle

6. In a right circular cone the axis is _____ to the diameter.

Ans : Perpendicular

7. The difference between the C.S.A and T.S.A of a cone is _____.

Ans : πr^2 i.e., area of base

8. When a sector of a circle is transformed to form a cone, then match the conversions taking place between the sector and the cone.

Sector	Cone
Radius	Circumference of the base
Area	Slant height
Arc length	Curved surface area

Ans :

Sector	Cone
Radius	Slant height
Area	Curved surface area
Arc length	Circumference of the base

9. Every section of a sphere by a plane is a _____.

Ans : circle.

10. The centre of a great circle is at the _____ of the sphere.

Ans : diameter.

11. The difference between the T.S.A and C.S.A of hemisphere is _____.

Ans : πr^2 . i.e., area of circle.

12. The ratio of surface area of a sphere and C.S.A of hemisphere is _____.

Ans : 3 : 2

13. A section of the sphere by a plane through any of its great circle is _____.

Ans : Largest.

14. The portion of a right circular cone intersected between two parallel planes is _____.

Ans : Frustum of a cone.

15. How many frustums can a right circular cone have?

Ans : Only one.

Thinking Corner

1. When 'h' coins each of radius 'r' units and thickness 1 unit is stacked one upon the other, what would be the solid object you get? Also find its C.S.A.

Ans : The solid is a cylinder

radius = r

height = h × 1 = h

C.S.A = $2\pi rh$ sq. units

2. When the radius of a cylinder is double its height, find the relation between its C.S.A and base area.

Ans :

Radius is double the height $r = 2h$

C.S.A $= 4\pi h^2$ as $2\pi(2h)h$

Base area $= 4\pi h^2$ as $\pi(2h)^2$

C.S.A $=$ Base area

3. Two circular cylinders are formed by rolling two rectangular aluminium sheets each of dimensions 12 m length and 5 m breadth, one by rolling along its length and the other along its width. Find the ratio of their curved surface areas.

Ans :

$$\left.\begin{matrix} \textit{ratio of curved} \\ \textit{surface areas} \end{matrix}\right\} = \frac{2\pi r(12)}{2\pi\, r(5)} = \frac{12}{5} = 12:5$$

4. Give practical example of solid cone.

Ans : Ice cream Cone

5. Find surface area of a cone in terms of its radius when height is equal to radius.

Ans :

$$h = r,$$

$$l = \sqrt{h^2 + r^2} = \sqrt{r^2 + r^2} = \sqrt{2}r$$

$$\text{Surface Area} = \pi r\,(l + r)$$

$$= \pi r\left(\sqrt{2}r + r\right)$$

$$= \pi r^2\left(\sqrt{2} + 1\right)$$

6. Compare the above surface area with the area of the base of the cone.

Ans :

Area of base of cone $= \pi r^2$

Surface area of the cone obtained in (5)

$$= \left(\sqrt{2}+1\right)\pi r^2 = \left(\sqrt{2}+1\right) \text{ times more}$$

7. Find the value of the radius of a sphere whose surface area is 36π square units.

Ans :

Surface area $= 36\pi$

$4\pi r^2 = 36\pi \Rightarrow r^2 = 9 \Rightarrow r = 3$

8. How many great circles can a sphere have?

Ans : Two circles

9. Find the surface area of the earth whose diameter is 12756 kms.

Ans :

Diameter of earth $= 12756$ kms

$$\text{Radius} \quad r = \frac{12756}{2} = 6378 \text{ kms}$$

$$\text{Surface Area} = 4\pi r^2 = 4 \times \frac{22}{7} \times (6378)^2$$

$$= \frac{3579741792}{7}$$

$$= 511391684.571 \text{ sq. km}$$

10. Shall we get a hemisphere when a sphere is cut along the small circle?

Ans : No, it is not possible to get the hemisphere, when a sphere is cut along the small circle.

11. T.S.A of a hemisphere is equal to how many times the area of its base?

Ans : 3 times.

12. How many hemispheres can be obtained from a given sphere?

Ans : 2 hemispheres.

13. Give two real life examples for a frustum of a cone.

Ans : Bucket, Table lamp.

14. Can a hemisphere be considered as a frustum of a sphere?

Ans : No.

Exercise 7.1

1. The radius and height of a cylinder are in the ratio 5 : 7 and its curved surface area is 5500 sq.cm. Find its radius and height.

Sol :

Given that radius and height of a cylinder are in the ratio 5 : 7

$$\text{i.e., } \frac{r}{h} = \frac{5}{7} \Rightarrow h = \frac{7r}{5}$$

Curved surface area $= 5500$ sq. cm

$$2\pi rh = 5500$$

$$\Rightarrow 2 \times \frac{22}{7} \times r \times \frac{7r}{5} = 5500$$

$$\Rightarrow \quad r^2 = \frac{5500 \times 5}{2 \times 22}$$

$$\Rightarrow \quad r^2 = 625 \Rightarrow r = 25$$

$$\therefore \quad h = \frac{7(25)}{5} = 35$$

radius = 25 cm, height = 35 cm

2. A solid iron cylinder has total surface area of 1848 sq. m. Its curved surface area is five - sixth of its total surface area. Find the radius and height of the iron cylinder.

Sol :

Given total surface area of cylinder $= 1848$ sq. m

i.e., $2\pi r\ (h + r) = 1848$

It is given that C.S.A $= \frac{5}{6}$ (T.S.A)

$$\text{C.S.A} = \frac{5}{6}(1848) = 1540$$

$$\text{C.S.A} = \frac{5}{6}(\text{T.S.A})$$

$$\Rightarrow \quad 2\pi rh = \frac{5}{6}(2\pi r\,(h+r))$$

$$\Rightarrow \quad h = 5r$$

we have C.S.A $= 1540$

$$2\pi rh = 1540$$

$$2 \times \frac{22}{7} \times r \times 5r = 1540$$

$$r^2 = \frac{1540 \times 7}{44 \times 5} = 49$$

$$r = 7$$

$$\therefore \quad h = 5r = 5(7) = 35$$

radius $= 7$ m, height $= 35$ m

3. The external radius and the length of a hallow wooden log are 16 cm and 13 cm respectively. If its thickness is 4 cm then find its T.S.A.

Sol :

External radius of hollow cylinder R = 16 cm

length $h = 13$ cm

Thickness $R - r = 4$

$$\Rightarrow \quad 16 - r = 4$$

$$r = 12 \text{ cm}$$

$$\left.\begin{array}{l}\textit{Total surface area}\\ \textit{of hollow cylinder}\end{array}\right\} = 2\pi\ (R+r)(R-r+h) \text{ sq. units}$$

$$= 2 \times \frac{22}{7} \times (16 + 12)\,(4 + 13)$$

$$= 2 \times \frac{22}{7} \times 28 \times 17$$

$$= 2992 \text{ sq. cm}$$

4. A right angled triangle PQR where $\angle Q = 90°$ is rotated about QR and PQ. If QR = 16 cm and PR = 20 cm, compare the curved surface areas of the right circular cones so formed by the triangle.

Sol :

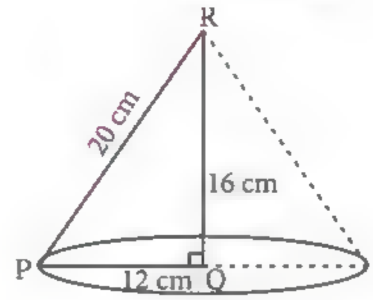
Right triangle PQR, right angled at Q and PR = 20 cm, QR = 16 cm

$$\therefore \quad PQ^2 = PR^2 - QR^2$$

$$= (20)^2 - (16)^2$$

$$= 400 - 256 = 144$$

$$\therefore \quad PQ = 12 \text{ cm}$$

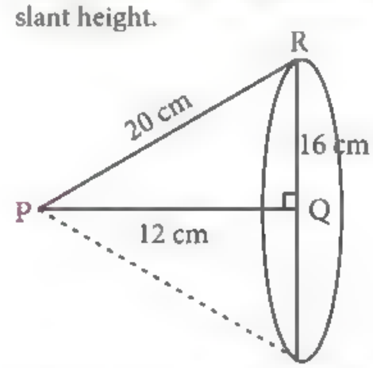


When right triangle PQR, rotates about QR, a right circular cone is formed with PQ = 12 cm as base radius and PR = 20 cm as slant height.

C.S.A of the Cone $= \pi rl$ sq. units

$$= \frac{22}{7} \times 12 \times 20 = 754.29 \text{ cm}^2$$

When right triangle PQR, rotates about PQ, a right circular cone is formed with QR = 16 cm as base radius and PR = 20 cm as slant height.



C.S.A of the Cone $= \pi rl$ sq.units

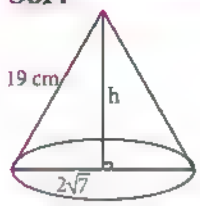
$$= \frac{22}{7} \times 16 \times 20$$

$$= 1005.71 \text{ cm}^2$$

Hence, C.S.A of the cone when rotates about PQ is larger.

5. 4 persons live in a conical tent whose slant height is 19 cm. If each person requires 22 cm² of the floor area, then find the height of the tent.

Sol :



Each person requires 22 cm² of floor area.

$\therefore$ Required base area $= 22 \times 4 = 88$ cm²

$$\Rightarrow \quad \pi r^2 = 88$$

$$r^2 = \frac{88 \times 7}{22} = 4 \times 7$$

$$r = 2\sqrt{7} \text{ cm}$$

$$\text{slant height} = 19 \text{ cm}$$

$$\therefore \text{ height of the tent, h} = \sqrt{l^2 - r^2}$$

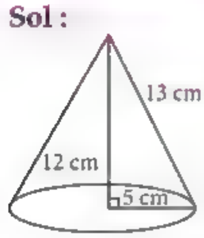
$$= \sqrt{(19)^2 - \left(2\sqrt{7}\right)^2}$$

$$= \sqrt{361 - 28} = \sqrt{330} \simeq 18.25 \text{ cm}$$

$\therefore$ Height of the tent = 18.25 cm.

6. A girl wishes to prepare birthday caps in the form of right circular cones for her birthday party, using a sheet of paper whose area is 5720 cm², how many caps can be made with radius 5 cm and height 12 cm?

Sol :



Area of the paper $= 5720$ cm²

Given radius of birthday cap r $= 5$ cm

height of birthday cap h $= 12$ cm

$$\therefore \text{ slant height } \quad l = \sqrt{h^2 + r^2}$$

$$= \sqrt{12^2 + 5^2} = \sqrt{144 + 25}$$

$$= \sqrt{169} = 13 \text{ cm}$$

CSA of conical cap $= \pi r l$ sq. units

$$= \frac{22}{7} \times 5 \times 13 = \frac{1430}{7}$$

$\therefore$ Number of birthday caps

$$= \frac{\textit{Area of paper sheet}}{\textit{CSA of conical cap}}$$

$$= \frac{5720}{1430} \times 7 = 28 \text{ caps}$$

7. The ratio of the radii of two right circular cones of same height is 1 : 3. Find the ratio of their curved surface area when the height of each cone is 3 times the radius of the smaller cone.

Sol :

Let the radii of two cones be r_1 and r_2 and heights be h_1 and h_2.

$$\text{Given ratio of their radii} = \frac{r_1}{r_2} = \frac{1}{3}$$

$$\Rightarrow \quad r_1 = \frac{r_2}{3}$$

$$\text{and} \quad h_1 = 3r_1, h_2 = 3r_1$$

[$\because$ r_1 is the radius of smaller cone]

$$\text{Slant heights} \quad l_1 = \sqrt{h_1^2 + r_1^2}$$

$$= \sqrt{9r_1^2 + r_1^2} = \sqrt{10}\ r_1$$

$$l_2 = \sqrt{h_2^2 + r_2^2}$$

$$= \sqrt{9r_1^2 + 9r_1^2} = \sqrt{18r_1^2} = 3\sqrt{2}\ r_1$$

Ratio of curved surface areas

$$= \frac{\textit{CSA of I cone}}{\textit{CSA of II cone}}$$

$$= \frac{\pi r_1 l_1}{\pi r_2 l_2} = \frac{r_1\left(\sqrt{10}\ r_1\right)}{(3r_1)\left(3\sqrt{2}\ r_1\right)}$$

$$= \frac{\sqrt{10}}{9\sqrt{2}} = \frac{\sqrt{5}\sqrt{2}}{9\sqrt{2}} = \frac{\sqrt{5}}{9}$$

Ratio of C.S.A $= \sqrt{5} : 9$

8. The radius of a sphere increases by 25%. Find the percentage increase in its surface area.

Sol :

Let the radius of the sphere be 'r' cm

Surface area $= 4\pi r^2$

when radius is increased by 25%, then new diameter $= r + 25\%$ of r

$$= r + \frac{25\,r}{100} = \frac{5r}{4}$$

Surface area of new sphere

$$= 4\pi\left(\frac{5r}{4}\right)^2$$

$$= 4\pi\left(\frac{25r^2}{16}\right)$$

$$= \frac{25\,\pi r^2}{4}$$

$$\text{Increase in surface area} = \frac{25\pi r^2}{4} - 4\pi r^2$$
$$= \frac{25\pi r^2 - 16\pi r^2}{4}$$
$$= \frac{9\pi r^2}{4}$$

$\therefore$ Percentage increase in surface area

$$= \frac{9\pi r^2/4}{4\pi r^2} \times 100\%$$
$$= \frac{900}{16}\% = 56.25\%$$

9. The internal and external diameters of a hollow hemispherical vessel are 20 cm and 28 cm respectively. Find the cost to paint the vessel all over at ₹ 0.14 per cm².

Sol :

Internal diameter = 20 cm
External diameter = 28 cm
Internal radius = 10 cm
External radius = 14 cm

$$\text{Total surface area} = \pi(3R^2 + r^2) \text{ sq. units}$$
$$= \frac{22}{7}(3(14)^2 + (10)^2)$$
$$= \frac{22}{7}[588 + 100]$$
$$= \frac{22}{7} \times 688$$
$$= \frac{15136}{7} \text{ cm}^2$$

Cost of painting per sq. cm = ₹ 0.14

$$\therefore \text{Total cost} = \frac{15136}{7} \times 0.14$$
$$= ₹\ 302.72$$

10. The frustum shaped outer portion of the table lamp has to be painted including the top part. Find the total cost of painting the lamp if the cost of painting 1 sq.cm is ₹ 2.

Sol :

From the figure $r = 6$ cm
$R = 12$ cm
$h = 8$ cm

$$l = \sqrt{h^2 + (R - r)^2}$$
$$= \sqrt{8^2 + (12 - 6)^2}$$
$$= \sqrt{64 + 36}$$
$$= \sqrt{100} = 10 \text{ cm}$$

Area to be painted = C.S.A + area of top circular region

$$= \pi(R + r)l + \pi r^2$$
$$= \frac{22}{7}(12 + 6)(10) + \frac{22}{7}(6)^2$$
$$= \frac{22}{7}(180) + \frac{22}{7}(36)$$
$$= \frac{22}{7}(180 + 36)$$
$$= \frac{22}{7}(216) = \frac{4752}{7} \simeq 678.86$$

Cost of painting per sq. cm = ₹ 2

$\therefore$ Total cost = 678.86 × 2 = ₹ 1357.72

VOLUME

Key Points

Volume refers to the amount of space occupied by an object. The volume is measured in cubic units.

- Volume of a cylinder = $\pi r^2 h$ cu. units.
- Volume of a hollow cylinder = $\pi(R^2 - r^2)h$ cu. units
- Volume of a cone = $\frac{1}{3}\pi r^2 h$ cu. units
- Volume of a sphere = $\frac{4}{3}\pi r^3$ cu. units
- Volume of a hollow sphere = $\frac{4}{3}\pi(R^3 - r^3)$ cu. units
- Volume of a solid hemisphere = $\frac{2}{3}\pi r^3$ cu. units
- Volume of a hollow hemisphere = $\frac{2}{3}\pi(R^3 - r^3)$ cu. units
- Volume of a frustum = $\frac{\pi h}{3}(R^2 + Rr + r^2)$ cu. units

Worked Examples

7.15 **Find the volume of a cylinder whose height is 2 m and whose base area is 250 m².**

Sol :

Let r and h be the radius and height of the cylinder respectively.

Given that, height h = 2 m, base area = 250 m²

Now, volume of a cylinder = $\pi r^2 h$ cu. units

$= \text{base area} \times h$

$= 250 \times 2 = 500 \text{ m}^3$

Therefore, volume of the cylinder = 500 m³.

7.16 **The volume of a cylindrical water tank is 1.078×10^6 litres. If the diameter of the tank is 7 m, find its height.**

Sol :

Let r and h be the radius and height of the cylinder respectively.

Given that,

volume of the tank= $1.078 \times 10^6 = 1078000$ litre

$= 1078 \text{ m}^3$

(since $1 \; litre = \frac{1}{1000} m^3$)

Diameter $= 7 \text{ m} \Rightarrow$ Radius $= \frac{7}{2}$ m

Volume of the tank $= \pi r^2 h$ cu. units

$1078 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times h$

Therefore, height of the tank h = 28 m.

7.17 **Find the volume of the iron used to make a hollow cylinder of height 9 cm and whose internal and external radii are 21 cm and 28 cm respectively.**

Sol :

Let r, R and h be the internal radius, external radius and height of the hollow cylinder respectively.

Given that, r = 21 cm, r = 28 cm, h = 9 cm

Now, volume of hollow cylinder

$= \pi(R^2 - r^2)h$ cu. units

$= \frac{22}{7}(28^2 - 21^2) \times 9$

$= \frac{22}{7}(784 - 441) \times 9 = 9702$

Therefore, volume of iron used = 9402 cm³.

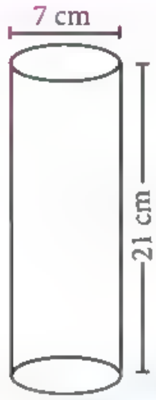
7.18 **For the cylinders A and B (Figure),**

(i) Find out the cylinder whose volume is greater?

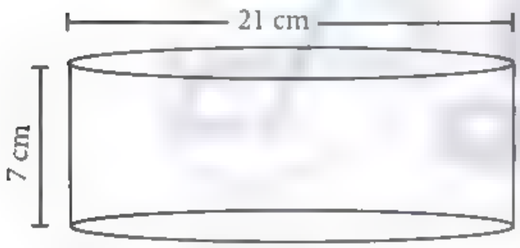
(ii) Verify whether the cylinder with greater volume has greater total surface area.

(iii) Find the ratios of the volumes of the cylinders A and B.

Sol :



Cylinder A



Cylinder B

(i) Volume of Cylinder= $\pi r^2 h$ Cu. units

$$\text{Volume of Cylinder A} = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 21$$

$$= 808.5 \text{ cm}^3$$

$$\text{Volume of Cylinder B} = \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 7$$

$$= 2425.5 \text{ cm}^3$$

Therefore, volume of cylinder B is greater than volume of cylinder A.

(ii) T.S.A of cylinder $= 2\pi r\,(h + r)$ sq. units

$$\text{T.S.A of Cylinder A} = 2 \times \frac{22}{7} \times \frac{7}{2} \times (21 + 3.5)$$

$$= 539 \text{ cm}^2$$

$$\text{T.S.A of Cylinder B} = 2 \times \frac{22}{7} \times \frac{21}{2} \times (7 + 10.5)$$

$$= 1155 \text{ cm}^2$$

Hence verified that Cylinder B with greater volume has a greater surface area.

(iii) $$\frac{\textit{Volume of cylinder A}}{\textit{Volume of cylinder B}} = \frac{808.5}{2425.5} = \frac{1}{3}$$

Therefore, ratio of the volumes of cylinders A and B is 1 : 3.

7.19 **The volume of a solid right circular cone is 11088 cm³. If its height is 24 cm then find the radius of the cone.**

Sol :

Let r and h be the radius and height of the cone respectively.

Given that, volume of the cone = 11088 cm³

$$\Rightarrow \quad \frac{1}{3}\pi r^2 h = 11088$$

$$\Rightarrow \quad \frac{1}{3} \times \frac{22}{7} \times r^2 \times 24 = 11088$$

$$\Rightarrow \quad r^2 = 441$$

Therefore, radius of the cone r = 21 cm.

7.20 **The ratio of the volumes of two cones is 2 : 3. Find the ratio of their radii if the height of second cone is double the height of the first.**

Sol :

Let r_1 and h_1 be the radius and height of the cone-I and let r_2 and h_2 be the radius and height of the cone-II.

Given $h_2 = 2h_1$ and $\dfrac{\textit{Volume of the cone I}}{\textit{Volume of the cone II}} = \dfrac{2}{3}$

$$\Rightarrow \quad \frac{\frac{1}{3}\pi r_1^2 h_1}{\frac{1}{3}\pi r_2^2 h_2} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{r_1^2}{r_2^2} \times \frac{h_1}{2h_1} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{r_1^2}{r_2^2} = \frac{4}{3} \Rightarrow \frac{r_1}{r_2} = \frac{2}{\sqrt{3}}$$

Therefore, ratio of their radii $= 2 : \sqrt{3}$

7.21 **The volume of a solid hemisphere is 29106 cm³. Another hemisphere whose volume is two-third of the above is carved out. Find the radius of the new hemisphere.**

Sol :

Let r be the radius of the hemisphere.

Given that,

volume of the hemisphere = 29106 cm³

Now, volume of new hemisphere

$$= \frac{2}{3} \times (\text{Volume of original sphere})$$

$$= \frac{2}{3} \times 29106$$

Volume of new hemisphere $= 19404 \text{ cm}^3$

$$\Rightarrow \quad \frac{2}{3}\pi r^3 = 19404$$

$$\Rightarrow \quad r^3 = \frac{19404 \times 3 \times 7}{2 \times 22} = 9261$$

$$\Rightarrow \quad r = \sqrt[3]{9261} = 21 \text{ cm}$$

Therefore, $r = 21$ cm

7.22 **Calculate the weight of a hollow brass sphere if the inner diameter is 14 cm and thickness is 1 mm and whose density is 17.3 g/cm³.**

Sol :

Let r and R the inner and outer radii of the hollow sphere.

Given that, inner diameter $d = 14$ cm;

Inner radius $r = 7$ cm;

$$\text{Thickness} = 1 \text{ mm} = \frac{1}{10} \text{ cm}$$

$$\text{Outer radius} \quad R = 7 + \frac{1}{10} = \frac{71}{10} = 7.1 \text{ cm}$$

$$\text{Volume of hollow sphere} = \frac{4}{3}\pi (R^3 - r^3) \text{ cu. cm}$$

$$= \frac{4}{3} \times \frac{22}{7} (357.91 - 343) = 62.48 \text{ cm}^3$$

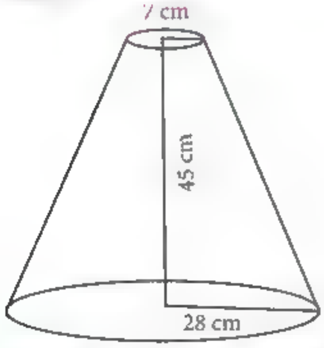
But, weight of brass in $1 \text{ cm}^3 = 17.3$ gm

Total weight $= 17.3 \times 62.48 = 1080.90$ gm

Therefore, Total weight = 1080.90 grams.

7.23 **If the radii of the circular ends of a frustum which is 45 cm high are 28 cm and 7 cm, find the volume of the frustum.**

Sol :



Let h, r and R be the height, top and bottom radii of the frustum.

Given that, $h = 45$ cm, $R = 28$ cm, $r = 7$ cm

$$\text{Now, Volume} = \frac{1}{3}\pi [R^2 + Rr + r^2]h \text{ cu. units}$$

$$= \frac{1}{3} \times \frac{22}{7} \times [28^2 + (28 \times 7) + 7^2] \times 45$$

$$= \frac{1}{3} \times \frac{22}{7} \times 1029 \times 45 = 48510$$

Therefore, volume of the frustum $= 48510 \text{ cm}^3$.

Progress Check

1. **Volume of a cone is the product of its base area and ____**

   Ans : One third of its height.

2. **If the radius of the cone is doubled, the new volume will be ____ times the original volume.**

   Ans : 4 times

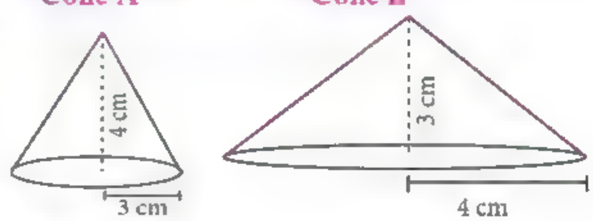
3. **Consider the cones given**

   (i) **Without doing any calculation, find out whose volume is greater?**

   (ii) **Verify whether the cone with greater volume has greater surface area.**

   (iii) **Volume of Cone A: Volume of Cone B = ?**

Cone A Cone B



Ans :

(i) Volume of cone B is greater as the radius is greater

(ii) **Volume of cone A** | **Volume of cone B**

$= \frac{1}{3}\pi(3)^2(4) = 12\pi$ | $= \frac{1}{3}\pi(4)^2(3) = 16\pi$

Surface area of cone A | **Surface area of cone B**

T.S.A $= \pi r(l + r) = \pi(3)(5+3) = 24\pi$ | T.S.A $= \pi(4)(5+4) = 36\pi$

(iii) **Surface area of cone B > Surface area of cone A**

$$\frac{\textit{Volume of cone A}}{\textit{Volume of cone B}} = \frac{12\pi}{16\pi} = 3:4$$

4. What is the ratio of volume to surface area of sphere?

Ans : $\dfrac{Volume\ of\ Sphere}{Volume\ of\ Surface\ area} = \dfrac{\frac{4}{3}\pi r^3}{4\pi r^2} = r:3.$

5. The relationship between the height and radius of the hemisphere is _____.

Ans : Height = radius

6. The volume of a sphere is the product of its surface area and _____.

Ans : One third of its radius

Thinking Corner

1. If the height is inversely proportional to the square of its radius, the volume of the cylinder is _____.

Ans :

$$\text{Height} \; \alpha \; \frac{1}{(\text{radius})^2}$$

$$\text{Height} = k\left(\frac{1}{(\text{radius})^2}\right) \quad [\because \text{ k – constant}]$$

$$\text{Volume} = \pi r^2 h$$

$$= \pi r^2\left(\frac{k}{r^2}\right) = k\pi.$$

2. What happens to the volume of the cylinder with radius r and height h, when its

(a) Radius is halved (b) Height is halved.

Ans : (a) radius is halved then radius $\frac{r}{2}$, height is h

$$\text{Volume} = \pi r^2 h = \pi\left(\frac{r}{2}\right)^2 h = \frac{\pi r^2 h}{4}$$

$$= \frac{1}{4} \text{ (Volume of original cylinder).}$$

(b) When height is halved, then height = $\frac{h}{2}$

$$\text{Volume} = \pi r^2\left(\frac{h}{2}\right) = \frac{\pi r^2 h}{2}$$

$$= \frac{1}{2} \text{ (Volume of original cylinder).}$$

3. Is it possible to find a right circular cone with equal.

(a) Height and slant height

(b) Radius and slant height

(c) Height and radius.

Ans :

(a) height = Slant height

i.e., h = l = cone is not possible

(b) r = l = cone is not possible

(c) h = r = cone is possible.

4. There are two cones with equal volumes. What will be the ratios of their radius and heights?

Ans : Ratios of radii and heights are same.

5. A cone, a hemisphere and a cylinder have equal bases. The heights of the cone and cylinder are equal and are same as the common radius. Are they equal in volume?

Ans : No.

6. Give any two real life examples of sphere and hemisphere.

Ans : Sphere – Globe, Ball

Hemisphere – Left side of the brain

7. A plane along a great circle will split the sphere into _____ parts.

Ans : Two parts.

8. If the volume and surface area of a sphere are numerically equal, then the radius of the sphere is _____.

Ans : $\frac{4}{3}\pi r^3 = 4\pi r^2 \Rightarrow r = 3.$

9. Is it possible to obtain the volume of the full cone when the volume of the frustum is known?

Ans : Not possible.

Exercise 7. 2

1. A 14 m deep well with inner diameter 10 m is dug and the earth taken out is evenly spread all around the well to form an embankment of width 5 m. Find the height of the embankment.

Sol : Radius of well = 5m

Depth of well = 14 m

Volume of earth taken out $= \pi r^2 h$

$$= \frac{22}{7} \times (5)^2 \times 14$$

$$= 1100 \text{ m}^3$$

Now, it is spread to form an embankment, which is in the form of hollow cylinder.

Inner radius $= 5$ m

Width of embankment $= 5$ m

$\therefore$ Outer radius $= 5 + 5 = 10$ m

height $= h$

Volume of hollow cylinder $= \pi h\,(R^2 - r^2)$

$$\therefore \quad \pi h\,(R^2 - r^2) = 1100$$

$$\frac{22}{7} \times h\,(10^2 - 5^2) = 1100$$

height of the embankment

$$h = \frac{1100 \times 7}{22 \times 75} = 4.67 \text{ m}$$

2. A cylindrical glass with diameter 20 cm has water to a height of 9 cm. A small cylindrical metal of radius 5 cm and height 4 cm is immersed it completely. Calculate the raise of the water in the glass.

Sol :

Diameter of Glass $= 20$ cm

$\therefore$ radius $= 10$ cm

water upto height $= 9$ cm

radius of cylindrical metal $= 5$ cm

height of cylindrical metal $= 4$ cm

Volume of water displaced = Volume of cylindrical metal

$$\pi r_1^2 h_1 = \pi r_2^2 h_2$$

$$(10)^2 h_1 = (5)^2 (4)$$

$$h_1 = \frac{100}{100} = 1 \text{ cm}$$

Hence, the increase in water level is 1 cm.

3. If the circumference of ■ conical wooden piece is 484 cm then find its volume when its height is 105 cm.

Sol :

Given circumference = 484 cm

$$2\pi r = 484$$

$$2 \times \frac{22}{7} \times r = 484$$

$$r = \frac{484 \times 7}{44} = 77 \text{ cm}$$

height $h = 105$ cm

Volume of cone $= \frac{1}{3}\pi r^2 h$ cu. units

$$= \frac{1}{3} \times \frac{22}{7} \times 77 \times 77 \times 105$$

$$= 652190 \text{ cm}^3$$

4. A conical container is fully filled with petrol. The radius is 10 m and the height is 15 m. If the container can release the petrol through its bottom at the rate of 25 cu. meter per minute, in how many minutes the container will be emptied. Round off your answer to the nearest minute.

Sol :

Radius of conical container $= 10$ m

Height of conical container $= 15$ m

Volume $= \frac{1}{3}\pi r^2 h$ cu. units

$$= \frac{1}{3} \times \frac{22}{7} \times 10 \times 10 \times 15$$

$$= \frac{11000}{7} m^3$$

water is released at the rate of 25 m^3/min

$\therefore$ Time required to empty the container

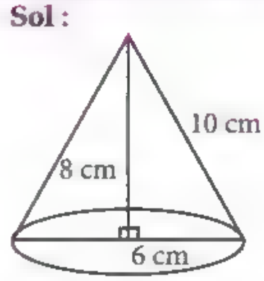
$$= \frac{\frac{11000}{7}}{25}$$

$$= \frac{11000}{7 \times 25} = 62.85$$

$= 63$ minutes (approx)

5. A right angled triangle whose sides are 6 cm, 8 cm and 10 cm is revolved about the sides containing the right angle in two ways. Find the difference in volumes of the two solids so formed.

Sol :



When right triangle is revolved about one of its sides (containing the right angle), a cone is formed

Now, Radius of the cone = 6 cm

Height of the cone = 8 cm

Slant height = 10 cm

Volume of the cone = $\frac{1}{3}\pi r^2 h$ cu. units

$= \frac{1}{3} \times \frac{22}{7} \times (6)^2 \times 8$

$= \frac{22 \times 12 \times 8}{7} = 301.71 \text{ cm}^3$

Now, Right triangle rotates about the side which is 8 cm in length

$\therefore$ Radius = 8 cm

Height = 6 cm

Slant height = 10 cm

Volume = $\frac{1}{3}\pi r^2 h$ cu. units

$= \frac{1}{3} \times \frac{22}{7} \times (8)^2 \times 6$

$= \frac{2816}{7} = 402.29 \text{ cm}^3$

Difference in volumes = 402.29 – 301.71

= 100.58 cm^3

6. The volumes of two cones of same base radius are 3600 cm³ and 5040 cm³. Find the ratio of heights.

Sol : Let r_1, r_2 be the radii of two cones,

Given $r_1 = r_2$ and let h_1, h_2 be the heights of two cones.

V_1 = Volumes of I cone $= \frac{1}{3}\pi r_1^2 h_1 = \frac{\pi}{3} r_1^2 h_1$

= 3600 cm^3

V_2 = Volume of II cone $= \frac{1}{3}\pi r_2^2 h_2 = 5040 \text{ cm}^3$

Now, $\frac{V_1}{V_2} = \frac{3600}{5040} \Rightarrow \frac{\frac{\pi}{3} r_1^2 h_1}{\frac{\pi}{3} r_2^2 h_2} = \frac{3600}{5040}$ [$\because r_1 = r_2$

$\therefore$ ratio of heights $\frac{h_1}{h_2} = \frac{5}{7} = 5 : 7$

7. If the ratio of radii of two spheres is 4 : 7, find the ratio of their volumes.

Sol :

Let r_1, r_2 be the radii of two spheres

Given $\frac{r_1}{r_2} = \frac{4}{7} \Rightarrow r_1 = \frac{4r_2}{7}$

Ratio of the volumes $= \frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3}$

$= \frac{\left(\frac{4r_2}{7}\right)^3}{r_2^3} = \frac{4^3}{7^3}$

Ratio of volumes $V_1 : V_2 = 64 : 343 = \frac{64}{343}$

8. A solid sphere and a solid hemisphere have equal total surface area. Prove that the ratio of their volume is $3\sqrt{3} : 4$.

Sol :

Let r_1 and r_2 be the radii of sphere and hemisphere respectively.

Given T.S.A of sphere = T.S.A of hemisphere

$4\pi r_1^2 = 3\pi r_2^2$

$\frac{r_1^2}{r_2^2} = \frac{3}{4} \Rightarrow \frac{r_1}{r_2} = \frac{\sqrt{3}}{2}$

Ratio of their volumes : $\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{2}{3}\pi r_2^3} = 2\left(\frac{r_1}{r_2}\right)^3$

$= 2\left(\frac{\sqrt{3}}{2}\right)^3$

$= \frac{3\sqrt{3}}{4} = 3\sqrt{3} : 4$

Ratio of their volumes = $3\sqrt{3} : 4$

9. The outer and the inner surface areas of a spherical copper shell are 576π cm² and 324π cm² respectively. Find the volume of the material required to make the shell.

Sol :

Let R, r be the outer and inner radii respectively.

Given outer surface area = $576\pi \, cm^2$

$4\pi R^2 = 576\pi$

$R^2 = 144$

R = 12 cm

Inner surface area = $324\pi \, cm^2$

$4\pi r^2 = 324\pi$

$r^2 = 81$

r = 9 cm

Volume of the material required

$$= \frac{4}{3}\pi(R^3 - r^3) \text{ cu. units}$$

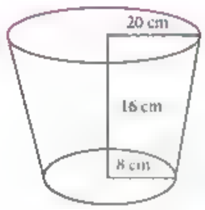
$$= \frac{4}{3} \times \frac{22}{7} \times (12^3 - 9^3)$$

$$= \frac{4}{3} \times \frac{22}{7} \times 999$$

$$= 4186.285 = 4186.29 \text{ cm}^3$$

10. A container open at the top is in the form of a frustum of a cone of height 16 cm with radii of its lower and upper ends are 8 cm and 20 cm respectively. Find the cost of milk which can completely fill a container at the rate of ₹ 40 per litre.

Sol :



Given radius of lower end $r = 8$ cm

radius of upper end $R = 20$ cm

height $h = 16$ cm

$$\text{Volume} = \frac{\pi h}{3}(R^2 + Rr + r^2) \text{ cu. units}$$

$$= \frac{22 \times 16}{7 \times 3}\left((20)^2 + (20)(8) + (8)^2\right)$$

$$= \frac{22 \times 16}{21}[400 + 160 + 64]$$

$$= \frac{22 \times 16}{21}(624) = 10459.43 \text{ cm}^3$$

$$= \frac{10459.43}{1000} \quad [\because 1000 \text{ cm}^3 = 1 \text{ litre}]$$

$$= 10.45943 \text{ litre}$$

Cost of milk per litre $=$ ₹ 40

$\therefore$ Total cost $= 10.459 \times 40$

$=$ ₹ 418.36

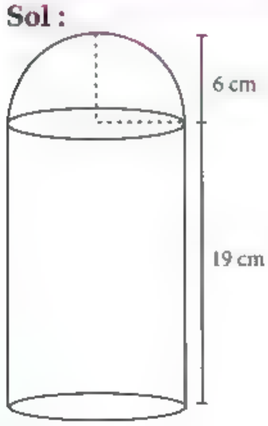
VOLUME AND SURFACE AREA OF COMBINED SOLIDS.

- A combined solid is said to be a solid formed by combining two or more solids.

Worked Examples

7.24 **A toy is in the shape of a cylinder surmounted by a hemisphere. The height of the toy is 25 cm. Find the total surface area of the toy if its common diameter is 12 cm.**

Sol :



Let r and h be the radius and height of the cylinder respectively.

Given that, diameter $d = 12$ cm,

radius $r = 6$ cm

Total height of the toy is 25 cm

Therefore, height of the cylindrical portion $= 25 - 6 = 19$ cm

T.S.A of the toy = C.S.A of the cylinder + C.S.A of the hemisphere + Base Area of the cylinder

$$= 2\pi rh + 2\pi r^2 + \pi r^2$$

$$= \pi r(2h + 3r) \text{ sq. units}$$

$$= \frac{22}{7} \times 6 \times (38 + 18)$$

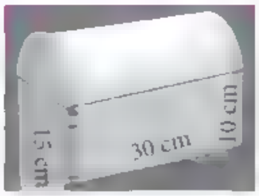
$$= \frac{22}{7} \times 6 \times 56 = 1056$$

Therefore, TSA of the toy $= 1056 \text{ cm}^2$.

7.25 **A jewEl box (Figure) is in the shape of a cuboid of dimensions 30 cm × 15 cm × 10 cm surmounted by a half part of a cylinder as shown in the figure. Find the volume and T.S.A of the box.**

Sol :

Let l, b and h_1 be the length, breadth and height of the cuboid. Also let us take r and h_2 be the radius and height of the cylinder.



Now, volume of the box

$$= \text{Volume of the cuboid} + \frac{1}{2}(\text{Volume of Cylinder})$$

$$= (l \times b \times h_1) + \frac{1}{2}\left(\pi r^2 h_2\right) \text{ cu. units}$$

$$= (30 \times 15 \times 10) + \frac{1}{2}\left(\frac{22}{7} \times \frac{15}{2} \times \frac{15}{2} \times 30\right)$$

$$= 4500 + 2651.79 \qquad = 7151.79$$

Therefore, volume of the box = 7151.79 cm^3

Now, T.S.A of the box = C.S.A of the cuboid + $\frac{1}{2}$ (C.S.A of the cylinder)

$$= 2(l+b)h_1 + \frac{1}{2}(2\pi r h_2)$$

$$= 2(45 \times 10) + \left(\frac{22}{7} \times \frac{15}{2} \times 30\right)$$

$$= 900 + 707.14 = 1607.14$$

Therefore, T.S.A of the box = 1607.14 cm^2.

7.26 **Arul has to make arrangements for the accommodation of 150 persons for his family function. For this purpose, he plans to build a tent which is in the shape of cylinder surmounted by a cone. Each person occupies 4 sq.m of the space on ground and 40 cu.meter of air to breathe. What should be the height of the conical part of the tent if the height of cylindrical part is 8 m?**

Sol :

Let h_1 and h_2 be the height of cylinder and cone respectively.

Area for one person = 4 sq. m

Total No. of persons = 150

Therefore total base area = 150×4

$$\pi r^2 = 600$$

$$r^2 = 600 \times \frac{7}{22} = \frac{2100}{11} \quad \dots (1)$$

Volume of air required for 1 person = 40 m^3

Total volume of air required for 150 persons $= 150 \times 40 = 6000 \text{ m}^3$.

$$\Rightarrow \pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2 = 6000$$

$$\Rightarrow \pi r^2\left(h_1 + \frac{1}{3}h_2\right) = 6000$$

$$\Rightarrow \frac{22}{7} \times \frac{2100}{11}\left(8 + \frac{1}{3}h_2\right) = 6000 \qquad [\text{using (1)}$$

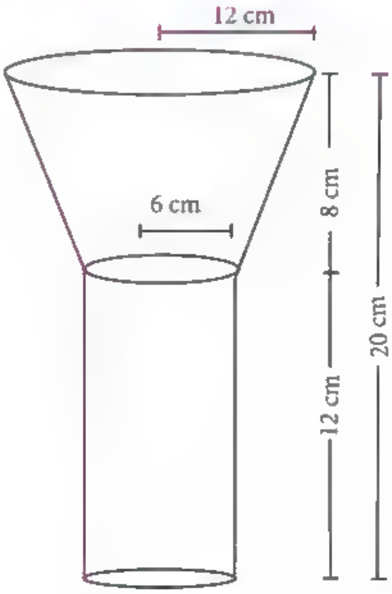
$$\Rightarrow 8 + \frac{1}{3}h_2 = \frac{6000 \times 7 \times 11}{22 \times 2100}$$

$$\Rightarrow \frac{1}{3}h_2 = 10 - 8 = 2$$

Therefore, the height of the conical tent h_2 is 6 m.

7.27 **A funnel consists of a frustum of a cone attached to a cylindrical portion 12 cm long attached at the bottom. If the total height be 20 cm, diameter of the cylindrical portion be 12 cm and the diameter of the top of the funnel be 24 cm. Find the outer surface area of the funnel.**

Sol :



Let R, r be the top and bottom radii of the frustum.

Let h_1, h_2 be the heights of the frustum and cylinder respectively.

Given that, $R = 12$ cm, $r = 6$ cm, $h_2 = 12$ cm

Now, $h_1 = 20 - 12 = 8$ cm

Here, Slant height of the frustum

$$l = \sqrt{(R-r)^2 + h_1^2} \text{ units}$$

$$= \sqrt{36+64}$$

$$l = 10 \text{ cm}$$

Outer surface area $= 2\pi r h_2 + \pi(R+r)\,l$ sq. units

$$= \pi\left[2rh_2 + (R+r)l\right]$$

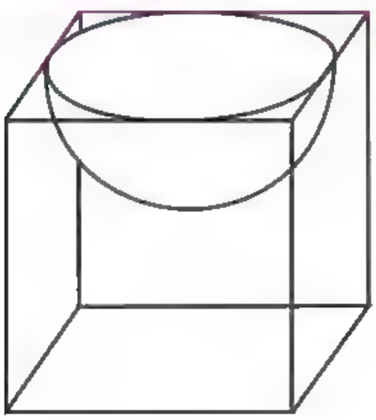
$$= \pi\left[(2\times 6\times 12) + (18\times 10)\right]$$

$$= \pi\,[144+180]$$

$$= \frac{22}{7}\times 324 = 1018.28$$

Therefore, Outer surface area of the funnel is 1018.28 cm^2.

7.28 **A hemispherical section is cut out from one face of a cubical block (figure) such that the diameter l of the hemisphere is equal to side length of the cube. Determine the surface area of the remaining solid.**



Sol :

Let r be the radius of the hemisphere.

Given that, diameter of the hemisphere = side of the cube = l

Radius of the hemisphere $= \dfrac{l}{2}$

T.S.A of the remaining solid = Surface area of the cubical part + C.S.A of the hemispherical part − Area of the base of the hemispherical part

$$= 6\times(\text{Edge})^2 + 2\pi r^2 - \pi r^2$$

$$= 6\times(\text{Edge})^2 + \pi r^2$$

$$= 6\times(l)^2 + \pi\left(\frac{l}{2}\right)^2 = \frac{1}{4}(24+\pi)l^2$$

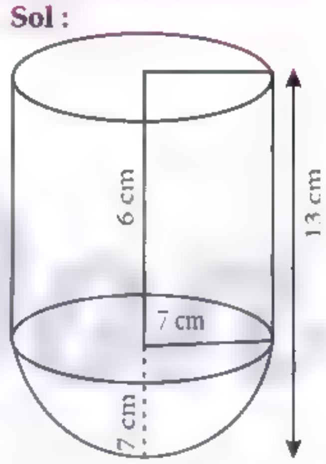
Total surface area of the remaining solid

$$= \frac{1}{4}(24+\pi)l^2 \text{ sq. units.}$$

Exercise 7. 3

1. A vessel is in form of a hemispherical bowl mounted by a hollow cylinder. The diameter is 14 cm and the height of the vessel is 13 cm. Find the capacity of the vessel.

Sol :



Diameter of the bowl = 14 cm

Radius r = 7 cm

Volume of hemisphere $= \dfrac{2}{3}\pi r^3$ cu. units

$$= \frac{2}{3}\times\frac{22}{7}\times 7\times 7\times 7$$

$$= \frac{2156}{3} = 718.67 \text{ cm}^3$$

Radius of cylinder 'r' = 7 cm

Height 'h' = 6 cm

Volume of cylinder $= \pi r^2 h$ cu. units

$$= \frac{22}{7}\times 7\times 7\times 6$$

$$= 924 \text{ cm}^3$$

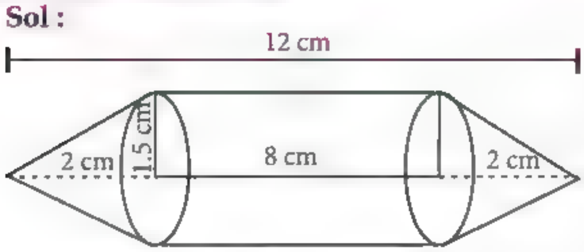
∴ Capacity of the vessel = Volume of hemisphere + Volume of cylinder

$$= 718.67 + 924$$

$$= 1642.67 \text{ cm}^3.$$

2. Nathan, an engineering student was asked to make a model shaped like a cylinder with two cones attached at its two ends. The diameter of the model is 3 cm and its length is 12 cm. If each cone has a height of 2 cm, find the volume of the model that Nathan made.

Sol :



From the figure, radius of cylinder = $\frac{3}{2}$ = 1.5 cm

Height = 8 cm

Volume of cylinder = $\pi r^2 h$ cu. units

$= \frac{22}{7} \times 1.5 \times 1.5 \times 8$

Radius of cone = 1.5 cm

Height of cone = 2 cm

Volume of 2 cones = $2\left(\frac{1}{3}\pi r^2 h\right)$ cu. units

$= \frac{2}{3} \times \frac{22}{7} \times 1.5 \times 1.5 \times 2$

$\therefore$ Volume of the model = Volume of cylinder + Volume of 2 cones

$= \frac{22}{7} \times (1.5)^2 \left[8 + \frac{4}{3}\right]$

$= \frac{22}{7} \times 2.25 \times \frac{28}{3}$

$= \frac{1386}{21} = 66 \text{ cm}^3$

3. From a solid cylinder whose height is 2.4 cm and the diameter 1.4 cm a cone of the same height and same diameter is carved out. Find the volume of the remaining solid to the nearest cm³.

Sol : Diameter of a solid cylinder = 1.4 cm

Radius of a solid cylinder = $\frac{1.4}{2}$ = 0.7 cm

Height of a solid cylinder = 2.4 cm

Volume of the cylinder = $\pi r^2 h$ cu. units

$= \frac{22}{7} \times 0.7 \times 0.7 \times 2.4$

Radius of cone = 0.7 cm

Height of cone = 2.4 cm

Volume of cone = $\frac{1}{3}\pi r^2 h$ cu. units

$= \frac{1}{3} \times \frac{22}{7} \times 0.7 \times 0.7 \times 2.4$

$\therefore$ Volume of the remaining solid = Volume of cylinder – Volume of cone

$= \frac{22}{7} \times 0.7 \times 0.7 \times 2.4 - \frac{1}{3} \times \frac{22}{7} \times 0.7 \times 0.7 \times 2.4$

$= \frac{22}{7} \times 0.7 \times 0.7 \times 2.4 \left(1 - \frac{1}{3}\right)$

$= \frac{22}{7} \times 0.7 \times 0.7 \times 2.4 \times \frac{2}{3}$

$= 2.464 \text{ cm}^3$

Aliter: Since the height and radius of the cylinder and cone are same,

Volume of the remaining solid = Volume of cylinder – Volume of cone

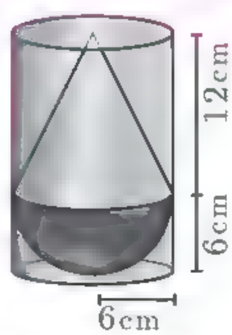
$= \pi r^2 h - \frac{1}{3}\pi r^2 h$

$= \frac{2}{3}\pi r^2 h$

$= \frac{2}{3} \times \frac{22}{7} \times (0.7)^2 \ (2.4)$

$= 2.464 \text{ cm}^3$

4. A solid consisting of a right circular cone of height 12 cm and radius 6 cm standing on a hemisphere of radius 6 cm is placed upright in a right circular cylinder full of water such that it touches the bottom. Find the volume of the water displaced out of the cylinder, if the radius of the cylinder is 6 cm and height is 18 cm.



Sol :

Radius of hemisphere = 6 cm

Volume of hemisphere = $\frac{2}{3}\pi r^3$ cu. units

$= \frac{2}{3}\pi(6)^3$

$= \frac{2}{3}\pi(216)$

$= 144\pi\ cm^3$

Radius of the base of cone $= 6$ cm

Height of the cone $= 12$ cm

Volume of the cone $= \frac{1}{3}\pi r^2 h$ cu. units

$= \frac{1}{3}\pi(6)^2\ (12)$

$= 144\pi\ cm^3$

Volume of the solid = Volume of cone + Volume of hemisphere

$= 144\pi + 144\pi = 288\pi$

Volume of water displaced

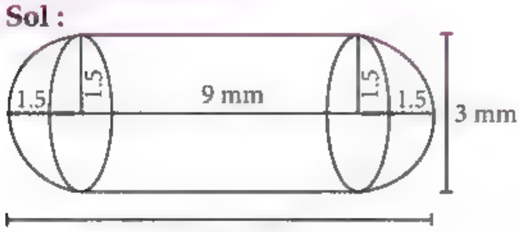
= Volume of the solid placed in the cylinder

$= 288\pi = 288 \times \frac{22}{7}$

$= 905.14\ cm^3$

5. A capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends. If the length of the entire capsule is 12 mm and the diameter of the capsule is 3 mm, how much medicine it can hold?

Sol :



From the figure,

Diameter of hemisphere $= 3$ mm

Radius of hemisphere $= 1.5$ mm

Volume of hemisphere $= \frac{2}{3}\pi r^3$ cu. units

$\therefore$ Volume of 2 hemispheres $= 2\left(\frac{2}{3}\pi r^3\right)$

$= \frac{4}{3}\pi(1.5)^3$

$= 4.5\pi\ mm^3$

Radius of cylinder $= 1.5$ mm

Height of cylinder $= 12 - 3 = 9$ mm

Volume of cylinder $= \pi r^2 h$ cu. units

$= \pi(1.5)^2\ (9)$

$= 20.25\pi\ mm^3$

$\therefore$ Amount of medicine that a capsule can hold

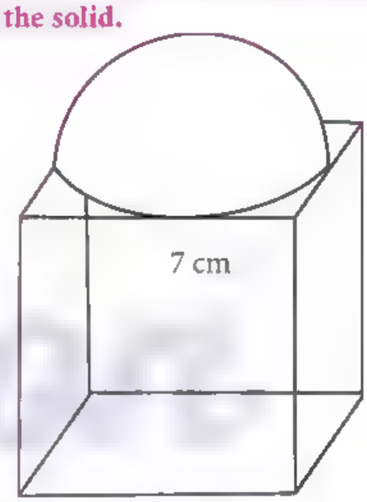
= Volume of cylinder + Volume of 2 hemispheres

$= 20.25\pi + 4.5\pi$

$= 24.75\pi\ mm^3$

$= 24.75 \times \frac{22}{7} = 77.785\ mm^3$

6. As shown in figure a cubical block of side 7 cm is surmounted by a hemisphere. Find the surface area of the solid.



Sol :

Edge of cube $= 7$ cm

surface area of a cube $= 6a^2$ sq. units

$= 6(7)^2$

$= 294\ cm^2$

radius of hemisphere $= \frac{7}{2}$ cm

[$\because$ Only C.S.A is considered as the hemisphere surmounted]

C.S.A of hemisphere $= 2\pi r^2$ sq. units

$= 2 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}$

$= 77\ cm^2$

Surface area of the solid = T. S a of cube + C.S.A of hemisphere – area of circular region (bottom of hemisphere)

$= 294 + 77 - \left(\frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right)$

$= 371 - 38.5$

$= 332.5\ cm^2$

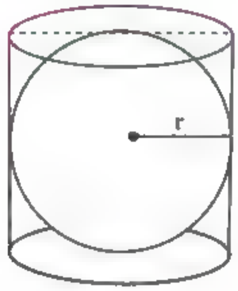
7. A right circular cylinder just enclose a sphere of radius r units. Calculate

(i) the surface area of the sphere

(ii) the curved surface area of the cylinder

(iii) the ratio of the areas obtained in (i) and (ii).

Sol :



(i) Surface area of a sphere

Radius of sphere $= r$

Surface area $= 4\pi r^2$ sq. units

(ii) Curved surface area of cylinder

Radius of cylinder $= r$

Height of cylinder $= r + r = 2r$

Curved surface area $= 2\pi rh$ sq. units

$= 2\pi r\,(2r)$

$= 4\pi r^2$ sq. units

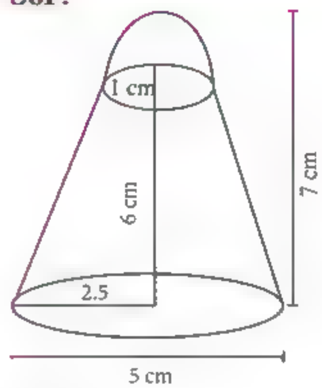
(iii) Ratio of the areas $= \dfrac{\textit{Surface area of sphere}}{\textit{CSA of cylinder}}$

$= \dfrac{4\pi r^2}{4\pi r^2} = \dfrac{1}{1}$

Ratio $= 1 : 1.$

8. A shuttlecock used for playing badminton has the shape of a frustum of a cone is mounted on a hemisphere. The diameters of the frustum are 5 cm and 2 cm. The height of the entire shuttle cock is 7 cm. Find its external surface area.

Sol :



Frustum

Radius of the base R $=$ 2.5 cm

Radius of the top r $=$ 1 cm

Height h $= 7 - 1 = 6$ cm

Slant height $l = \sqrt{h^2 + (R-r)^2}$

$= \sqrt{(6)^2 + (2.5-1)^2}$

$= \sqrt{36 + 2.25}$

$= \sqrt{38.25} = 6.18$ cm

Curved surface area $= \pi\,(R+r)l$ sq. units

$= \dfrac{22}{7}(2.5 + 1)\,(6.18)$

$= \dfrac{22}{7}(3.5)\,(6.18)$

$= 67.98\text{ cm}^2$

Radius of hemisphere $=$ 1 cm

C.S.A of hemisphere $= 2\pi r^2$ sq. units

$= 2 \times \dfrac{22}{7} \times (1)^2$

$= 6.29\text{ cm}^2$

$\therefore$ External surface area of shuttlecock

= C.S.A Frustum + C.S.A of hemisphere

$= 67.98 + 6.29 = 74.27\text{ cm}^2$

CONVERSION OF SOLIDS FROM ONE SHAPE TO ANOTHER WITH NO CHANGE IN VOLUME.

Key Points

- When converting one solid to another solid, the volumes are equal but they differ in surface area.
- In melting and casting problems, Since the volumes are equal, No need to take $\pi = \dfrac{22}{7}$ as they will get cancelled when equated.

Worked Examples

7.29 A metallic sphere of radius 16 cm is melted and recast into small spheres each of radius 2 cm. How many small spheres can be obtained?

Sol :

Let the number of small spheres obtained be n.

Let r be the radius of each small sphere and R be the radius of metallic sphere.

Here, R = 16 cm, r = 2 cm

Now, n × (Volume of a small sphere) = Volume of big metallic sphere

$$n\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$$

$$\Rightarrow n\left(\frac{4}{3}\pi \times 2^3\right) = \frac{4}{3}\pi \times 16^3$$

$$\Rightarrow \quad 8n = 4096 \Rightarrow n = 512$$

Therefore, there will be 512 small spheres.

7.30 A cone of height 24 cm is made up of modeling clay. A child reshapes it in the form of a cylinder of same radius as cone. Find the height of the cylinder.

Sol :

Let h_1 and h_2 be the heights of a cone and cylinder respectively.

Also, let r be the radius of the cone.

Given that, height of the cone h_1 = 24 cm; Radius of the cone and cylinder r = 6 cm

Since, Volume of cylinder = Volume of cone

$$\pi r^2 h_2 = \frac{1}{3}\pi r^2 h_1$$

$$\Rightarrow \quad h_2 = \frac{1}{3} \times h_1 \Rightarrow h_2 = \frac{1}{3} \times 24 = 8$$

Therefore, height of cylinder h_2 = 8 cm.

7.31 A right circular cylindrical container of base radius 6 cm and height 15 cm is full of ice cream. The ice cream is to be filled in cones of height 9 cm and base radius 3 cm, having a hemispherical cap. Find the number of cones needed to empty the container.

Sol :

Let h and r be the height and radius of the cylinder respectively.

Given that, h = 15 cm, r = 6 cm

Volume of the container V $= \pi r^2 h$ cubic. units

$$= \frac{22}{7} \times 6 \times 6 \times 15$$

Let r_1 = 3 cm, h_1 = 9 cm be the radius and height of the cone.

Also, r_1 = 3 cm is the radius of the hemispherical cap.

Volume of one ice cream cone = (Volume of the cone + Volume of the hemispherical cap)

$$= \left[\frac{1}{3}\pi r_1^2 h_1 + \frac{2}{3}\pi r_1^3\right]$$

$$= \left[\frac{1}{3} \times \frac{22}{7} \times 3 \times 3 \times 9 + \frac{2}{3} \times \frac{22}{7} \times 3 \times 3 \times 3\right]$$

$$= \left[\frac{22}{7} \times 9\,(3+2)\right] = \frac{22}{7} \times 45$$

$$\text{Number of cones} = \frac{\textit{volume of cylinder}}{\textit{volume of one ice cream cone}}$$

Number of Ice cream cones needed

$$= \frac{\frac{22}{7} \times 6 \times 6 \times 15}{\frac{22}{7} \times 45} = 12$$

Thus 12 ice cream cones are required to empty the cylindrical container.

Exercise 7.4

1. An aluminium sphere of radius 12 cm is melted to make a cylinder of radius 8 cm. Find the height of the cylinder.

Sol :

Radius of sphere = 12 cm

Volume of sphere $= \frac{4}{3}\pi r^3$ cu. units

$$= \frac{4}{3}\pi(12)^3$$

$$= 2304\,\pi \text{ cm}^3$$

Radius of cylinder = 8 cm

height = h cm

Volume of cylinder $= \pi r^2 h$ cu . units

$$= \pi(8)^2 h$$

$= 64\pi\ h\ cm^3$

Given that sphere is melted and cast into a cylinder

$\therefore$ Volume of cylinder = Volume of sphere

$$64\pi h = 2304\pi$$

$$h = \frac{2304\pi}{64\pi} = 36$$

$\therefore$ Height of the cylinder = 36 cm.

2. Water is flowing at the rate of 15 km per hour through a pipe of diameter 14 cm into a rectangular tank which is 50 m long and 44 m wide. Find the time in which the level of water in the tanks will rise by 21 cm.

Sol :

Diameter of cylindrical pipe = 14 cm

Radius = 7 cm

Length of the pipe = Speed of the water

= 15 km = 15000 m

Length of the water tank = 50 m

Width of the water tank = 44 m

Height of the water tank = Water level

= 21 cm

= 0.21 m

Volume of water tank = $l \times b \times h$ cu. units

$= 50 \times 44 \times 0.21 = 462\ m^3$

Volume of cylindrical pipe

= Volume of Rectangular tank

$$\pi r^2 h = 462$$

$$\frac{22}{7} \times 0.07 \times 0.07 \times h = 462$$

$$\therefore \quad h = \frac{462 \times 7}{22 \times 0.07 \times 0.07}$$

$$= \frac{3234}{0.1078} = 30000$$

$$\text{Time required} = \frac{30000}{15000} = 2 \text{ hrs.}$$

3. A conical flask is full of water. The flask has base radius r units and height h units, the water poured into a cylindrical flask of base radius xr units. Find the height of water in the cylindrical flask.

Sol :

Radius of conical flask = 'r' units

Height of conical flask = 'h' units

Volume of conical flask = Volume of water

$= \frac{1}{3}\pi r^2 h$ cu. units

Since, water is poured into the cylindrical flask

$\therefore$ Volume of cylinder = Volume of water

$$\pi(xr)^2 H = \frac{1}{3}\pi r^2 h$$

[xr – radius of cylinder, H – height]

$$x^2 r^2 H = \frac{r^2}{3} h$$

Height of the water in cylinder flask

$$H = \frac{h}{3x^2}$$

4. A solid right circular cone of diameter 14 cm and height 8 cm is melted to form a hollow sphere. If the external diameter of the sphere is 10 cm, find the internal diameter.

Sol :

Diameter of cone = 14 cm

Radius of cone = 7 cm

Height of cone = 8 cm

Volume of cone $= \frac{1}{3}\pi r^2 h$ cu. units

$$= \frac{1}{3} \times \pi \times 7 \times 7 \times 8$$

$$= \frac{392\pi}{3}\ cm^3$$

External diameter of sphere = 10 cm

External radius of sphere 'R' = 5 cm

Internal radius = 'r'

Given, Right Circular Cone is melted to form a hollow sphere.

i.e., Volume of hollow sphere = Volume of cone

$$\frac{4}{3}\pi(5^3 - r^3) = \frac{392\pi}{3}$$

$$5^3 - r^3 = \frac{392}{4} = 98$$

$$r^3 = 125 - 98 = 27$$

Radius r = 3 cm

$\therefore$ Internal diameter = 2r = 6 cm.

5. Seenu's house has an overhead tank in the shape of a cylinder. This is filled by pumping water from a sump (underground tank) which is in the shape of a cuboid. The sump has dimensions 2 m × 1.5 m × 1 m. The overhead tank has its radius of 60 cm and height 105 cm. Find the volume of the water left in sump after the overhead tank has been completely filled with water from the sump which has been full, initially.

Sol :

Cuboidal Tank

length = 2 m = 200 cm

width = 1.5 m = 150 cm

height = 1 m = 100 cm

Volume of cuboidal tank = $l \times b \times h$

$= 200 \times 150 \times 100$

$= 30{,}00{,}000 \text{ cm}^3$

Overhead cylindrical Tank

radius = 60 cm

height = 105 cm

volume = $\pi r^2 h$ cu. units

$= \frac{22}{7} \times 60 \times 60 \times 105$

$= 11{,}88{,}000 \text{ cm}^3$

∴ Volume of water left in the sump = Volume of Cuboidal Tank – Volume of Cylindrical Tank

$= 30{,}00{,}000 - 11{,}88{,}000$

$= 18{,}12{,}000 \text{ cm}^3.$

6. The internal and external diameter of a hollow hemispherical shell are 6 cm and 10 cm respectively. If it is melted and recast into a solid cylinder of diameter 14 cm, then find the height of the cylinder.

Sol :

Hollow Hemisphere

Internal diameter = 6 cm

Internal radius 'r' = 3 cm

External diameter = 10 cm

External radius 'R' = 5 cm

$\left.\begin{array}{l}\text{Volume of hemisphere (or)}\\ \text{Volume of material used}\end{array}\right\} = \frac{2}{3}\pi (R^3 - r^3)$ cu. units

$= \frac{2}{3}\pi (5^3 - 3^3)$

$= \frac{2}{3}\pi (125 - 27) = \frac{196\pi}{3} \text{ cm}^3$

Cylinder

Diameter = 14 cm

radius = 7 cm

height = h

Volume of cylinder = $\pi r^2 h$ cu. units

$= \pi(7)^2 h$

$= 49\,\pi h \text{ cm}^3$

Given that hollow hemisphere is melted and cast into a solid cylinder

∴ Volume of cylinder = Volume of hollow hemisphere

$$49\,\pi h = \frac{196\,\pi}{3}$$

$$h = \frac{196}{3 \times 49} = \frac{4}{3} = 1.33$$

∴ Height of the cylinder = 1.33 cm.

7. A solid sphere of radius 6 cm is melted into a hollow cylinder of uniform thickness. If the external radius of the base of the cylinder is 5 cm and its height is 32 cm, then find the thickness of the cylinder.

Sol :

Solid sphere

radius = 6 cm

volume = $\frac{4}{3}\pi r^3$ cu. units

$= \frac{4}{3}\pi(6)^3$

$= \frac{4}{3}\pi(216) = 288\,\pi \text{ cm}^3$

Hollow cylinder

Internal radius = 'r'

External radius = 'R' = 5 cm

Height h = 32 cm

Volume of Hollow Cylinder

$= \pi h (R^2 - r^2)$ cu. units

$= \pi (32)(25 - r^2) \text{ cm}^3$

Given that solid sphere is melted to form a hollow cylinder.

∴ Volume of Hollow Cylinder = Volume of Sphere

$$32\pi(25 - r^2) = 288\,\pi$$

$$25 - r^2 = \frac{288}{32} = 9$$

$$r^2 = 25 - 9 = 16$$

Internal radius r = 4 cm

∴ Thickness = External radius – Internal radius

= R – r = 5 – 4 = 1 cm.

8. A hemispherical bowl is filled to the brim with juice. The juice is poured into a cylindrical vessel whose radius is 50% more than its height. If the diameter is same for both the bowl and the cylinder then find the percentage of juice that can be transferred from the bowl into the cylindrical vessel.

Sol : Let the radius of hemispherical bowl = r

∴ Volume of hemispherical bowl

$$= \frac{2}{3}\pi r^3 \text{ cu. units}$$

Let the height of cylindrical vessel = h

Given $r = h + h\frac{50}{100} \Rightarrow r = h\left(1+\frac{50}{100}\right)$

$$h = \frac{2}{3}r$$

Now, Volume of cylindrical vessel

$$= \pi r^2\left(\frac{2r}{3}\right) = \frac{2}{3}\pi r^3$$

Hence, Volume of juice in the cylindrical vessel

$$= \frac{\frac{2}{3}\pi r^3}{\frac{2}{3}\pi r^3} \times 100\% = 100\ \%$$

Exercise 7. 5

Multiple Choice Questions:

1. The curved surface area of a right circular cone of height 15 cm and base diameter 16 cm is

(1) 60π cm^2 (2) 68π cm^2
(3) 120π cm^2 (4) 136π cm **[Ans (4)]**

Sol :

Base diameter = 16 cm, height = 15 cm
Base radius = 8 cm
C.S.A of cone = πrl sq. units

$$= \pi(8)\sqrt{15^2+8^2}$$
$$= \pi(8)\sqrt{289}$$
$$= 8\pi(17) = 136\,\pi \text{ cm}^2$$

2. If two solid hemisphere of same base radius r units are joined together along their bases, then curved surface area of this new solid is

(1) $4\pi r^2$ sq. units (2) $6\pi r^2$ sq. units
(3) $3\pi r^2$ sq. units (4) $8\pi r^2$ sq. units

[Ans (1)]

Sol :

Two hemispheres of same base radius 'r' units are joined, then it is a sphere.

∴ CSA of new solid = $4\pi r^2$ sq. units

3. The height of a right circular cone whose radius is 5 cm and slant height is 13 cm will be

(1) 12 cm (2) 10 cm
(3) 13 cm (4) 5 cm **[Ans (1)]**

Sol :

Given r = 5 cm, l = 13 cm

∴ height $h = \sqrt{l^2 - r^2} = \sqrt{13^2 - 5^2}$

$$= \sqrt{169-25} = \sqrt{144} = 12 \text{ cm}$$

4. If the radius of the base of a right circular cylinder is halved keeping the same height, then the ratio of the volume of the cylinder thus obtained to the volume of original cylinder is

(1) 1 : 2 (2) 1 : 4
(3) 1 : 6 (4) 1 : 8 **[Ans (2)]**

Sol :

Radius of the cylinder = r
Height of the cylinder = h
Volume of the original cylinder = $\pi r^2 h$ sq. units

Radius is halved, then radius = $\frac{r}{2}$

Volume of new cylinder $= \pi\left(\frac{r}{2}\right)^2 h = \frac{\pi r^2 h}{4}$

Now, ratio $= \frac{\textit{Volume of new cylinder}}{\textit{Volume of original cylinder}}$

$$= \frac{\frac{\pi r^2 h}{4}}{\pi r^2 h} = \frac{1}{4}$$

Ratio = 1 : 4

5. The total surface area of a cylinder whose radius is $\frac{1}{3}$ of its height is

(1) $\frac{9\pi h^2}{8}$ sq. units (2) $24\pi h^2$ sq. units

(3) $\frac{8\pi h^2}{9}$ sq. units (4) $\frac{56\pi h^2}{9}$ sq. units

[Ans (3)]

Sol : height = h, radius = $\frac{1}{3}h$

T.S.A of cylinder= $2\pi r$ (h + r)

$$= 2\pi \frac{h}{3}\left(h+\frac{h}{3}\right)$$
$$= 2\pi \frac{h}{3}\left(\frac{4h}{3}\right)$$
$$= \frac{8\pi h^2}{9} \text{ sq. units}$$

6. In a hollow cylinder, the sum of the external and internal radii is 14 cm and the width is 4 cm. If its height is 20 cm, the volume of the material in it is

(1) $5600\pi \text{ cm}^3$ (2) $1120\pi \text{ cm}^3$
(3) $56\pi \text{ cm}^3$ (4) $3600\pi \text{ cm}^3$

[Ans (2)]

Sol :

Internal radius = r, External radius = R

Given, sum = R + r = 14, width R − r = 4,
height h = 20 cm

Volume of material used

$$= \pi(R^2 - r^2)h \text{ cu. units}$$
$$= \pi(R + r)(R - r)h$$
$$= \pi(14)(4)20$$
$$= 1120\pi \text{ cm}^3$$

7. If the radius of the base of a cone is tripled and the height is doubled then the volume is

(1) made 6 times (2) made 18 times
(3) made 12 times (4) unchanged **[Ans (2)]**

Sol :

Radius is tripled, then new radius is 3r
Height is doubled, then new height is 2h

Volume of cone $= \frac{1}{3}\pi r^2 h$ cu. units

$$= \frac{1}{3}\pi(3r)^2(2h)$$
$$= \frac{\pi}{3}(9r)^2(2h)$$
$$= 18\left(\frac{1}{3}\pi r^2 h\right)$$

= 18 (Volume of cone) = 18 times

8. The total surface area of a hemi-sphere is how many times the square of its radius.

(1) π (2) 4π
(3) 3π (4) 2π **[Ans (3)]**

Sol :

TSA of hemisphere $= 3\pi r^2$
$= 3\pi$ (Square of radius)
$= 3\pi$ times

9. A solid sphere of radius x cm is melted and cast into a shape of a solid cone of same radius. The height of the cone is

(1) $3x$ cm (2) x cm
(3) $4x$ cm (4) $2x$ cm **[Ans (3)]**

Sol :

Radius of sphere = 'x' cm

Volume of sphere $= \frac{4}{3}\pi x^3 \text{ cm}^3$

Radius of the cone = 'x' cm

Height = h

Volume of cone $= \frac{1}{3}\pi x^2 h$

Volume of cone = volume of sphere

[∵ sphere is melted and cast into a cone]

$$\frac{1}{3}\pi x^2 h = \frac{4}{3}\pi x^3$$
$$\Rightarrow h = 4x \text{ cm}$$

10. A frustum of a right circular cone is of height 16 cm with radii of its ends as 8 cm and 20 cm. Then, the volume of the frustum is

(1) $3328\pi \text{ cm}^3$ (2) $3228\pi \text{ cm}^3$
(3) $3240\pi \text{ cm}^3$ (4) $3340\pi \text{ cm}^3$ **[Ans (1)]**

Sol :

Height of frustum h = 16 cm
Radii are 8 cm and 20 cm
i.e., r = 8 cm, R = 20 cm
Volume of frustum

$$= \frac{\pi h}{3}(R^2 + Rr + r^2) \text{ cu. units}$$
$$= \frac{\pi(16)}{3}(400 + 160 + 64)$$
$$= \frac{16\pi}{3}(624) = 3328\pi \text{ cm}^3$$

11. A shuttlecock used for playing badminton has the shape of the combination of

(1) a cylinder and a sphere
(2) a hemisphere and a cone
(3) a sphere and a cone
(4) frustum of a cone and a hemisphere

[Ans (4)]

12. A spherical ball of radius r_1 units is melted to make 8 new identical balls each of radius r_2 units. Then $r_1 : r_2$ is

(1) 2 : 1 (2) 1 : 2
(3) 4 : 1 (4) 1 : 4 **[Ans (1)]**

Sol :

Volume of spherical ball of radius r_1

$$= \frac{4}{3}\pi r_1^3$$

Volume of 8 spherical balls of radius r_2

$$= 8\left(\frac{4}{3}\pi r_2^3\right)$$

Now, $\frac{\text{Volume of Sphere}}{\text{Volume of 8 New Spheres}}$

$$= \frac{\frac{4}{3}\pi r_1^3}{8\left(\frac{4}{3}\pi r_2^3\right)} = \frac{r_1^3}{8r_2^3} = 2:1$$

13. The volume (in cm³) of the greatest sphere that can be cut off from a cylindrical log of wood of base radius 1 cm and height 5 cm is

(1) $\frac{4}{3}\pi$ (2) $\frac{10}{3}\pi$
(3) 5π (4) $\frac{20}{3}\pi$ **[Ans (1)]**

Sol :

Radius of cylindrical log of wood = 1 cm
Height of cylindrical log of wood = 5 cm

$\left.\begin{array}{l}\text{Diameter of Sphere of greatest volume}\\ \text{which can be cut off from cylinder}\end{array}\right\} = 2 \text{ cm}$

$\therefore$ radius = 1 cm

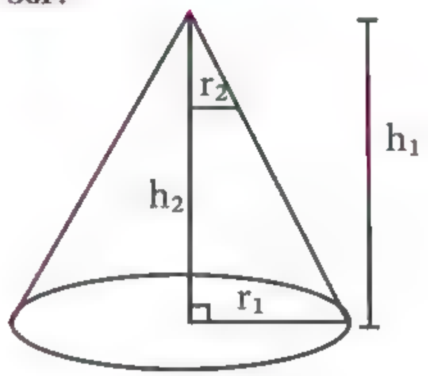
$$\text{Volume} = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi \text{ cm}^3$$

14. The height and radius of the cone of which the frustum is a part are h_1 units and r_1 units respectively. Height of the frustum is h_2 units and radius of the smaller base is r_2 units. If $h_2 : h_1 = 1 : 2$ then $r_2 : r_1$ is

(1) 1 : 3 (2) 1 : 2
(3) 2 : 1 (4) 3 : 1 **[Ans (2)]**

Sol :



Given $\frac{h_2}{h_1} = \frac{1}{2}$

$$\frac{h_2}{h_1} = \frac{r_2}{r_1} = \frac{1}{2}$$

$$r_2 : r_1 = 1:2$$

15. The ratio of the volumes of a cylinder, a cone and a sphere, if each has the same diameter and same height is

(1) 1 : 2 : 3 (2) 2 : 1 : 3
(3) 1 : 3 : 2 (4) 3 : 1 : 2 **[Ans (4)]**

Sol :

Diameter of cylinder = Diameter of cone = Diameter of sphere

Height of cylinder = Height of cone = Height of sphere

Volume of cylinder = $V_1 = \pi r^2 h$

Volume of cone = $V_2 = \frac{1}{3}\pi r^2 h$

Volume of sphere = $V_3 = \frac{4}{3}\pi r^3$

Radius of sphere = 'R', Diameter = 2R

Height of cylinder = Height of cone = 2R

Ratio of volumes $\Rightarrow V_1 : V_2 : V_3$

$$= \pi R^2(2R) : \frac{1}{3}\pi R^2(2R) : \frac{4}{3}\pi R^3$$

$$= 1 : \frac{1}{3} : \frac{2}{3} = 3:1:2$$

UNIT EXERCISE - 7

1. The barrel of a fountain-pen cylindrical in shape is 7 cm long and 5 mm in diameter. A full barrel of ink in the pen will be used on writing 330 words on an average. How many words can be written using a bottle of ink containing one fifth of a litre?

Sol :

Height of the barrel = h = 7 cm

Diameter = 5 mm

radius r = $\frac{5}{2}$ = 2.5 mm = 0.25 cm

Volume of cylindrical barrel = $\pi r^2 h$

$= \frac{22}{7} \times 0.25 \times 0.25 \times 7$

$= 1.375 \text{ cm}^3$

Given 1.375 cm^3 of ink is used for writing 330 words.

∴ Number of words that can be written with one - fifth of a litre

[∵ 1000 cm^3 = 1 ltr; $\frac{1}{5} \times 1000 \text{ cm}^3$; 200 cm^3]

$= \frac{330}{1.375} \times 200 = 48000$ words

2. A hemi-spherical tank of radius of 1.75 m is full of water. It is connected with a pipe which empties the tank at the rate of 7 litre per second. How much time will it take to empty the tank completely?

Sol :

Radius of hemispherical tank 'r' = 1.75 m

Volume of hemispherical tank = $\frac{2}{3}\pi r^3$ cu. units

$= \frac{2}{3} \times \frac{22}{7} \times (1.75)^3$

$= 11.225 \text{ m}^3$

= 11225 litre

Given that cylindrical pipe empties the tank at the rate of 7 litre per second.

∴ Time Required to empty the tank completely

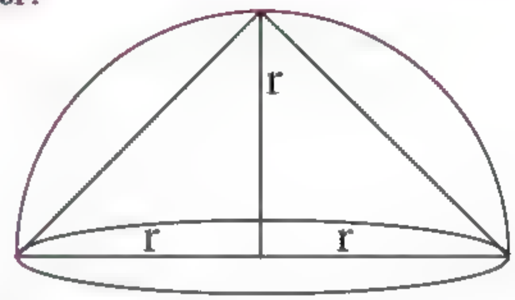
$= \frac{Volume}{Rate}$

$= \frac{11225}{7} = 1604$ sec (app)

= 27 min (app)

3. Find the maximum volume of a cone that can be carved out of a solid hemisphere of radius r units.

Sol :



Radius of hemisphere = r units

Radius of cone = Radius of hemisphere

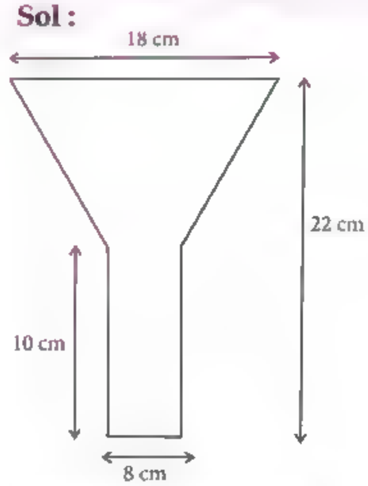
Height of cone = Radius of hemisphere

∴ Maximum volume of cone = $\frac{1}{3}\pi r^2 h$ cu. units

$= \frac{1}{3}\pi(r^2)r = \frac{1}{3}\pi r^3$ cu. units

4. An oil funnel of tin sheet consists of a cylindrical portion 10 cm long attached to a frustum of a cone. If the total height is 22 cm, the diameter of the cylindrical portion be 8 cm and the diameter of the top of the funnel be 18 cm, then find the area of the tin sheet required to make the funnel.

Sol :



Area of tin sheet required

= C.S.A of cylinder + C.S.A of frustum

Cylinder:

Radius = 4 cm

Height = 10 cm

C.S.A = $2\pi rh$ sq. units

$$= 2 \times \frac{22}{7} \times 4 \times 10 = \frac{1760}{7} \text{ sq. units}$$

Frustum of a cone

r_1 = radius of top = 9 cm

r_2 = radius of bottom = 4 cm

Height = 22 – 10 = 12 cm

$$\text{Slant height } l = \sqrt{h^2 + (r_1 - r_2)^2}$$

$$= \sqrt{12^2 + (9-4)^2}$$

$$= \sqrt{144 + 25} = \sqrt{169} = 13 \text{ cm}$$

$$\text{C.S.A} = \pi(r_1 + r_2)l \text{ sq. units}$$

$$= \frac{22}{7}(9+4)(13)$$

$$= \frac{3718}{7} \text{ sq. units}$$

$$\therefore \text{ Area of tin sheet} = \frac{1760}{7} + \frac{3718}{7}$$

$$= \frac{5478}{7} = 782.57 \text{ cm}^2.$$

5. Find the number of coins, 1.5 cm in diameter and 2 mm thickness, to be melted to form a right circular cylinder of height 10 cm and diameter 4.5 cm.

Sol :

Coin is in the form of a cylinder

Diameter of the coin = 1.5 cm

$$\text{Radius of the coin} = \frac{1.5}{2}$$

$$\text{Thickness} = \text{height} = 2 \text{ mm} = \frac{2}{10} = 0.2 \text{ cm}$$

$$\text{Volume of coin (cylinder)} = \pi r^2 h$$

$$= \pi\left(\frac{1.5}{2}\right)^2 (0.2)$$

$$= 0.1125\,\pi \text{ cm}^3$$

Diameter of cylinder = 4.5 cm

$$\text{radius} = \frac{4.5}{2} = 2.25 \text{ cm}$$

height = 10 cm

$$\text{volume} = \pi r^2 h \text{ sq. units}$$

$$= \pi(2.25)^2(10)$$

$$= 50.625\,\pi$$

$$\therefore \text{ No. of coins} = \frac{\textit{Volume of cylinder}}{\textit{Volume of Coin}}$$

$$= \frac{50.625\,\pi}{0.1125\,\pi} = 450 \text{ coins.}$$

6. A hollow metallic cylinder whose external radius is 4.3 cm and internal radius is 1.1 cm and whole length is 4 cm is melted and recast into a solid cylinder of 12 cm long. Find the diameter of solid cylinder.

Sol :

Hollow metallic cylinder

External radius R = 4.3 cm

Internal radius r = 1.1 cm

Length = height = h = 4 cm

$$\text{Volume} = \pi(R^2 - r^2)h \text{ cu. units}$$

$$= \pi\left((4.3)^2 - (1.1)^2\right)(4)$$

$$= \pi\,(18.49 - 1.21)\,4$$

$$= 69.12\,\pi \text{ cm3}$$

Solid cylinder

height h = 12 cm

radius r = ?

$$\text{volume} = \pi r^2 h \text{ sq. units}$$

$$= \pi r^2 (12)$$

Given, Hollow cylinder is melted to form solid cylinder

∴ Volume of cylinder = Volume of hollow cylinder

$$\pi r^2(12) = 69.12\,\pi$$

$$r^2 = \frac{69.12}{12} = 5.76$$

$$r = 2.4 \text{ cm}$$

Diameter of cylinder = 2r = 4.8 cm

7. The slant height of a frustum of a cone is 4 m and the perimeter of circular ends are 18 m and 16 m. Find the cost of painting its curved surface area at ₹ 100 per sq. m.

Sol :

Slant height l = 4 m

Perimeter of larger circle = $2\pi R$ = 18 cm

$$R = \frac{9}{\pi} = \frac{63}{22} \text{ m}$$

Perimeter of smaller circle = $2\pi r$ = 16 m

$$r = \frac{8}{\pi} = \frac{56}{22} \text{ m}$$

$$\text{C.S.A of Frustum of cone} = \pi l\,(R + r) \text{ sq. units}$$

$$= \frac{22}{7} \times 4 \times \left(\frac{63}{22} + \frac{56}{22}\right)$$

$$= \frac{4}{7}(119) = 68 \text{ m}^2$$

Cost of painting per sq. m = Rs. 100

$\therefore$ Cost of painting for 68 sq. m = 68×100
= Rs. 6800

8. A hemi - spherical hollow bowl has material of volume $\frac{436\pi}{3}$ cubic cm. Its external diameter is 14 cm. Find its thickness.

Sol :

External diameter of hollow hemisphere $= 2R = 14$ cm

$\therefore$ External radius R = 7 cm

$$\text{Given volume} = \frac{436\pi}{3} \text{ cm}^3$$

$$\frac{2}{3}\pi(R^3 - r^3) = \frac{436\pi}{3}$$

$$R^3 - r^3 = 218$$

$$(7)^3 - r^3 = 218$$

$\therefore r^3 = 343 - 218 = 125 = 5$ cm

$\therefore$ Thickness $= R - r = 7 - 5 = 2$ cm

9. The volume of a cone is $1005\frac{5}{7}$ cu.cm. The area of its base is $201\frac{1}{7}$ sq. cm. Find the slant height of the cone.

Sol :

$$\text{Volume of a cone} = 1005\frac{5}{7} \text{ cu. cm}$$

$$\text{i.e., } \frac{1}{3}\pi r^2 h = 1005\frac{5}{7} \quad \text{... (1)}$$

area of base = area of circle

$$= 201\frac{1}{7} \text{ sq. units}$$

$$\text{i.e., } \pi r^2 = 201\frac{1}{7} \Rightarrow r^2 = 64$$

Substituting in (1), r = 8 cm

$$\frac{1}{3}\left(201\frac{1}{7}\right)h = 1005\frac{5}{7}$$

$$\frac{1}{3}\left(\frac{1408}{7}\right)h = \frac{7040}{7}$$

$$h = \frac{7040}{7} \times \frac{7}{1408} \times 3 = 15 \text{ cm}$$

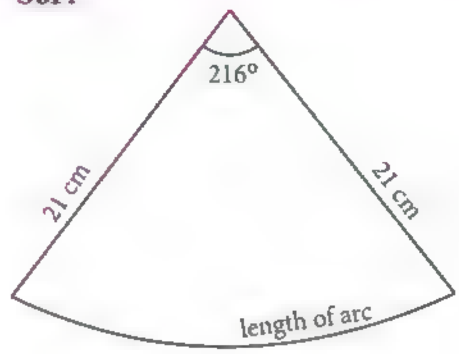
$\therefore$ Slant height of cone $l = \sqrt{h^2 + r^2}$

$$= \sqrt{15^2 + 8^2} = \sqrt{225 + 64}$$

$$= \sqrt{289} = 17 \text{ cm}$$

10. A metallic sheet in the form of a sector of a circle of radius 21 cm has central angle of 216°. The sector is made into a cone by bringing the bounding radii together. Find the volume of the cone formed.

Sol :



$$\text{Length of arc of a sector} = \frac{\theta}{360^\circ} \times 2\pi r_1$$

$$= \frac{216^\circ}{360^\circ} \times 2\pi \times 21$$

$$= \frac{3}{5}(2\pi)(21) = \frac{126\pi}{5} \text{ cm}$$

Since, sector is made into a cone by bringing the bounding radii together.

$$\left.\begin{matrix}\text{Circumference of base} \\ \text{of the cone}\end{matrix}\right\} = \left\{\begin{matrix}\text{length of arc} \\ \text{of a sector}\end{matrix}\right.$$

$$\therefore 2\pi r = \frac{126\pi}{5}$$

$$\text{radius of base of a cone, } r = \frac{63}{5} \text{ cm}$$

and radius of sector = slant height of cone

$$21 = l$$

$$\text{Height of cone, } h = \sqrt{l^2 - r^2} = \sqrt{(21)^2 - \left(\frac{63}{5}\right)^2}$$

$$= \sqrt{441 - 158.76}$$

$$= \sqrt{282.24} = 16.8 \text{ cm}$$

$$\therefore \text{Volume of cone} = \frac{1}{3}\pi r^2 h \text{ cu. units}$$

$$= \frac{1}{3} \times \frac{22}{7} \times 12.6 \times 12.6 \times 16.8$$

$$= \frac{58677.696}{21}$$

$$= 2794.176 = 2794.18 \text{ cm}^3$$

CREATIVE QUESTIONS

I. Multiple Choice Questions

Surface area

1. The lateral surface area of a cylinder is developed into a square whose diagonal is $2\sqrt{2}$ cm. The area of the base of the cylinder (in cm^2) is

(1) 3π (2) $\frac{1}{\pi}$
(3) π (4) 6π [Ans (2)]

Sol :

$$\text{diagonal } \sqrt{2}a = 2\sqrt{2}$$
$$\text{Side of the square (a)} = \frac{2\sqrt{2}}{\sqrt{2}} = 2 \text{ cm}$$
$$\text{Side of the square} = \text{Base perimeter} = 2 \text{ cm}$$
$$\text{i.e., } 2\pi r = 2 \Rightarrow r = \frac{1}{\pi}$$
$$\therefore \text{ Base area of cylinder} = \pi r^2 = \pi\left(\frac{1}{\pi^2}\right) = \frac{1}{\pi}$$

2. How many metres of cloth 2.5 m wide will be required to make a conical tent whose radius is 7 m and height is 24 m?

(1) 210 m (2) 220 m
(3) 230 m (4) 240 m [Ans (2)]

Sol :

$$\text{Slant height } l = \sqrt{r^2 + h^2} = \sqrt{24^2 + 7^2} = \sqrt{625} = 25 \text{ m}$$
$$\text{Area of cloth required} = \text{CSA of Cone} = \pi rl = \frac{22}{7}\times 7\times 25 = 550 \text{ m}^2$$
$$\therefore \text{ Length of cloth} = \frac{550}{2.5} = 220 \text{ m}$$

3. The total surface area of a hemisphere of radius 10 cm is

(1) 942.86 cm^2 (2) 900 cm^2
(3) 300 cm^2 (4) 592.86 cm^2 [Ans (1)]

Sol :

$$\text{TSA of hemisphere} = 3\pi r^2 = 3\times\frac{22}{7}\times 10\times 10 = \frac{6600}{7} = 942.86 \text{ cm}^2$$

3. The Curved Surface area of a right circular cone of radius 11.3 cm is 355 cm^2. What is its slant height?

(1) 8 cm (2) 9 cm
(3) 10 cm (4) 11 cm [Ans (3)]

Sol :

$$r = 11.3, \quad \text{CSA} = 355$$
$$\pi rl = 355$$
$$\frac{22}{5}\times 11.3\times l = 355$$
$$l = \frac{355\times 7}{22\times 11.3} = 10 \text{ (app.)}$$

4. The Curved Surface area of a right circular cone of height 15 cm and base diameter 16 cm is

(1) 146 π (2) 116 π
(3) 126 π (4) 136 π [Ans (4)]

Sol :

$$h = 15, \ r = \frac{16}{2} = 8$$
$$l = \sqrt{r^2 + h^2} = \sqrt{64 + 225} = \sqrt{289} = 17$$
$$\text{C.S.A} = \pi rl = \pi\times 8\times 17 = 136\pi \text{ cm}^2$$

5. The ratio of total surface area to the lateral surface area of a cylinder with base radius 80 cm and height 20 cm is

(1) 1 : 5 (2) 2 : 3
(3) 5 : 1 (4) 3 : 2 [Ans (3)]

Sol :

$$\text{LSA} = 2\pi rh,$$
$$\text{TSA} = 2\pi r(h + r)$$
$$\text{Ratio} = \frac{TSA}{LSA} = \frac{2\pi r(h+r)}{2\pi rh} = \frac{h+r}{h}$$

$$= \frac{20+80}{20}$$

$$= \frac{100}{20} = \frac{5}{1}$$

6. The surface area of a sphere of diameter 'r' is (sq.units)

(1) $2\pi r^2$ (2) πr^2

(3) $\frac{\pi r^2}{2}$ (4) $\frac{\pi r^2}{4}$ **[Ans (2)]**

Sol :

$$\text{Diameter} = r, \text{radius} = \frac{r}{2}$$

$$\text{Surface area} = 4\pi\left(\frac{r}{2}\right)^2$$

$$= 4\pi\left(\frac{r^2}{4}\right) = \pi r^2$$

7. The total surface area of a cone whose radius is $\frac{r}{2}$ and slant height $2l$ is (sq.units)

(1) $2\pi r(l+r)$ (2) $\pi r\left(l+\frac{r}{4}\right)$

(3) $\pi r(l+r)$ (4) $2\pi rl$ **[Ans (2)]**

Sol :

$$\text{TSA} = \pi R(L+R)$$

$$= \pi\left(\frac{r}{2}\right)\left(2l+\frac{r}{2}\right) = \pi r\left(l+\frac{r}{4}\right)$$

8. The slant height of the frustum of a cone is 4 cm and the circumference of its circular ends are 18 cm and 6 cm, then the curved surface area of the frustum is (cm^2)

(1) 12 (2) 24

(3) 48 (4) 54 **[Ans (3)]**

Sol :

$$\text{Circumference } 2\pi R = 18 \Rightarrow \pi R = 9$$

$$2\pi r = 6 \Rightarrow \pi r = 3$$

$$\text{CSA of frustum} = \pi l(R+r)$$

$$= l(\pi R + \pi r)$$

$$= 4(9+3)$$

$$= 48 \text{ cm}^2$$

9. If two solid hemispheres of same base radius 'r' are joined together along their bases, then the curved surface area of this new solid is (sq.units)

(1) $4\pi r^2$ (2) $6\pi r^2$

(3) $3\pi r^2$ (4) $8\pi r^2$ **[Ans (1)]**

Sol :

When the hemispheres are joined, then the new solid is a sphere.

$$\text{CSA of a sphere} = 4\pi r^2$$

Volume

10. x and y are two cylinders of the same height. The base of x has diameter that is half the diameter of the base of y. If the height of x is doubled, the volume of x becomes

(1) equal to the volume of y

(2) double the volume of y

(3) half the volume of y

(4) greater than the volume of y **[Ans (3)]**

Sol :

Let the height of x and y be 'h' and their radii be 'r' and '2r' respectively.

$$\text{Then volume of x} = \pi r^2 h \text{ and}$$

$$\text{volume of y} = \pi(2r)^2 h = 4\pi r^2 h$$

$$\text{New volume of x} = \pi r^2(2h)$$

$$= 2\pi r^2 h$$

$$= \frac{1}{2}(\text{volume of y})$$

11. The Area of the base of a right circular cone is 78.5 cm^2 and its height is 12 cm. Find the volume

(1) 341 cm^3 (2) 314 cm^3

(3) 301 cm^3 (4) 304 cm^3 **[Ans (2)]**

Sol :

$$\text{Base Area} = 78.5$$

$$\text{i.e., } \pi r^2 = 78.5$$

$$\text{and h} = 12$$

$$\text{Volume} = \frac{1}{3}(\pi r^2)h$$

$$= \frac{1}{3}(78.5)(12)$$

$$= 314 \text{ cm}^3$$

12. The curved surface area of a cylindrical pillar is 264 m^2 and its volume is 924 m^3, then its diameter is

(1) 12 m (2) 13 m

(3) 14 m (4) 15 m **[Ans (3)]**

Sol :

$$\text{Given CSA} = 2\pi rh = 264 \quad ...(1)$$

$$\text{Volume} = \pi^2 rh = 924 \quad ...(2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{\pi r^2 h}{2\pi rh} = \frac{924}{264} \Rightarrow r = 7$$

$$\therefore \text{ Diameter} = 2(7) = 14 \text{ m}$$

13. The volume of the sphere in 38808 cm^3, then its surface area is

(1) 5544 cm^2 (2) 4455 cm^2
(3) 4545 cm^2 (4) 5454 cm^2 **[Ans (1)]**

Sol :

$$\text{Volume} = \frac{4}{3}\pi r^3 = 38808$$

$$r^3 = 9261 = (21)^3$$

$$r = 21$$

$$\therefore \text{ Surface area} = 4\pi r^2 = 4 \times \frac{22}{7} \times 21 \times 21$$

$$= 5544 \text{ cm}^2$$

14. The volumes of two cylinders are as a : b and their heights are as c : d, then the ratio of their diameters is (cubic units)

(1) $\frac{ad}{bc}$ (2) $\frac{ad^2}{bc^2}$

(3) $\sqrt{\frac{ad}{bc}}$ (4) $\sqrt{\frac{a}{b} \times \frac{c}{d}}$ **[Ans (3)]**

Sol :

Let r_1 and r_2 be the radii of two cylinders respectively, and

Let the heights of two cylinders be ck and dk respectively.

$$\therefore \text{ Ratio of volumes} = \frac{\pi r_1^2 ck}{\pi r_2^2 dk} = \frac{a}{b} \text{ (given)}$$

$$\Rightarrow \frac{r_1}{r_2} = \sqrt{\frac{ad}{bc}}$$

15. A hemispherical container with radius 6 cm contains 325 ml of milk. Then the volume of milk that is needed to fill the container completely is

(1) 124.75 ml (2) 127.45 ml
(3) 217.45 ml (4) 117.45 ml **[Ans (2)]**

Sol :

$$\text{Volume of Container} = \frac{2}{3}\pi r^3 \text{ cubic units}$$

$$= \frac{2}{3} \times \frac{22}{7} \times (6)^3 \text{ cm}^3$$

$$= 452.45 \text{ ml}$$

$\therefore$ Volume of milk needed to fill the container

$$= 452.45 - 325$$

$$= 127.45 \text{ ml}$$

16. The external and internal diameters of a hemispherical bowl are 10 cm and 8 cm respectively, then the volume is (cm^3)

(1) 121.87 (2) 121.78
(3) 128.71 (4) 127.81 **[Ans (4)]**

Sol :

$$\text{External radius R} = \frac{10}{2} = 5 \text{ cm}$$

$$\text{Internal radius r} = \frac{8}{2} = 4 \text{ cm}$$

$$\text{Volume} = \frac{2}{3}\pi(R^3 - r^3)$$

$$= \frac{2}{3} \times \frac{22}{7}(5^3 - 4^3)$$

$$= \frac{2}{3} \times \frac{22}{7}(61) = 127.81 \text{ cm}^3$$

17. If the volume and surface area are numerically equal then its radius is

(1) 2 units (2) 3 units
(3) 4 units (4) 5 units **[Ans (2)]**

Sol :

$$\text{Volume} = \text{Surface area}$$

$$\frac{4}{3}\pi r^3 = 4\pi r^2$$

$$r = 3$$

18. If the radius of cone is reduced to half, then the new volume would be

(1) $\frac{1}{3}\left(\frac{1}{3}\pi r^2 h\right)$ (2) $\frac{1}{3}\pi\left(\frac{r}{2}\right)^2 h$

(3) $\frac{1}{3}\pi\left(\frac{r}{9}\right)^2 h$ (4) $\frac{1}{3}\pi\left(\frac{r^2}{4}\right)\left(\frac{h}{2}\right)$ **[Ans (2)]**

Combination of Solids

19. A right circular cylinder of radius 'r' cm and height 'h' cm (h > 2r) just encloses a sphere of diameter

(1) r cm (2) 2r cm
(3) h cm (4) 2h cm **[Ans (2)]**

Sol :

Diameter of the sphere = Diameter of base of cylinder

= 2r

20. Two cones with same base radius 8 cm and height 15 cm are joined together along their bases. Then the surface area of the shape so formed is

(1) 854 cm^2 (2) 860 cm^2

(3) 864 cm^2 (4) 870 cm^2 **[Ans (1)]**

Sol :

$$\text{Slant height } l = \sqrt{r^2+h^2} = \sqrt{8^2+15^2} = \sqrt{289} = 17$$

$$\text{Surface area} = 2(\pi r l) = 2 \times \frac{22}{7} \times 8 \times 17 = 854.08$$

21. A cylinder circumscribes a sphere. The ratio of their volumes is

(1) 1 : 2 (2) 3 : 2

(3) 4 : 3 (4) 5 : 6 **[Ans (2)]**

Sol :

Let 'h' be the height of the cylinder then the radius of sphere $= \frac{h}{2}$ = radius of base of cylinder

$$\text{Now } \frac{V_1}{V_2} = \frac{\pi\left(\frac{h}{2}\right)^2 h}{\frac{4}{3}\pi\left(\frac{h}{2}\right)^3} = \frac{\pi\frac{h^3}{4}}{4\pi\frac{h^3}{24}} = 3:2$$

22. A sphere and a cube have the same surface area. The ratio of their volumes is

(1) $\sqrt{6}:\sqrt{\pi}$ (2) $\sqrt{3}:\sqrt{\pi}$

(3) $\pi:\sqrt{2}$ (4) None of these

[Ans (1)]

Sol :

S.A of Sphere = S.A. of cube

$$4\pi r^2 = 6a^2$$

$$r^2 = \frac{6a^2}{4\pi} = \frac{3a^2}{2\pi}$$

$$r = \frac{\sqrt{3}a}{\sqrt{2}\sqrt{\pi}}$$

$$\text{Ratio of volumes} = \frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r^3}{a^3} = \frac{\frac{4}{3}\pi\left(\frac{\sqrt{3}a}{\sqrt{2}\sqrt{\pi}}\right)^3}{a^3}$$

$$= \frac{4\pi 3\sqrt{3}}{3.2\sqrt{2}\pi\sqrt{\pi}}$$

$$= \frac{2\sqrt{3}}{\sqrt{2}\sqrt{\pi}} = \frac{\sqrt{2}\sqrt{3}}{\sqrt{\pi}} = \frac{\sqrt{6}}{\sqrt{\pi}}$$

23. A sphere of radius R has volume equal to that of a cone of radius R, the height of the cone is

(1) R (2) 2R

(3) 3R (4) 4R **[Ans (4)]**

Conversion of Solids

24. A cone is 8.4 cm high and radius of its base is 2.1 cm. It is melted and recast into sphere. The radius of the sphere is

(1) 4.2 cm (2) 2.1 cm

(3) 2.4 cm (4) 1.6 cm **[Ans (2)]**

Sol :

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2.1)^2(8.4)$$

$$\text{Volume of sphere} = \frac{4}{3}\pi r_1^3$$

$$\text{Given } \frac{4}{3}\pi r_1^3 = \frac{1}{3}\pi(2.1)^2(8.4)$$

$$r_1^3 = (2.1)^3$$

$$r_1 = 2.1 \text{ cm}$$

25. A spherical iron ball is dropped into a vessel of base diameter 14 cm, containing water. The water level is increased by $9\frac{1}{3}$ cm. What is the radius of the ball?

(1) 3.5 cm (2) 7 cm

(3) 9 cm (4) 12 cm **[Ans (2)]**

Sol :

$$\text{Volume of spherical ball} = \frac{4}{3}\pi r^3$$

$$\text{Now } \frac{4}{3}\pi r^3 = \pi(7)^2\left(\frac{28}{3}\right)$$

$$r^3 = 7^3$$

$$\Rightarrow r = 7$$

26. Three solid spheres of gold whose radii are 1 cm, 6 cm and 8 cm respectively are melted into a single solid sphere. Then the radius of the sphere is

(1) 7 cm (2) 8 cm

(3) 9 cm (4) 10 cm **[Ans (3)]**

Sol :

$$\text{Radius of the new sphere} = \left[(1^3+6^3+8^3)\right]^{\frac{1}{3}}$$

$$= (729)^{\frac{1}{3}} = 9$$

27. A copper sphere of diameter 18 cm is drawn into a wire of diameter 4 mm. Then the length of the wire is

(1) 143 m (2) 243 m
(3) 343 m (4) 443 m **[Ans (2)]**

Sol :

$$\text{Volume of sphere} = \text{volume of wire (cylinder)}$$

$$\frac{4}{3}\pi(9\times9\times9) = \pi\times0.2\times0.2\times h$$

$$\therefore\ h = \frac{972\times5\times5}{100} = 243 \text{ m}$$

28. A hemispherical bowl of internal radius 9 cm contains a liquid. This liquid is to be filled into cylindrical shaped small bottles of diameter 3 cm and height 4 cm. How many bottles will be needed to empty the bowl?

(1) 24 (2) 34
(3) 44 (4) 54 **[Ans (4)]**

Sol :

$$\text{Volume of bowl} = \frac{2}{3}\pi(9)^3 = 486\pi$$

$$\text{Volume of 1 bottle} = \pi\times\frac{3}{2}\times\frac{3}{2}\times4 = 9\pi$$

$$\therefore\ \text{No. of bottles} = \frac{486\pi}{9\pi} = 54$$

II. Very Short Answer Questions

1. The curved surface area of a right circular cylinder of height 14 cm is 88 cm². Find the diameter of the base of the cylinder.

Sol :

$$\text{Radius of the cylinder} = r \text{ cm}$$

$$\text{Height 'h'} = 14 \text{ cm}$$

$$\text{C.S.A} = 2\pi rh \text{ sq. units}$$

$$\therefore\ 2\pi rh = 88$$

$$2\times\frac{22}{7}\times r\times14 = 88$$

$$r = 1 \text{ cm}$$

$$\text{Diameter of the base} = 2\times1 = 2 \text{ cm.}$$

2. In the hot water heating system, there is a cylindrical pipe of length 28 m and diameter 5 cm. Find the total radiating surface in the system.

Sol :

$$\text{Total radiating surface} = \text{Curved surface area of cylindrical pipe.}$$

$$= 2\pi rh$$

$$\text{Length } h = 28 \text{ m}$$

$$\text{Radius } r = \frac{5}{2} cm$$

$$= 2.5 \text{ cm} = 0.025 \text{ m}$$

$$\therefore \text{Total radiating surface} = 2\pi rh$$

$$= 2\times\frac{22}{7}\times0.025\times28$$

$$= 4.4 \text{ m}^2$$

3. Find the total surface area of a cone, if its slant height is 21 m and diameter of its base is 24 m.

Sol :

$$\text{Diameter} = 24 \text{ m},$$

$$\text{radius } r = \frac{24}{2} = 12 \text{ m}$$

$$\text{Slant height } l = 21 \text{ m}$$

$$\text{Total surface area} = \pi r(l+r) \text{ sq. units}$$

$$= \frac{22}{7}\times12\times(21+12)$$

$$= 1244.57 \text{ m}^2 \text{ (app).}$$

4. Find the radius of a sphere whose surface area is 154 cm².

Sol :

$$\text{Total surface area} = 4\pi r^2$$

$$4\pi r^2 = 154$$

$$4\times\frac{22}{7}\times r^2 = 154$$

$$r^2 = \frac{154\times7}{4\times22} = 12.25$$

$$r = \sqrt{12.25} = 3.5$$

$$\text{Radius} = 3.5 \text{ cm.}$$

5. A hemispherical bowl is made of steel, 0.25 cm thick. The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl.

Sol :

$$\text{Inner radius } r = 5 \text{ cm}$$

$$\text{Thickness} = 0.25 \text{ cm}$$

$$\text{outer radius } R = 5+0.25 = 5.25 \text{ cm}$$

$\therefore$ Outer curved surface area $= 2\pi R^2$ Sq. units

$$= 2 \times \frac{22}{7} \times (5.25)^2$$

$$= 173.25 \text{ cm}^2$$

6. The radii of the circular ends of a bucket of height 24 cm are 15 cm and 5 cm. Find the area of its curved surface.

Sol :

Given R = 15 cm, r = 5 cm, h = 24 cm

$$l = \sqrt{h^2 + (R-r)^2}$$

$$= \sqrt{(24)^2 + (15-5)^2}$$

$$= \sqrt{676} = 26 \text{ cm}$$

$$\text{C.S.A} = \pi l (R + r)$$

$$= \frac{22}{7} \times 26 \times (15 + 5)$$

$$= \frac{11440}{7} = 1634.28 \text{ cm}^2$$

7. The circumference of the base of a cylindrical vessel is 132 cm and its height is 25 cm. How many litres of water it can hold?

Sol :

Radius = r, height = h.

Circumference $2\pi r = 132$

$\Rightarrow r = 21$ cm

Volume $= \pi r^2 h$ cu. units

$$= \frac{22}{7} \times 21 \times 21 \times 25$$

$$= 34650 \text{ cm}^3$$

Quantity of water $= \frac{34650}{1000}$ litres

$= 34.65$ litres.

[$\because$ 1000 cm^3 = 1 ltr.]

8. The height of the cone is 15 cm. If its volume is 1570 cm^3, find the radius of the base.

Sol :

h = 15 cm, radius = r

Volume $\frac{1}{3}\pi r^2 h = 1570$

$$\frac{1}{3} \times \frac{22}{7} \times r^2 \times 15 = 1570$$

$$r = \sqrt{100}$$

radius of the base $r = \sqrt{100} = 10$ cm.

9. Find the amount of water displaced by a solid spherical ball of diameter 0.21 cm.

Sol :

Diameter = 0.21 cm;

Radius $= \frac{0.21}{2} = 0.105$ cm

Amount of water displaced = Volume of the ball

$$= \frac{4}{3}\pi r^3$$

$$= \frac{4}{3} \times \frac{22}{7} \times (0.105)^3$$

$$= 0.004851 \text{ cm}^3.$$

10. A drinking glass is in the shape of a frustum of a cone of height 14 cm. The diameters of its two circular ends are 4 cm and 2 cm. Find the capacity of the glass.

Sol :

Given R = 2 cm, r = 1 cm

Capacity of glass = Volume of frustum

$$= \frac{\pi h}{3}(R^2 + Rr + r^2) \text{ cu. units.}$$

$$= \frac{22}{7} \times \frac{14}{3}((2)^2 \times (2 \times 1) + (1)^2)$$

$$= \frac{44}{3}(4 + 2 + 1)$$

$$= \frac{308}{3}$$

$$= 102.67 \text{ cm}^3$$

11. The radius of a cone is 20 cm. If its volume is 8800 cm^3, find the height of the base.

Sol :

Let Height h = h cm,

radius = 20 cm

Volume = 8800 cm^3

Volume $= \frac{1}{3}\pi r^2 h$

$$\frac{1}{3}\pi r^2 h = 8800$$

$$\frac{1}{3} \times \frac{22}{7} \times 20 \times 20 \times h = 8800$$

$$h = \frac{8800 \times 3 \times 7}{22 \times 20 \times 20}$$

height = 21 cm

III. Short Answer Questions:

1. The diameter of a roller is 84 cm and its length is 120 cm. It takes 500 complete revolutions to move once over to level a playground. Find the area of the play ground (in sq. m).

Sol :

Diameter of roller = 84 cm

$$\text{radius} = \frac{84}{2} = 42 \text{ cm} = 0.42 \text{ m}$$

$$\text{length} = h = 120 \text{ cm} = 1.2 \text{ m}$$

Area covered by roller in one revolution = C.S.A of the roller

$$= 2\pi rh$$

$$= 2 \times \frac{22}{7} \times 0.42 \times 1.2$$

$$= 3.168 \text{ m}^2$$

$\therefore$ Area covered in 500 revolutions = Area of the playground

$$= 500 \times 3.168 = 1584 \text{ sq. m}$$

2. The slant height and base diameter of a conical tomb are 25 m and 14 m respectively. Find the cost of white washing its curved surface area at the rate of ₹ 210 per 100 m²?

Sol :

diameter = 14 m; radius = 7 cm;

Slant height l = 25 m

$$\text{C.S.A} = \pi rl = \frac{22}{7} \times 7 \times 25$$

$$= 550 \text{ m}^2$$

Given, Cost of white washing per 100 m² is ₹ 210

$\therefore$ Cost of white washing the conical tomb

$$= 550 \times \frac{210}{100} = ₹\ 1155$$

3. The diameter of the moon is approximately one fourth of the diameter of the Earth. Find the ratio of their surface areas.

Sol :

Let the diameter of the Earth be 'R'

$$\text{Radius of the Earth} = \frac{R}{2}$$

$$\text{Diameter of the Moon} = \frac{1}{4}R$$

$$\text{Radius of the Moon} = \frac{R}{8}$$

$\therefore$ Ratio of surface areas of moon and earth

$$= \frac{4\pi\left(\frac{R}{8}\right)^2}{4\pi\left(\frac{R}{2}\right)^2}$$

$$\therefore \text{Ratio} = \frac{1}{16} = 1 : 16$$

4. The radii of circular ends of a solid frustum of a cone are 33 cm and 27 cm and its slant height is 10 cm. Find its total surface area.

Sol :

Given R = 33 cm, r = 27 cm and l = 10 cm

$$\therefore \text{T.S.A of frustum} = \pi(R^2 + r^2 + l(R+r))$$

$$= \frac{22}{7}((33)^2 + (27)^2 + 10(33+27))$$

$$= \frac{22}{7}(1089 + 729 + 600)$$

$$= \frac{53196}{7} = 7599.43 \text{ cm}^2$$

5. The sum of the radius of the base and the height of a solid cylinder is 37 m. If the total surface area of the solid cylinder is 1628 m², find the circumference its base and volume of the cylinder.

Sol :

radius = r, height = h

T.S.A = 1628

$$2\pi r(h+r) = 1628$$

$$2\pi r(37) = 1628$$

$$\Rightarrow r = 7 \text{ m}$$

$$\text{and } 7 + h = 37$$

$$\Rightarrow h = 37 - 7 = 30 \text{ m}$$

$$\text{Circumference} = 2\pi r$$

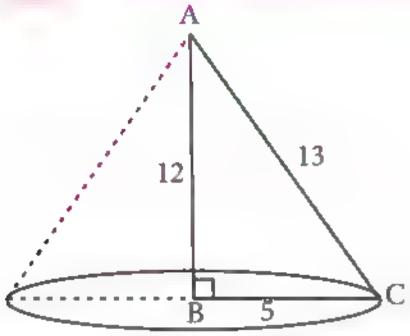
$$= 2 \times \frac{22}{7} \times 7 = 44 \text{ m}$$

$$\text{Volume} = \pi r^2 h$$

$$= \frac{22}{7} \times 7 \times 7 \times 30 = 4620 \text{ m}^3.$$

6. A right triangle ABC with sides 5 cm, 12 cm and 13 cm is revolved about the side 12 cm. Find the volume of the solid so obtained.

Sol :



Triangle ABC is revolved about the side AB

$= 12$ cm

Volume of cone $= \frac{1}{3}\pi r^2 h$ cu. units

$= \frac{1}{3}\pi(5)^2(12)$

$= 100 \times \frac{22}{7}$

$= 314$ cm^3 (app)

7. The diameter of a metallic ball is 4.2 cm. What is the mass of ball if the density of the metal is 8.9 g per cm^3?

Sol :

Diameter $= 4.2$ cm;

Radius $= 2.1$ cm

Volume $= \frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times(2.1)^3$

$= 38.808$ cm^3

Mass of the ball $=$ Volume $\times$ Density

$= 38.808 \times 8.9 = 345.3912$ g.

8. A bucket is in the form of a cone. Its depth is 24 cm and the diameters of the top and bottom ends are 30 cm and 10 cm respectively. Find the capacity of the bucket.

Sol :

Given: $R = \frac{30}{2} = 15$ cm, $r = \frac{10}{2} = 5$ cm and

$h = 24$ cm

Volume $= \frac{1}{3}\pi h(R^2 + Rr + r^2)$ cubic units.

$= \frac{1}{3}\times\frac{22}{7}\times 24\times(15^2+15\times5+5^2)$

$= \frac{22}{7}\times 8\times(225+75+25)$

$= \frac{57200}{7} = 8171.42$ cm^3

IV. Long Answer Questions

1. A factory manufactures 1,20,000 pencils daily. The pencils are cylindric in shape, each of length 25 cm and circumference 1.5 cm. Determine the cost of colouring the curved surface of the pencils manufactured in one day at ₹ 0.05 per dm^2.

Sol :

Let the radius of the base be 'r' cm.

Circumference $= 2\pi r$

$2\pi r = 1.5$ cm

$r = \frac{1.5}{2\pi} = \frac{10.5}{44}\, cm$

Curved surface area of pencil

$= 2\pi rh$

$= 2\times\frac{22}{7}\times\frac{10.5}{44}\times 25$

Cost of colouring

100 cm^2 $=$ ₹. 0.05

[$\because$ 1. sq. dm = 100 sq. cm]

Cost of colouring $= \frac{0.05}{100}\times 2\times\frac{22}{7}\times\frac{10.5}{44}\times 25$

$=$ ₹ $\frac{3}{160}$

Hence, cost of colouring 1,20,000 pencils

$= \frac{3}{160}\times 120000 =$ ₹ 2250.

2. There are two cones. The curved surface area of one is twice that of the other. The slant height of the later is twice that of the former. Find the ratio of their radii.

Sol :

Let r_1 be the radius and l_1 be the slant height of first cone and r_2 be the radius and l_2 be the slant height of the second cone.

C.S.A of I cone $= \pi r_1 l_1$

C.S.A of II cone $= \pi r_2 l_2$

Given $\pi r_1 l_1 = 2\pi r_2 l_2$

$\Rightarrow r_1 l_1 = 2r_2 l_2$

and $r_1 l_1 = 2r_2(2l_1)$

$r_1 l_1 = 4r_2 l_1$

$r_1 = 4r_2$

$\frac{r_1}{r_2} = \frac{4}{1}$

Hence, the ratio of their radii is 4 : 1.

3. The diameter of a sphere is decreased by 25%. By what percent does its curved surface area decrease?

Sol :

Let the diameter be 'x' units

$$\text{radius} = \frac{x}{2} \text{ units}$$

$$\therefore \text{ C.S.A of sphere} = 4\pi r^2 \text{ sq. units.}$$

$$= 4\pi\left(\frac{x}{2}\right)^2$$

$$= 4\pi\left(\frac{x^2}{4}\right) = \pi x^2$$

Given that diameter is decreased by 25%.

$$\text{New diameter} = x - 25\% \text{ of } x$$

$$= x\left(1-\frac{25}{100}\right) = \frac{3}{4}x$$

$$\text{New Radius} = \frac{3x}{8}$$

$$\text{C.S.A of new sphere} = 4\pi\left(\frac{3x}{8}\right)^2 = \frac{9\pi x^2}{16}$$

$$\text{Decrease in C.S.A} = \pi x^2 - \frac{9\pi x^2}{16} = \frac{7\pi x^2}{16}$$

Hence, percentage in decrease

$$= \frac{\frac{7\pi x^2}{16}}{\pi x^2} \times 100\%$$

$$= 43.75\%.$$

4. A solid cylinder has total surface area of 462 sq. cm. Its curved surface area is one-third its total surface area. Find the volume of cylinder.

Sol :

$$\text{radius} = r, \text{ height} = h$$

$$\text{T.S.A} = 2\pi r(h+r) \text{ sq. units}$$

$$\text{Given, } 2\pi r(h+r) = 462$$

$$r(h+r) = \frac{462\times 7}{2\times 22} = \frac{147}{2} \quad \dots (1)$$

$$\text{C.S.A} = 2\pi rh \text{ sq. units}$$

$$\text{Given, } 2\pi rh = \frac{1}{3}(2\pi r(h+r))$$

$$\Rightarrow h = \frac{h+r}{3} \Rightarrow r = 2h \quad \dots (2)$$

Substituting (2) in (1)

$$2h(h+2h) = \frac{147}{2}$$

$$6h^2 = \frac{147}{2}$$

$$h^2 = \frac{49}{4} \Rightarrow h = \frac{7}{2}$$

$$\therefore r = 2h = 2\left(\frac{7}{2}\right) = 7 \text{ cm}$$

$$\therefore \text{ Volume} = \pi r^2 h = \frac{22}{7}\times(7)^2\left(\frac{7}{2}\right)$$

$$= 539 \text{ cm}^3.$$

5. A cone of height 24 cm has a curved surface area 550 cm². Find its volume.

Sol :

$$\text{radius} = r,$$

$$\text{slant height} = l$$

$$\text{and } l^2 = r^2 + h^2$$

$$= r^2 + 24^2 = r^2 + 576$$

$$\text{C.S.A} = \pi rl$$

$$= \frac{22}{7}\times r\times\sqrt{r^2+576} \text{ cm}^2$$

$$\frac{22}{7}\times r\times\sqrt{r^2+576} = 550$$

$$r\sqrt{r^2+576} = 175$$

Squaring both sides

$$r^2(r^2+576) = (175)^2$$

$$\Rightarrow r^4 + 576r^2 - (175)^2 = 0$$

$$\Rightarrow (r^2-49)(r^2+625) = 0$$

$$r^2+625 \neq 0, \therefore r^2 - 49 = 0$$

$$r^2 = 49$$

$$r = 7 \text{ cm.}$$

$$\therefore \text{ Volume} = \frac{1}{3}\pi r^2 h \text{ cubic units.}$$

$$= \frac{1}{3}\times\frac{22}{7}\times 7\times 7\times 24$$

$$= 1232 \text{ cm}^3.$$

6. A hemispherical tank is made up of an iron sheet 1 cm thick. If the inner radius is 1 m, then find the volume of the iron used to make the tank.

Sol :

Internal radius = 'r' m = 1 m

External radius = 'R' m = 1 + 0.01 = 1.01 m

Volume of iron used = External volume – Internal volume

$$= \frac{2}{3}\pi(R^3 - r^3) \text{ cubic units}$$

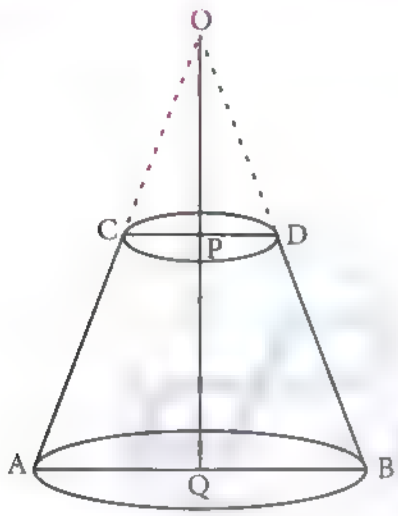
$$= \frac{2}{3}\pi((1.01)^3 - 1^3)$$

$$= \frac{2}{3} \times \frac{22}{7} \times 0.030301$$

$$= 0.06348 \text{ m}^3 \text{ (app).}$$

7. The height of a cone is 30 cm. A small cone is cut off at the top by a plane parallel to the base. If its volume is $\frac{1}{27}$ of the volume of the given cone, at what height above the base is the section made?

Sol :



Volume of the original cone OAB

$$= \frac{1}{3}\pi R^2 H$$

$$= \frac{1}{3}\pi (R^2)(30) = 10\pi R^2 \text{ cm}^3$$

Volume of small cone OCD

$$= \frac{1}{3}\pi r^2 h \text{ cubic units.}$$

Given Volume of cone OCD

$$= \frac{1}{27} \text{ (Volume of cone OAB)}$$

$$\frac{1}{3}\pi r^2 h = \frac{1}{27}(10\pi R^2)$$

$$h = \frac{10\pi R^2}{27}\left(\frac{3}{\pi r^2}\right)$$

$$= \frac{10}{9}\left(\frac{R}{r}\right)^2$$

From similar triangles OQB and OPD.

We get $\frac{QB}{PD} = \frac{OQ}{OP} = \frac{30}{h}$

$$\Rightarrow \quad \frac{R}{r} = \frac{30}{h} \quad \text{... (2)}$$

Substituting (2) in (1),

$$h = \frac{10}{9}\left(\frac{30}{h}\right)^2$$

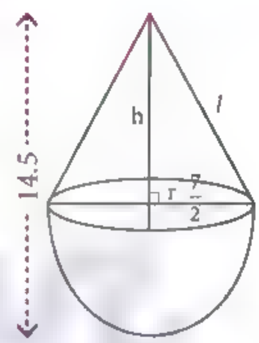
$$h^3 = 1000$$

$$h = 10 \text{ cm}$$

Hence, at $(30 - 10) = 20$ cm above the base, the section is made.

8. A toy is in the form of a cone on a hemisphere of diameter 7 cm. The total height of the toy is 14.5 cm. Find the volume and the total surface area of the toy.

Sol :



Radius of hemisphere 'r' $= \frac{7}{2} = 3.5$ cm

Radius of base of cone = 3.5 cm

Total height of toy = 14.5 cm

Height of conical part = 14.5 – 3.5 = 11 cm

Slant height of cone $l = \sqrt{h^2 + r^2}$

$$= \sqrt{11^2 + 3.5^2}$$

$$= \sqrt{121 + 12.25}$$

$$= \sqrt{133.25} = 11.54 \text{ cm}$$

Volume of toy = Volume of cone + Volume of hemisphere

$$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \frac{\pi r^2}{3}(h + 2r)$$

$$= \frac{22}{3 \times 7} \times 12.25 \times 18$$

$$= 231 \text{ cm}^3$$

Total surface area of toy = C.S.A of cone + C.S.A of hemisphere

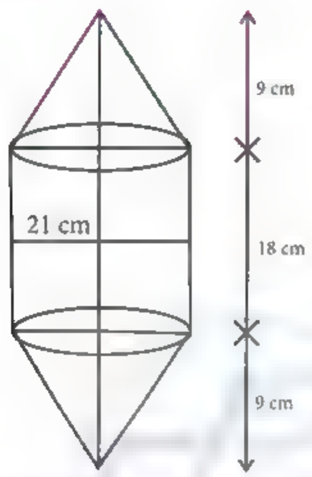
$= \pi rl + 2\pi r^2 = \pi r(l + 2r)$

$= \frac{22}{7} \times 3.5\ (11.54 + 2 \times 3.5)$

$= 22 \times 0.5 \times 18.54 = 203.94\ cm^2$

9. A petrol tank is a cylinder of base diameter 21 cm and length 18 cm fitted with conical ends each of axis length 9 cm. Determine the capacity of the tank.

Sol :



Volume of Cylindrical Portion

$= \pi r^2 h$

$= \frac{22}{7} \times \left(\frac{21}{2}\right)^2 \times 18$

$= 6237\ cm^3$

Volume of two Conical ends

$= 2\left(\frac{1}{3}\pi r^2 h\right)$

$= \frac{2}{3} \times \frac{22}{7} \times \left(\frac{21}{2}\right)^2 \times 9\ cm^3$

$= \frac{174636}{84}$

$= 2079\ cm^3$

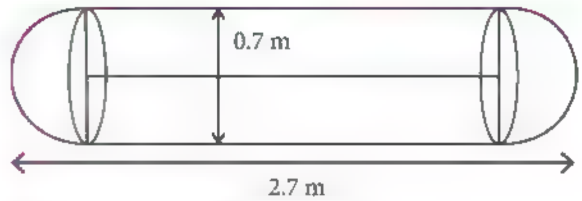
$\therefore$ Capacity of the tank = Volume of Cylinder + Volume of 2 Cones

$= 6237 + 2079$

$= 8316\ cm^3$

10. Find the volume of a solid in the form of a right circular cylinder with hemispherical ends whose total length is 2.7 m and the diameter of each hemispherical end is 0.7 m.

Sol :



radius of hemispherical ends

$= \frac{1}{2} \times 0.7$

$= \frac{0.7}{2} m = \frac{7}{20} m$

Total length of Solid = 2.7 m

Volume of two hemispheres

$= 2\left(\frac{2}{3}\pi r^3\right)$ cu. units

$= \frac{4}{3} \times \frac{22}{7} \times \left(\frac{7}{20}\right)^3$

$= 0.1797\ m^3$

Volume of cylinder $= \pi r^2 h$

$= \frac{22}{7} \times \left(\frac{7}{20}\right)^2 \times 2$

$= 0.77\ m^3$

$\therefore$ Volume of solid $= 0.1797 + 0.77$

$= 0.95\ m^3$ (app)

11. A hemispherical bowl of internal diameter 36 cm contains a liquid. This liquid is to be filled into cylindrical bottles of radius 3 cm and height 6 cm. How many such bottles are required to empty the bowl?

Sol :

Radius of hemispherical bowl $= \frac{36}{2} = 18$ cm

Volume of hemispherical bowl $= \frac{2}{3}\pi r^3$ cubic units

$= \frac{2}{3}\pi (18)^3\ cm^3$

Height of cylindrical bottle = 6 cm

Radius of cylindrical bottle = 3 cm

Volume of cylindrical bottle $= \pi r^2 h$ cu. units

$= \pi (3)^2 (6)$

$\therefore$ Number of bottles required $= \frac{\text{Volume of hemispherical bowl}}{\text{Volume of a bottle}}$

$\therefore$ Number of bottles required $= \dfrac{\frac{2}{3}\pi(18)^3}{\pi(3)^2(6)} = 72.$

12. A vessel in the form of an inverted cone. Its height is 8 cm and the radius is 5 cm. It is filled with water upto the brim. When lead shots each of which is a sphere of radius 0.5 cm are dropped into the vessel, one fourth of the water flows out. Find the number of lead shots dropped into the vessel.

Sol :

radius of cone r = 5 cm, height h = 8 cm

$$\text{volume of cone} = \frac{1}{3}\pi r^2 h \text{ cubic units}$$

$$= \frac{1}{3}\pi(5)^2(8) = \frac{200\pi}{3} cm^3$$

Given that the cone is filled to the brim. When lead shots are dropped, one fourth of the water flown out. The volume of water flown out

$$= \frac{1}{4}\times\frac{200\pi}{3} = \frac{50\pi}{3} cm^3$$

$$\text{Volume of lead shot} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{1}{2}\right)^3$$

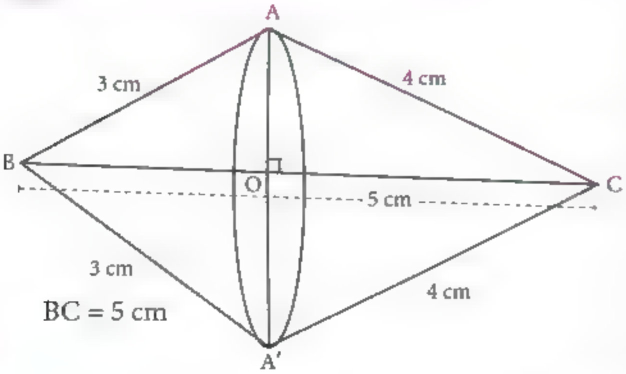
$$= \frac{\pi}{6} cm^3$$

$\therefore$ Number of lead shots dropped into the vessel

$$= \frac{50\pi/3}{\pi/6} = 100.$$

13. A right triangle with sides 3 cm and 4 cm is revolved around its hypotenuse. Find the volume of the double cone thus formed.

Sol :



BC = 5 cm

Let ΔABC be the right triangle, right angled at A whose sides are AB and AC measure 3 cm and 4 cm respectively.

The length of the side BC (hypotenuse)

$$BC = \sqrt{3^2+4^2}$$

$$= \sqrt{25}$$

$$= 5 \text{ cm}$$

By revolving, ΔABC around its hypotenuse BC, the double Cone is formed.

This solid consists two cones namely BAA' and CAA'

AO or $A'O$ is the common radius.

Height of the Cone CAA' is CO and slant height is 4 cm.

Height of the Cone BAA' is BO and slant height is 3 cm.

Now ΔAOB is similar to ΔCAB

$\therefore$ Corresponding sides are proportional

$$\text{i.e., } \frac{AO}{AC} = \frac{AB}{BC} = \frac{BO}{AB}$$

$$\Rightarrow \frac{AO}{4} = \frac{3}{5} \Rightarrow AO = \frac{12}{5} cm$$

$$\text{Similarly, } \frac{BO}{3} = \frac{3}{5} \Rightarrow BO = \frac{9}{5} cm$$

$$CO = BC - BO$$

$$= 5 - \frac{9}{5} = \frac{16}{5} cm$$

$\therefore$ Volume of double Cone

$$v = \frac{12}{5}, h_1 = BO = \frac{9}{5}, h_2 = CO = \frac{16}{5}$$

$$= \left(\frac{1}{3}\pi r^2 \times BO\right) + \left(\frac{1}{3}\pi r^2 \times CO\right)$$

$$= \frac{1}{3}\pi r^2 (BO + CO)$$

$$= \frac{22}{7\times 3}\left(\frac{12}{5}\right)^2\left(\frac{9}{5}+\frac{16}{5}\right)$$

$$= \frac{22}{7\times 3}\times\frac{12}{5}\times\frac{12}{5}\times 5$$

$$= 30.17 \text{ cm}^3$$

UNIT 8

STATISTICS AND PROBABILITY

MIND MAP

- STATISTICS AND PROBABILITY
  - Statistics
    - Measures of Central Tendency
      - Mean
        Median
        Mode
    - Measures of Dispersion
      - Range = L − S
      - Standard Deviation
      - Co-efficient of variation $C.V = \frac{\sigma}{\bar{x}} \times 100\%$
      - Variance = σ^2 = $(S.D.)^2$
  - Probability
    - Probability of Events
      - S-Sample space
        - Tree diagram
      - $P(E) = \frac{n(E)}{n(S)}$
    - Algebra of Events
      - (1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
      - (2) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$
      - (3) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

MEASURES OF DISPERSION

Key Points

Prasanta Chandra Mahalanobis is the "Father of Indian Statistics".

- He was awarded the Padma Vibhushan in 1968 for introducing innovative techniques for conducting large scale sample surveys.
- The Government of India has designated 29th June every year, coinciding with his birth anniversary as "National Statistics Day".
- STATISTICS is derived from the Latin word status which means a Political State.
- The study of statistics is concerned with scientific methods for collecting, organising, summarising, presenting, analysing data and making meaningful decisions.

Recalling Measures of Central Tendency

- A number that represents the whole data is called a Measures of Central Tendency or an average.
- The Measures of Central Tendency usually will be near to the middle value of the data.
- The Most Common Measures of Central Tendency are

(i) Arithmetic mean

(ii) Median

(iii) Mode

Arithmetic mean $\bar{x} = \dfrac{\textit{Sum of all observations}}{\textit{Number of observations}}$

Methods of Finding Mean

- **Ungrouped data**
  - **Direct method**: $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$
- **Grouped data**
  - **Direct method**: $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$
  - **Assumed Mean Method**: $\bar{x} = A + \dfrac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$, Where $d_i = x_i - A$
  - **Step Deviation Method**: $\bar{x} = A + C \times \dfrac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$, Where $d_i = \dfrac{x_i - A}{C}$

- The representation of facts numerically is called Data.
- Each entry in the data is called an Observation.
- The quantities which are being considered in a survey are called Variables. Variables are generally denoted by $x_1, x_2, x_3, \ldots$
- The number of times a variable occurs in a given data is called the Frequency of that variable. It is generally denoted as $f_1, f_2, f_3, f_4 \ldots$
- Dispersion is a measure which gives an idea about the scatteredness of the values.
- Measures of Variation (or) Dispersion of a data provide an idea of how observations spread out (or) Scattered throughout the data.
- Different Measures of Dispersion are
  - Range
  - Quartile Deviation
  - Variance
  - Mean Deviation
  - Standard Deviation
  - Co-efficient of Variation

Range:

- The difference between the largest value and the smallest value is called Range.

$$\text{Range } R = L - S$$

- $$\text{Co-efficient of Range} = \frac{L-S}{L+S}$$

  where L – Largest Value
  S – Smallest value
- If the frequency of the initial class is zero, then the next class will be considered for the calculation of range.
- We know that the squares of deviations from the mean $(\bar{x})$, i.e., $(x_i - \bar{x})^2 \geq 0$ for all observations x_i, i = 1, 2, 3,..., n.
- If the deviations from the mean $x_i - \bar{x}$ are small, then the squares of the deviations will be very small.

Variance:

- The mean of the squares of the deviations from the mean is called Variance. It is denoted by σ^2.
- Variance = Mean of squares of deviations.
- Variance $\sigma^2 = \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$

Standard Deviation:

- The positive square root of the variance is called Standard Deviation.
- Standard deviation is the positive square root of the mean of the squares of deviations of the given values from their mean.
- Standard deviation gives the clear idea about how far the values are spreading or deviating from their mean.
- Standard deviation $\sigma = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$
- Karl Pearson was the first person to use the word standard deviation.
- German mathematician Gauss used the word mean error.
- The standard deviation and mean have same units in which the data are given.

CALCULATION OF STANDARD DEVIATION

- **Ungrouped data**
  - **Direct Method**: $\sigma = \sqrt{\dfrac{\Sigma x_i^2}{n} - \left(\dfrac{\Sigma x_i}{n}\right)^2}$
  - **Mean Method**: $\sigma = \sqrt{\dfrac{\Sigma d_i^2}{n}}$, where $d_i = x_i - \bar{x}$
  - **Assumed Mean Method**: $\sigma = \sqrt{\dfrac{\Sigma d_i^2}{n} - \left(\dfrac{\Sigma d_i}{n}\right)^2}$, where $d_i = x_i - A$, A – assumed mean
  - **Step Deviation Method**: $\sigma = C \times \sqrt{\dfrac{\Sigma d_i^2}{n} - \left(\dfrac{\Sigma d_i}{n}\right)^2}$, where $d_i = \dfrac{x_i - A}{C}$; C – Common divisor; A – Assumed mean
- **Grouped data (Both discrete and continuous)**
  - **Mean Method**: $\sigma = \sqrt{\dfrac{\Sigma f_i d_i^2}{N}}$, where $N = \sum_{i=1}^{n} f_i$, $d_i = x_i - \bar{x}$
  - **Assumed Mean Method**: $\sigma = \sqrt{\dfrac{\Sigma f_i d_i^2}{N} - \left(\dfrac{\Sigma f_i d_i}{N}\right)^2}$, where $d_i = x_i - A$, A – Assumed mean
    - $\sigma = \sqrt{\dfrac{\Sigma f_i (x_i - \bar{x})^2}{N}}$ if continuous
  - **Step Deviation Method**: $\sigma = C \times \sqrt{\dfrac{\Sigma f_i d_i^2}{N} - \left(\dfrac{\Sigma f_i d_i}{N}\right)^2}$, $d_i = \dfrac{x_i - A}{C}$

- If the data values are given directly then to find standard deviation we use

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

- If the squares of the deviations from the mean of each observation is given then to find the standard deviation we use $\sigma = \sqrt{\dfrac{\Sigma (x_i - \bar{x})^2}{n}}$
- When the mean value is not an integer, then we use Assumed mean method to find standard deviation.
- The standard deviation will not change when we add or subtract some fixed constant to all the values.
- When we multiply or divide each data by a fixed constant then the standard deviation is also get multiplied or divided by the constant.

Worked Examples

8.1 **Find the range and co-efficient of range of the following data: 25, 67, 48, 53, 18, 39, 44.**

Sol :

Largest value L = 67; Smallest value S = 18

Range R = L – S = 67 – 18 = 49

Co-efficient of range= $\frac{L-S}{L+S}$

Co-efficient of range= $\frac{67-18}{67+18} = \frac{49}{85} = 0.576$

8.2 **Find the range of the following distribution.**

Age (in years)	Number of students
16-18	0
18-20	4
20-22	6
22-24	8
24-26	2
26-28	2

Sol :

Here Largest value L = 28

Smallest value S = 18

Range R = L – S = 28 –18

= 10 Years

8.3 **The range of a set of data is 13.67 and the largest value is 70.08. Find the smallest value.**

Sol :

Range R = 13.67

Largest value L = 70.08

Range R = L – S

13.67 = 70.08 – S

S = 70.08 – 13.67 = 56.41

Therefore, the smallest value is 56.41.

8.4 **The number of televisions sold in each day of a week are 13, 8, 4, 9, 7, 12, 10. Find its standard deviation.**

Sol :

x_i	x_i^2
13	169
8	64
4	16
9	81
7	49
12	144
10	100
$\Sigma x_i = 63$	$\Sigma x_i^2 = 623$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{623}{7} - \left(\frac{63}{7}\right)^2}$$

$$= \sqrt{89-81} = \sqrt{8}$$

Hence $\sigma \simeq 2.83$

8.5 **The amount of rainfall in a particular season for 6 days are given as 17.8 cm, 19.2 cm, 16.3 cm, 12.5 cm, 12.8 cm and 11.4 cm. Find its standard deviation.**

Sol :

Arranging the numbers in ascending order we get 11.4, 12.5, 12.8, 16.3, 17.8, 19.2.

Number of observations n = 6

$$\text{Mean} = \frac{11.4+12.5+12.8+16.3+17.8+19.2}{6}$$

$$= \frac{90}{6} = 15$$

x_i	$d_i = x_i - \bar{x}$ $= x - 15$	d_i^2
11.4	– 3.6	12.96
12.5	– 2.5	6.25
12.8	– 2.2	4.84
16.3	1.3	1.69
17.8	2.8	7.84
19.2	4.2	17.64
		$\Sigma d_i^2 = 51.22$

Standard deviation $\sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$

$$= \sqrt{\frac{51.22}{6}} = \sqrt{8.53}$$

Hence $\sigma \simeq 2.9$

8.6 **The marks scored by 10 students in a class test are 25, 29, 30, 33, 35, 37, 38,40, 44, 48. Find the standard deviation.**

Sol :

The mean of marks is 35.9 which is not an integer. Hence we take assumed mean A = 35, n = 10.

x_i	$d_i = x_i - A$ $d_i = x_i - 35$	d_i^2
25	− 10	100
29	− 6	36
30	− 5	25
33	− 2	4
35	0	0
37	2	4
38	3	9
40	5	25
44	9	81
48	13	169
	$\Sigma d_i = 9$	$\Sigma d_i^2 = 453$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\frac{453}{10} - \left(\frac{9}{10}\right)^2}$$

$$= \sqrt{45.3 - 0.81}$$

$$= \sqrt{44.49}$$

Henec $\sigma \simeq 6.67$

8.7 **The amount that the children have spent for purchasing some eatables in one day trip of a school are 5, 10, 15, 20, 25, 30, 35, 40. Using step deviation method, find the standard deviation of the amount they have spent.**

Sol :

We note that all the observations are divisible by 5. Hence we can use the step deviation method. Let Assumed mean A = 20, n = 8.

x_i	$d_i = x_i - A$ $d_i = x_i - 20$	$d_i = \frac{x_i - A}{c}$ $c = 5$	d_i^2
5	− 15	− 3	9
10	− 10	− 2	4
15	− 5	− 1	1
20	0	0	0
25	5	1	1
30	10	2	4
35	15	3	9
40	20	4	16
		$\Sigma d_i = 4$	$\Sigma d_i^2 = 44$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} \times c$$

$$= \sqrt{\frac{44}{8} - \left(\frac{4}{8}\right)^2} \times 5 = \sqrt{\frac{11}{2} - \frac{1}{4}} \times 5$$

$$= \sqrt{5.5 - 0.25} \times 5 = 2.29 \times 5$$

$$\sigma \simeq 11.45$$

8.8. **Find the standard deviation of the following data 7, 4, 8, 10, 11. Add 3 to all the values then find the standard deviation for the new values.**

Sol :

Arranging the values in ascending order we get 4, 7, 8, 10, 11 and n = 5

x_i	x_i^2
4	16
7	49
8	64
10	100
11	121
$\Sigma x_i = 40$	$\Sigma x_i^2 = 350$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{350}{5} - \left(\frac{40}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

When we add 3 to all the values, we get the new values as 7, 10, 11, 13, 14.

x_i	x_i^2
7	9
10	100
11	121
13	169
14	196
$\Sigma x_i = 55$	$\Sigma x_i^2 = 635$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{635}{5} - \left(\frac{55}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

From the above, we see that the standard deviation will not change when we add some fixed constant to all the values.

8.9 **Find the standard deviation of the data 2, 3, 5, 7, 8. Multiply each data by 4. Find the standard deviation of the new values.**

Sol :

Given, n = 5

x_i	x_i^2
2	4
3	9
5	25
7	49
8	64
$\Sigma x_i = 25$	$\Sigma x_i^2 = 151$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{151}{5} - \left(\frac{25}{5}\right)^2}$$

$$= \sqrt{30.2 - 25}$$

$$= \sqrt{5.2} \simeq 2.28$$

When we multiply each data by 4, we get the new values as 8, 12, 20, 28, 32.

x_i	x_i^2
8	64
12	144
20	400
28	784
32	1024
$\Sigma x_i = 100$	$\Sigma x_i^2 = 2416$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{2416}{5} - \left(\frac{100}{5}\right)^2}$$

$$= \sqrt{483.2 - 400} = \sqrt{83.2}$$

$$\sigma = \sqrt{16 \times 5.2} = 4\sqrt{5.2} \simeq 9.12$$

From the above, we see that when we multiply each data by 4 the standard deviation is also get multiplied by 4.

8.10 **Find the mean and variance of the first n natural numbers.**

Sol :

$$\text{Mean } \bar{x} = \frac{\textit{Sum of all the observations}}{\textit{Number of observations}}$$

$$= \frac{\Sigma x_i}{n} = \frac{1+2+3+.....+n}{n} = \frac{n(n+1)}{2 \times n}$$

$$\text{Mean } \bar{x} = \frac{n+1}{2}$$

$$\text{Variance } \sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

$$\begin{bmatrix} \Sigma x_i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 \\ (\Sigma x_i)^2 = (1+2+3+.....+n)^2 \end{bmatrix}$$

$$= \frac{n(n+1)(2n+1)}{6 \times n} - \left[\frac{n(n+1)}{2 \times n}\right]^2$$

$$= \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$= \frac{4n^2+6n+2-3n^2-6n-3}{12}$$

$$\text{Variance } \sigma^2 = \frac{n^2-1}{12}.$$

8.11 **48 students were asked to write the total number of hours per week they spent for watching television. With this information find the standard deviation of hours spent for watching television.**

x	6	7	8	9	10	11	12
f	3	6	9	13	8	5	4

Sol :

x_i	f_i	x_if_i	$d_i = x_i - \bar{x}$	d_i^2	$f_i d_i^2$
6	3	18	– 3	9	27
7	6	42	– 2	4	24
8	9	72	– 1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	N = 48	$\Sigma x_i f_i = 432$	$\Sigma d_i = 0$		$\Sigma f_i d_i^2 = 124$

Mean $\bar{x} = \frac{\Sigma x_i f_i}{N} = \frac{432}{48} = 9$ (Since $N = \Sigma f_i$)

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}} = \sqrt{\frac{124}{48}} = \sqrt{2.58}$$

$$\sigma \simeq 1.6$$

8.12 The marks scored by the students in a slip test are given below.

x	4	6	8	10	12
f	7	3	5	9	5

Find the standard deviation of their marks.

Sol :

Let the assumed mean, A = 8

x_i	f_i	$d_i = x_i - A$	$f_i d_i$	$f_i d_i^2$
4	7	– 4	– 28	112
6	3	– 2	– 6	12
8	5	0	0	0
10	9	2	18	36
12	5	4	20	80
	N = 29		$\Sigma f_i d_i = 4$	$\Sigma f_i d_i^2 = 240$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{240}{29} - \left(\frac{4}{29}\right)^2}$$

$$= \sqrt{\frac{240 \times 29 - 16}{29 \times 29}}$$

$$\sigma = \sqrt{\frac{6944}{29 \times 29}} \simeq 2.87$$

8.13 Marks of the students in a particular subject of a class are given below.

Marks	Number of students
0-10	8
10–20	12
20–30	17
30–40	14
40–50	9
50–60	7
60–70	4

Find its standard deviation.

Sol :

Let Assumed mean, A = 35, c = 10

Marks	Mid value(x_i)	f_i	$d_i = x_i - A$	$d_i = \frac{x_i - A}{C}$	$f_i d_i$	$f_i d_i^2$
0-10	5	8	– 30	– 3	– 24	72
10-20	15	12	– 20	– 2	– 24	48
20-30	25	17	– 10	– 1	– 17	17
30-40	35	14	0	0	0	0
40-50	45	9	10	1	9	9
50-60	55	7	20	2	14	28
60-70	65	4	30	3	12	36
		N = 71			$\Sigma f_i d_i = -30$	$\Sigma f_i d_i^2 = 210$

Standard deviation

$$\sigma = c \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$\sigma = 10 \times \sqrt{\frac{210}{71} - \left(-\frac{30}{71}\right)^2}$$

$$\sigma = 10 \times \sqrt{\frac{210}{71} - \frac{900}{5041}}$$

$$\sigma = 10 \times \sqrt{2.779} \ ; \ \simeq 16.67$$

8.14 The mean and standard deviation of 15 observations are found to be 10 and 5 respectively. On rechecking it was found that one of the observation with value 8 was incorrect. Calculate the correct mean and standard deviation if the correct observation value was 23.

Sol :

$n = 15,\ \bar{x} = 10,\ \sigma = 5;\ \bar{x} = \frac{\Sigma x}{n};$

$\Sigma x = 15 \times 10 = 150$

Wrong observation value = 8.

Correct observation value = 23.

Correct total = 150 – 8 + 23 = 165

Correct mean $\bar{x} = \frac{165}{15} = 11$

Standard deviation $\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$

Incorrect value of $\sigma = 5 = \sqrt{\frac{\Sigma x^2}{15} - (10)^2}$

$25 = \frac{\Sigma x^2}{15} - 100 \Rightarrow \frac{\Sigma x^2}{15} = 125$

Incorrect value of $\Sigma x^2 = 1875$

Correct value of $\Sigma x^2 = 1875 - 8^2 + 23^2 = 2340$

Correct standard deviation $\sigma = \sqrt{\frac{2340}{15} - (11)^2}$

$\sigma = \sqrt{156 - 121} = \sqrt{35} \simeq 5.9$

Progress Check

1. _____ is a value that represent a set of data

Ans : An observation

2. The sum of all the observations divided by number of observations is _____

Ans : Arithmetic Mean

3. If the sum of 10 data values is 265 then their mean is _____

Ans : 26.5

4. If the sum and mean of a data are 407 and 11 respectively, then the number of observations in the data are _____

Ans : 37

5. The range of first 10 prime numbers is _____

Ans : 27

Range = L – S = 29 – 2 = 27

6. If the variance is 0.49, then the standard deviation is _____

Ans : 0.7

Thinking Corner

1. Does the mean, median and mode are same for a given data?

Ans : In a perfectly symmetrical distribution, the mean and median are the same. If the distribution has one mode, the mode is same as the mean and median.

That is a symmetrical distribution which is unimodal has its mean, median and mode the same.

2. What is the difference between the arithmetic mean and average?

Ans : The measures of Central Tendency Mean, median and mode are called averages. Arithmetic mean is the sum of all observation divided by total number of observations. So. Arithmetic mean is one type of averages.

3. The mean of n observation is $\bar{x}$, if first term is increased by 1 second term is increased by 2 and so on. What will be the new mean?

Ans : Given the mean of n observations is $\bar{x}$. Let x_1, x_2, x_3, ..., x_n be the n observations.

Then $\frac{x_1 + x_2 + x_3 + ... + x_n}{n} = \bar{x}$.

If 1 is added to the first term, 2 to the second term and so on.

Then the terms will be $x_1 + 1, x_2 + 2, x_3 + 3, ... x_n + n$.

Now the mean

$= \frac{x_1 + 1 + x_2 + 2 + x_3 + 3 + ... + x_n + n}{n}$

$= \frac{x_1 + x_2 + x_3 + ... + x_n}{n} + \frac{1 + 2 + 3 + ... + n}{n}$

$= \bar{x} + \frac{\frac{n(n+1)}{2}}{n}$

$$= \bar{x} + \frac{n(n+1)}{2} \times \frac{1}{n}$$

$$= \bar{x} + \frac{(n+1)}{2}$$

4. Can variance be negative?

Ans : Variance cannot be negative, because it is the mean of the squares of the deviations from the mean. Any squared value never be negative.

5. Can the standard deviation be more than the variance?

Ans : The variance is the square of the standard deviation.

So if the variance less than 1, then the standard deviation be more than the variance, otherwise variance has the greater value.

For example if variance = 0.001, then the standard deviation = $\sqrt{0.01}$ = 0.1

6. For any collection of n values.

(i) $\Sigma(x_i - \bar{x}) =$ _____

(ii) $(\Sigma x_i) - \bar{x} =$ _____

(i) $\Sigma(x_i - \bar{x}) = \Sigma d_i$, the sum of deviations from the mean of each observations.

(ii) $\Sigma x_i - \bar{x} = \Sigma x_i - \frac{\Sigma x_i}{n}$ $\left[\because \bar{x} = \frac{\Sigma x_i}{n}\right]$

$$= \frac{(n-1)}{n}\Sigma x_i$$

7. (i) The standard deviation of a data is 2.8, if 5 is added to all the data values then the new standard deviation is _____

Ans : 2.8

(ii) If S is the standard deviation p, q, r then standard deviation of p – 3, q – 3, r – 3 is _____

Ans : S

Exercise 8.1

1. Find the range and coefficient of range of the following data.

(i) 63, 89, 98, 125, 79, 108, 117, 68

(ii) 43.5, 13.6, 18.9, 38.4, 61.4, 29.8

Sol :

(i) 63, 89, 98, 125, 79, 108, 117, 68

Largest value L = 125

Smallest value S = 63

Range R = L – S = 125 – 63 = 62

Coefficient of range $= \frac{L-S}{L+S} = \frac{125-63}{125+63}$

$= \frac{62}{188} = 0.329 = 0.33$

$\therefore$ Range = 62 ; coefficient of range = 0.33

(ii) 43.5, 13.6, 18.9, 38.4, 61.4, 29.8

Largest value L = 61.4

Smallest value S = 13.6

Range = L – S

= 61.4 – 13.6 = 47.8

Coefficient of range $= \frac{61.4-13.6}{61.4+13.6}$

$= \frac{47.8}{75} = 0.637 = 0.64$

$\therefore$ Range = 47.8 ; co-efficient of range = 0.64.

2. If the range and the smallest value of a set of data are 36.8 and 13.4 respectively, then find the largest value.

Sol :

Given range = 36.8

Smallest value = 13.4

Range R = L – S

36.8 = L – 13.4

36.8 + 13.4 = L

L = 50.2

$\therefore$ The largest value L = 50.2

3. Calculate the range of the following data.

Income	400-450	450-500	500-550	550-600	600-650
Number of workers	8	12	30	21	6

Sol :

Here the Largest value L = 650

Smallest value S = 400

$\therefore$ Range R = L – S

R = 650 – 400 = 250

$\therefore$ Range R = 250

4. A teacher asked the students to complete 60 pages of a record note book. Eight students have completed only 32, 35, 37, 30, 33, 36, 35 and 37 pages. Find the standard deviation of the pages yet to be completed by them.

Sol :

Total pages = 60

Completed pages by the students are

32, 35, 37, 30, 33, 36, 35, 37

Let the pages yet to be completed by 8 students be x_i

Completed Pages	Pages yet to be completed (x_i) $x_i = 60 -$ completed pages	${x^2}_i$
32	28	784
35	25	625
37	23	529
30	30	900
33	27	729
36	24	576
35	25	625
37	23	529
	$\Sigma {x_i}^2 = 205$	$\Sigma x_i^2 = 5297$

Standard deviation $\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$

$= \sqrt{\frac{9497}{8} - \left(\frac{275}{8}\right)^2}$

$= \sqrt{1187.125 - (34.375)^2} = \sqrt{1187.125 - 1181.64063}$

$= \sqrt{5.48437} = 2.34 \quad \therefore \sigma \simeq 2.34$

5. Find the variance and standard deviation of the wages of 9 workers given below: ₹ 310, ₹ 290, ₹ 320, ₹ 280, ₹ 300, ₹ 290, ₹ 320, ₹ 310, ₹ 280.

Sol :

Arranging the numbers in ascending order

₹ 280, ₹ 280, ₹ 290, ₹ 290, ₹ 300, ₹ 310, ₹ 310, ₹ 320, ₹ 320

Mean= $\frac{280+280+290+290+300+310+310+320+320}{9}$

$\bar{x} = \frac{2700}{9} = 300$

x_i	$d_i = x_i - \bar{x} = x_i - 300$	d_i^2
280	– 20	400
280	– 20	400
290	– 10	100
290	– 10	100
300	0	0
310	10	100
310	10	100
320	20	400
320	20	400
		$\Sigma d_i^2 = 2000$

Standard Deviation

$\sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$

$\sigma = \sqrt{\frac{2000}{9}}$

$\sigma = \sqrt{222.222...} = 14.907 = 14.91$

We have $\sigma = \sqrt{222.222}$

Variance $\sigma^2 = 222.22$

$\therefore$ Variance = 222.22 ;

Standard deviation = 14.91

6. A wall clock strikes the bell once at 1 O'clock, 2 times at 2 O'clock, 3 times at 3 O'clock and so on. How many times will it strike in a particular day? Find the standard deviation of the number of strikes the bell make a day.

Sol :

Clock strikes once at 1, twice at 2, thrice at 3 and so on.

But it only strikes 12 times at most and then it repeats. So, Number of times clock strikes =

2 (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12)

=78 × 2 = 156 times.

$\therefore$ In a particular day the clock strikes 156 times.

Also standard deviation for first n natural numbers is $\sqrt{\frac{n^2-1}{12}}$

$\therefore$ Standard deviation of the number of strikes

$= 2\sqrt{\frac{n^2-1}{12}}$

$= 2\sqrt{\frac{12^2-1}{12}} = 2\sqrt{\frac{144-1}{12}}$

$$= 2\sqrt{\frac{143}{12}} = 2\sqrt{11.92}$$

$$= 2 \times 3.45 = 6.9$$

$\therefore$ Standard deviation of the number of strikes the bell make a day = 6.9

7. Find the standard deviation of first 21 natural numbers.

Sol : Standard deviation of first n natural numbers

$$= \sqrt{\frac{n^2 - 1}{12}}$$

$\therefore$ SD of first 21 natural numbers

$$= \sqrt{\frac{21^2 - 1}{12}}$$

$$= \sqrt{\frac{441 - 1}{12}} = \sqrt{\frac{440}{12}}$$

$$= \sqrt{36.6666} = 6.05$$

Standard deviation of first 21 natural numbers = 6.05

8. If the standard deviation of a data is 4.5 and if each value of the data is decreased by 5, then find the new standard deviation.

Sol : The standard deviation of a given data is 4.5. If we subtract some fixed constant from all the data, the standard deviation will not change.

$\therefore$ Each value of the data decreased by 5, the new standard deviation will not change.

$\therefore$ New standard deviation = 4.5

9. If the standard deviation of a data is 3.6 and each value of the data is divided by 3, then find the new variance and standard deviation.

Sol :

Standard deviation of a data = 3.6.

When each value of the data is divided by a fixed constant, then the new standard deviation is also get divided by the constant.

$\therefore$ If each value is divided by 3, then

$$\text{New standard deviation} = \frac{3.6}{3} = 1.2$$

$$\text{New variance} = \sigma^2 = (1.2)^2$$

$$\therefore \text{New variance} = 1.44$$

New standard deviation= 1.2

10. The rainfall recorded in various places of five districts in a week are given below.

Rainfall (in mm)	45	50	55	60	65	70
Number of places	5	13	4	9	5	4

Find its standard deviation

Sol : Let the assumed mean A = 55

x_i	f_i	$d_i = x_i - A$ $d_i = x_i - 55$	$f_i d_i$	d_i^2	$f_i d_i^2$
45	5	– 10	– 50	100	500
50	13	– 5	– 65	25	325
55	4	0	0	0	0
60	9	5	45	25	225
65	5	10	50	100	500
70	4	15	60	225	900
	$\Sigma f_i = 40$		$\Sigma f_i d_i = 40$		$\Sigma f_i d_i^2 = 2450$

$\Sigma f_i = N = 40$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{2450}{40} - \left(\frac{40}{40}\right)^2} = \sqrt{61.25 - 1^2}$$

$$= \sqrt{61.25 - 1} = \sqrt{60.25}$$

Standard deviation σ = 7.76

11. In a study about viral fever, the number of people affected in a town were noted as

Age in years	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Number of people affected	3	5	16	18	12	7	4

Find its standard deviation.

Sol :

Let us take the assumed mean A = 20 and C = 5

Ages	Mid value x_i	f_i	$d_i = x_i - A$ $d_i = x_i - 20$	$d_i = \frac{x_i - A}{C}$ $d_i = \frac{x_i - A}{5}$	$f_i d_i$	$f_i d_i^2$
0-10	5	3	– 15	– 3	– 9	27
10-20	15	5	– 5	– 1	– 5	5
20-30	25	16	5	1	16	16
30-40	35	18	15	3	54	162
40-50	45	12	25	5	60	300
50-60	55	7	35	7	49	343
60-70	65	4	45	9	36	324
		$\Sigma f_i = N = 65$			$\Sigma f_i d_i = 201$	$\Sigma f_i d_i^2 = 1177$

Standard deviation $\sigma = C \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$

$\sigma = 5 \times \sqrt{\frac{1177}{65} - \left(\frac{201}{65}\right)^2}$

$\sigma = 5 \times \sqrt{\frac{1177}{65} - \frac{40401}{4225}}$

$\sigma = 5 \times \sqrt{\frac{76505 - 40401}{4225}}$

$\sigma = 5 \times \sqrt{8.54}$

$\sigma = 5 \times 2.92 = 14.6$

Standard deviation $\sigma = 14.6$

12. The measurements of the diameters (in cms) of the plates prepared in a factory are given below. Find its standard deviation.

Diameter (cm)	21-24	25-28	29-32	33-36	37-40	41-44
Number of plates	15	18	20	16	8	7

Sol : Let the assumed mean A = 34.5

Diameters (cm)	Mid value x_i	f_i	$d_i = \frac{x_i - A}{2}$, $d_i = \frac{x_i - 34.5}{2}$	$f_i d_i$	d_i^2	$f_i d_i^2$
20.5–24.5	22.5	15	– 6	– 90	36	540
24.5–28.5	26.5	18	– 4	– 72	16	288
28.5–32.5	30.5	20	– 2	– 40	4	80
32.5–36.5	34.5	16	0	0	0	0
36.5–40.5	38.5	8	2	16	4	32
40.5–44.5	42.5	7	4	28	16	112
		Σf_i = N = 84		$\Sigma f_i d_i$ = – 158		$\Sigma f_i d_i^2$ = 1052

Standard deviation $\sigma = C \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$

$\sigma = 2 \times \sqrt{\frac{1052}{84} - \left(\frac{-158}{84}\right)^2}$

$\sigma = 2 \times \sqrt{\frac{22092}{1764} - \frac{6241}{1764}}$

$\sigma = 2 \times \sqrt{\frac{15851}{1764}} = 2 \times \sqrt{8.98}$

$\sigma = 2 \times 2.99 = 5.99$

$\therefore$ Standard deviation $\sigma \simeq 5.99 \simeq 6$

13. The time taken by 50 students to complete a 100 meter race are given below. Find its standard deviation

Time taken (sec)	8.5 - 9.5	9.5 - 10.5	10.5 - 11.5	11.5 - 12.5	12.5 - 13.5
Number of students	6	8	17	10	9

Sol :

Let x_i are the midvalues of the given set

Assumed mean A = 11

Time taken (sec)	x_i	f_i	$d_i = x_i - 11$	d^2	fd	fd^2
8.5–9.5	9	6	– 2	4	– 12	24
9.5–10.5	10	8	– 1	1	– 8	8
10.5–11.5	11	17	0	0	0	0
11.5–12.5	12	10	1	1	10	10
12.5–13.5	13	9	2	4	18	36
		Σf_i = N = 50			$\Sigma f_i d_i$ = 8	$\Sigma f_i d_i^2$ = 78

Standard deviation

$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$

$= \sqrt{\frac{78}{50} - \left(\frac{8}{50}\right)^2}$

$= \sqrt{\frac{78}{50} - \frac{64}{2500}}$

$= \sqrt{\frac{3900 - 64}{2500}} = \sqrt{\frac{3836}{2500}}$

$= \sqrt{1.5344} \simeq 1.238 \simeq 1.24$

$\therefore$ Standard deviation $\sigma \simeq 1.24$

14. For a group of 100 candidates the mean and standard deviation of their marks were found to be 60 and 15 respectively. Later on it was found that the scores 45 and 72 were wrongly entered as 40 and 27. Find the correct mean and standard deviation.

Sol :

Mean $\bar{x}$ = 60

Standard deviation σ = 15

Wrong scores = 40 and 27

Correct scores = 45 and 72.

$$\text{Old Mean} = \frac{\Sigma x_i}{n} = 60$$

$$\frac{\Sigma x_i}{100} = 60$$

$$\text{Old } \Sigma x_i = 60 \times 100 = 6000$$

$$\therefore \text{Correct } \Sigma x_i = 6000 - \text{wrong scores} + \text{correct scores}$$

$$= 6000 - (40 + 27) + (45 + 72)$$

$$= 6000 - 67 + 117$$

$$\therefore \text{Correct } \Sigma x_i = 6050$$

$$\text{Correct Mean} = \text{Correct } \frac{\Sigma x_i}{n} = \frac{6050}{100}$$

$$\text{Correct Mean} = 60.5$$

Old Standard deviation $\sigma = 15$

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$15 = \sqrt{\frac{\Sigma x_i^2}{n} - (60)^2}$$

Squaring on both sides

$$225 = \frac{\Sigma x_i^2}{n} - 3600$$

$$225 + 3600 = \frac{\Sigma x_i^2}{100}$$

$$\frac{old\ \Sigma x_i^2}{100} = 3825$$

$$\text{Old } \Sigma x_i^2 = 3825 \times 100$$

$$\text{Old } \Sigma x_i^2 = 382500$$

$$\text{Correct } \Sigma x_i^2 = 382500 - (\text{wrong scores})^2 + (\text{correct scores})^2$$

$$= 382500 - 40^2 - 27^2 + 45^2 + 72^2$$

$$= 382500 - 1600 - 729 + 2025 + 5184$$

$$= 382500 - 2329 + 2025 + 5184$$

$$= 389709 - 2329$$

$$\text{Correct } \Sigma x_i^2 = 387380$$

$$\text{Correct } \sigma = \sqrt{\frac{387380}{100} - (60.5)^2}$$

$$= \sqrt{3873.80 - 3660.25}$$

$$= \sqrt{213.55}$$

$$\sigma = 14.61$$

$\therefore$ Correct σ = 14.61; Correct Mean = 60.5

15. The mean and variance of seven observations are 8 and 16 respectively. If five of these are 2 , 4, 10, 12 and 14, then find the remaining two observations.

Sol :

Mean = 8

Variance = 16.

Let a and b be the missing observations

$$\frac{\textit{Sum of observations}}{\textit{Number of observations}} = \text{Mean}$$

$$\frac{2+4+10+12+14+a+b}{7} = 8$$

$$42 + a + b = 56$$

$$a + b = 56 - 42 = 14 \qquad \text{......(1)}$$

$$b = 14 - a$$

$$\text{Variance} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$$

$$= \frac{\sum_{i=1}^{7}(x_i - \bar{x})^2}{7}$$

$$= \frac{(2-8)^2 + (4-8)^2 + (10-8)^2 + (12-8)^2 + (14-8)^2 + (a-8)^2 + (b-8)^2}{7}$$

$$16 = \frac{36+16+4+16+36+(a-8)^2+(b-8)^2}{7}$$

$$108 + (a-8)^2 + (b-8)^2 = 112$$

$$(a-8)^2 + (b-8)^2 = 4 \qquad \text{... (2)}$$

Put b = 14 – a in (2)

$$(a-8)^2 + (b-8)^2 = 4$$

$$(a-8)^2 + ((14-a)-8)^2 = 4$$

$$(a-8)^2 + (6-a)^2 = 4$$

$$a^2 + 64 - 16a + 36 + a^2 - 12a - 4 = 0$$

$$2a^2 - 28a + 96 = 0$$

Divided by 2,

$$a^2 - 14a + 48 = 0$$

$$(a-6)(a-8) = 0$$

$$a = 6 \text{ or } a = 8$$

$$a = 6 \Rightarrow b = 14 - a = 14 - 6 = 8$$

$$a = 6 \text{ and } b = 8$$

$\therefore$ Remaining numbers are 6 and 8.

CO-EFFICIENT OF VARIATION

Key Points

Co-efficient of variation is used for comparing two or more data for corresponding changes even if they are in different units.

- It is expressed in percentage
- Co-efficient of variation C.V $= \frac{\sigma}{\bar{x}} \times 100\%$
- The set of data showing less co-efficient of variation is more stable than the other set which shows higher co-efficient of variation.

Worked Examples

8.15 The mean of a data is 25.6 and its co-efficient of variation is 18.75. Find the standard deviation.

Sol :

Mean $\bar{x} = 25.6$, Co-efficient of variation, C.V. $= 18.75$

Co-efficient of variation, C.V. $= \frac{\sigma}{\bar{x}} \times 100\%$

$$18.75 = \frac{\sigma}{25.6} \times 100 ; \Rightarrow \sigma = 4.8$$

8.16 The following table gives the values of mean and variance of heights and weights of the 10th standard students of a school.

	Height	Weight
Mean	155 cm	46.50 kg
Variance	72.25 cm^2	28.09 kg^2

Which is more varying than the other?

Sol :

For comparing the two data, first we have to find their Co-efficient of Variations

Mean $\bar{x}_1 = 155$ cm, variance $\sigma_1^2 = 72.25$ cm^2

Therefore standard deviation $\sigma_1 = 8.5$

Co-efficient of variation C.V$_1$ $= \frac{\sigma_1}{\bar{x}_1} \times 100\%$

C.V$_1$ $= \frac{8.5}{155} \times 100\% = 5.48\%$ (for heights)

Mean $\bar{x}_2 = 46.50$ kg, Variance $\sigma_2^2 = 28.09$ kg^2

Standard deviation $\sigma_2 = 5.3$ kg

Coefficient of variation C.V$_2$ $= \frac{\sigma_2}{\bar{x}_2} \times 100\%$

C.V$_2$ $= \frac{5.3}{46.50} \times 100\% = 11.40\%$ (for weights)

C.V$_1$ $= 5.48\%$ and C.V$_2 = 11.40\%$

Since C.V$_2$ > C.V$_1$, the weight of the students is more varying than the height.

8.17 The consumption of number of guava and orange on a particular week by a family are given below.

Number of Guavas	3	5	6	4	3	5	4
Number of Oranges	1	3	7	9	2	6	2

Which fruit is consistently consumed by the family?

Sol : First we find the Coefficient of variation for guavas and oranges separately.

Number of guavas n = 7

x_i	x_i^2
3	9
5	25
6	36
4	16
3	9
5	25
4	16
$\Sigma x_i = 30$	$\Sigma x_i^2 = 136$

$$\text{Mean } \bar{x}_1 = \frac{30}{7} = 4.29$$

$$\text{Standard deviation } \sigma_1 = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\sigma_1 = \sqrt{\frac{136}{7} - \left(\frac{30}{7}\right)^2}$$

$$= \sqrt{19.43 - 18.40} = 1.01$$

Co-efficient of Variation for guavas

$$C.V_1 = \frac{\sigma_1}{\bar{x}_1} \times 100\% = \frac{1.01}{4.29} \times 100\% = 23.54\%$$

Number of oranges n = 7

x_i	x_i^2
1	1
3	9
7	49
9	81
2	4
6	36
2	4
$\Sigma x_i = 30$	$\Sigma x_i^2 = 184$

$$\text{Mean } \bar{x}_2 = \frac{30}{7} = 4.29$$

$$\text{Standard deviation } \sigma_2 = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\sigma_2 = \sqrt{\frac{184}{7} - \left(\frac{30}{7}\right)^2}$$

$$= \sqrt{26.29 - 18.40} = 2.81$$

Coefficient of Variation for Oranges

$$C.V_2 = \frac{\sigma_2}{\bar{x}_2} \times 100\% = \frac{2.81}{4.29} \times 100\%$$

$$= 65.50\%$$

$C.V_1 = 23.54\%$, $C.V_2 = 65.50\%$. Since, $C.V_1 < C.V_2$, we can conclude that the consumption of guavas is more consistent than oranges.

Progress Check

1. Co-efficient of variation is a relative measure of ____.

Ans : Standard deviation

2. When the standard deviation is divided by the mean we get ____.

Ans : Co-efficient of variation

3. The coefficient of variation depends upon ____ and ____.

Ans : Mean and Standard deviation

4. If the mean and standard deviation of a data are 8 and 2 respectively then the coefficient of variation is ____.

Ans : 25%

5. When comparing two data, the data with ____ coefficient of variation is inconsistent.

Ans : More

Exercise 8. 2

1. The standard deviation and mean of a data are 6.5 and 12.5 respectively. Find the coefficient of variation.

Sol :

Standard deviation $\sigma = 6.5$

Mean $\bar{x} = 12.5$

$$\text{Coefficient of variation C.V} = \frac{\sigma}{\bar{x}} \times 100\%$$

$$= \frac{6.5}{12.5} \times 100\%$$

$$= \frac{65}{125} \times 100\%$$

$$= \frac{13}{25} \times 100\%$$

$$= 52\%$$

$\therefore$ Co-efficient of variation is 52%

2. The standard deviation and coefficient of variation of a data are 1.2 and 25.6 respectively. Find the value of mean.

Sol :

Standard deviation = 1.2

Coefficient of variation = 25.6

$$\text{Coefficient of variation C.V} = \frac{\sigma}{\bar{x}} \times 100\%$$

$$25.6 = \frac{1.2}{\bar{x}} \times 100\%$$

$$\bar{x} = \frac{1.2}{25.6} \times 100 = 4.687$$

$\therefore$ Mean $\bar{x} = 4.69$

3. If the mean and coefficient of variation of a data are 15 and 48 respectively, then find the value of standard deviation.

Sol :

Mean $\bar{x} = 15$

Coefficient of variation (C.V) = 48

$$\text{Coefficient of variation C.V} = \frac{\sigma}{\bar{x}} \times 100\%$$

$$48 = \frac{\sigma}{15} \times 100\%$$

$$\frac{48 \times 15}{100} = \sigma$$

$$\sigma = \frac{36}{5} = 7.2$$

$\therefore$ Standard deviation $\sigma = 7.2$

4. If $n = 5$, $\bar{x} = 6$, $\Sigma x^2 = 765$, then calculate the coefficient of variation.

Sol :

To find the coefficient of variation we need standard deviation (σ)

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\frac{\Sigma x^2}{n} = \frac{765}{5} = 153$$

$$\left(\frac{\Sigma x}{n}\right)^2 = (\bar{x})^2 = 6^2 = 36$$

$$\therefore \quad \sigma = \sqrt{(153) - 36} = \sqrt{117}$$

$$= \sqrt{3 \times 3 \times 13}$$

$$\sigma = 3\sqrt{13}$$

$$\text{Coefficient of variation} = \frac{\sigma}{\bar{x}} \times 100\%$$

$$= \frac{3\sqrt{13}}{6} \times 100\% = \frac{\sqrt{13}}{2} \times 100\% = \frac{3.60555}{2} \times 100\%$$

$= 1.80277 \times 100\% = 180.277\ \%$

$\therefore$ Coefficient of variation = 180.28 %

5. Find the coefficient of variation of 24, 26, 33, 37, 29, 31.

Sol :

Ascending order : 24, 26, 29, 31, 33, 37

$$\text{Standard deviation} \quad \sigma = \sqrt{\frac{\Sigma d_i^2}{N} - \left(\frac{\Sigma d_i}{N}\right)^2}$$

Let the assumed mean A = 31

x_i	$d_i = x_i - A$ $d_i = x_i - 31$	d_i^2
24	– 7	49
26	– 5	25
29	– 2	4
31	0	0
33	2	4
37	6	36
	$\Sigma d_i = -6$	$\Sigma d_i^2 = 118$

$$\therefore \quad \sigma = \sqrt{\frac{118}{6} - \left(\frac{-6}{6}\right)^2}$$

$$\sigma = \sqrt{19.67 - (-1)^2} = \sqrt{19.7 - 1}$$

$$\sigma = \sqrt{18.67} = 4.3$$

$$\text{Mean} \quad \bar{x} = \frac{\Sigma x_i}{N}$$

$$\text{Mean} \quad \bar{x} = \frac{24 + 26 + 29 + 31 + 33 + 37}{6}$$

$$\bar{x} = \frac{180}{6} = 30$$

$$\text{Now co-efficient of variation C.V} = \frac{\sigma}{\bar{x}} \times 100\%$$

$$\text{C.V} = \frac{4.3}{30} \times 100\% = 0.1433 \times 100\%$$

$$= 14.33\%$$

Co-efficient of variation of the given data
= 14.33%

6. The time taken (in minutes) to complete a homework by 8 students in a day are given by 38, 40, 47, 44, 46, 43, 49, 53. Find the coefficient of variation.

Sol :

Given data are 38, 40, 43, 44, 46, 47, 49, 53.

Standard deviation $\sigma = \sqrt{\frac{\Sigma d_i^2}{N} - \left(\frac{\Sigma d_i}{N}\right)^2}$

Let us take the assumed mean A = 44

x_i	$d_i = x_i - A$ $d_i = x_i - 44$	d_i^2
38	– 6	36
40	– 4	16
43	– 1	1
44	0	0
46	2	4
47	3	9
49	5	25
53	9	81
	$\Sigma d_i = 8$	$\Sigma d_i^2 = 172$

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{N} - \left(\frac{\Sigma d_i}{N}\right)^2}$$

$$\sigma = \sqrt{\frac{172}{8} - \left(\frac{8}{8}\right)^2} = \sqrt{21.5-(1)^2}$$

$$\sigma = \sqrt{21.5-1} = \sqrt{20.5} = 4.53$$

$$\text{Mean } \bar{x} = \frac{\textit{Sum of the given observations}}{\textit{Number of observations}}$$

$$\bar{x} = \frac{38+40+43+44+46+47+49+53}{8}$$

$$\bar{x} = \frac{360}{8} = 45$$

Co-efficient of variation C.V = $\frac{\sigma}{\bar{x}} \times 100\%$

$$= \frac{4.53}{45} \times 100\% = \frac{453}{45}$$

$\therefore$ Co-efficient of variation = 10.07%

7. The total marks scored by two students Sathya and Vidhya in 5 subjects are 460 and 480 with standard deviation 4.6 and 2.4 respectively. Who is more consistent in performance?

Sol :

(i) Sathya

Sum of marks in 5 subjects = 460

$\therefore$ Mean marks of sathya $\bar{x} = \frac{\Sigma x_i}{N} = \frac{460}{5} = 92$

Standard deviation σ = 4.6

$\therefore$ Co-efficient of variation C.V = $\frac{\sigma}{\bar{x}} \times 100\%$

$$= \frac{4.6}{92} \times 100\% = \frac{460}{92}\%$$

$$= \frac{10}{2}\% = 5\%$$

C.V of Sathya = 5%.

(ii) Vidhya

Sum of the marks in 5 subjects = 480

$\therefore$ Mean marks of Vidhya $\bar{x} = \frac{\Sigma x_i}{N} = \frac{480}{5} = 96$

Standard deviation σ = 2.4

Co-efficient of variation C.V. = $\frac{\sigma}{\bar{x}} \times 100\%$

$$= \frac{2.4}{96} \times 100\% = \frac{240}{96}\%$$

$$= \frac{10}{4}\% = 2.5\%$$

$\therefore$ C. V. of Vidhya = 2.5%

C. V. of Vidhya < C. V. of Sathya

$\therefore$ Vidhya is more consistent in her performance.

8. The mean and standard deviation of marks obtained by 40 students of a class in three subjects Mathematics, Science and Social Science are given below.

Subjects	Mean	SD
Mathematics	56	12
Science	65	14
Social Science	60	10

Which of the three subjects shows highest variation and which shows lowest variation in marks?

Sol :

Mathematics:

Mean $\bar{x}$ = 56; SD σ = 12

Co-efficient of variation C. V = $\frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{12}{56} \times 100\% = \frac{1200}{56}\% = 21.43\%$

Science:

Mean $\bar{x} = 65$; SD $\sigma = 14$

Co-efficient of variation C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{14}{65} \times 100\% = \frac{1400}{65}\%$

$= 21.538\% = 21.54\%$

Social Science:

Mean $\bar{x} = 60$; SD $\sigma = 10$

Co-efficient of variation C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{10}{60} \times 100\% = \frac{100}{6}\%$

$= 16.666\%$

$= 16.67\%$

The highest variation is in the subject science and the lowest variation is in the subject social science.

9. The temperature of two cities A and B in a winter season are given below.

Temperature of city A (in degree Celsius)	18	20	22	24	26
Temperature of city B (in degree Celsius)	11	14	15	17	18

Find which city is more consistent in temperature changes?

Sol :

Standard deviation $\sigma = \sqrt{\frac{\Sigma d_i^2}{N} - \left(\frac{\Sigma d_i}{N}\right)^2}$

For the city A

Let the assumed mean A = 22

x_i	$d_i = x_i - A$ $d_i = x_i - 22$	d_i^2
18	– 4	16
20	– 2	4
22	0	0
24	2	4
26	4	16
	$\Sigma d_i = 0$	$\Sigma d_i^2 = 40$

$\sigma = \sqrt{\frac{40}{5} - (0)^2} = \sqrt{8}$

$\sigma = 2.828 = 2.83$

Mean $\bar{x} = \frac{\Sigma x_i}{N} = \frac{18+20+22+24+26}{5}$

$= \frac{110}{5} = 22$

Co-efficient of variation C. V. $= \frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{2.83}{22} \times 100\% = 12.86\%$

For the city B

Let the Assumed mean B = 15

x_i	$d_i = x_i - B$ $d_i = x_i - 15$	d_i^2
11	– 4	16
14	– 1	1
15	0	0
17	2	4
18	3	9
	$\Sigma d_i = 0$	$\Sigma d_i^2 = 30$

Standard deviation $\sigma = \sqrt{\frac{30}{5} - \left(\frac{0}{5}\right)^2}$

$= \sqrt{6 - 0} = \sqrt{6} = 2.449$

Mean $\bar{x} = \frac{\Sigma x_i}{N}$

$= \frac{11+14+15+17+18}{5}$

$\bar{x} = \frac{75}{5} = 15.$

$\therefore$ Co-efficient of variation C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{2.4495}{15} \times 100\%$

$= \frac{244.95}{15}\% = 16.33\%$

C. V. of temperature of city A < C. V of temperature of city B.

$\therefore$ City A is more consistent in temperature change.

PROBABILITY OF AN EVENT

Key Points

1. Random Experiment:

A random experiment is an experiment in which

(i) The set of all possible outcomes are known.

(ii) Exact outcome is not known.

Example: Tossing a coin, rolling a die.

2. Sample Space:

The set of all possible outcomes in a random experiment is called a sample space. It is generally denoted by S.

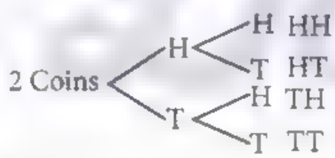
Example: When tossing a coin the possible outcomes are {H,T}.

∴ Sample space S = {H, T}

3. Tree Diagram:

Tree diagram allow us to see visually all the possible outcomes of an experiment. Each branch of a tree diagram represent a possible outcome.

Example: When tossing two coins.



∴ Sample space S = {HH, HT, TH, TT}

4. Event:

In a random experiment, each possible outcome is called an event.

Thus an event will be a subset of the sample space.

Example: Getting two heads on tossing two coins.

5. Trial:

Performing an experiment once is called a trial.

6. Equally likely events:

Two or more events are said to be equally likely if each one of them has an equal chance of occurring.

Example: Head and tail are equally likely events in tossing a coin.

7. Certain events:

In an experiment, the event which surely occur is called certain event.

Example: When we roll a die, the event of getting any natural number from 1 to 6 is a certain event.

8. **Impossible event:**

In an experiment if an event has no scope to occur then it is called an impossible event.

Example: When we toss two coins, the event of getting three heads is an impossible event.

9. **Mutually exclusive events:**

Events A and B are said to be mutually exclusive if $A \cap B = \phi$

Example: On rolling a die the events of getting odd numbers and even numbers are mutually exclusive events.

10. **Exhaustive Events:**

The collection of events whose union is the whole sample space are called exhaustive events.

Example: On rolling a die the events of getting odd numbers and even numbers are exhaustive events.

11. **Complementary events:**

The complement of an event A is the event representing collection of sample points not in A.

It is denoted by A' or A^C or $\overline{A}$.

The event A and its complement A' are mutually exclusive and exhaustive.

12. **Elementary event:**

If an event E consists of only one outcome then it is called an elementary event.

Example: In tossing a coin, event of getting a head is an elementary event.

13. **Probability of an event:**

In a random experiment, let S be the sample space $E \subseteq S$. Then E is an event.

The probability of occurance of E is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to occurance of } E}{\text{Number of all possible outcomes}}$$

$$P(E) = \frac{n(E)}{n(S)}$$

14. **Results:**

(i) $P(E) = \frac{n(E)}{n(S)}$ (ii) $P(S) = \frac{n(S)}{n(S)} = 1$

i.e., The probability of sure event is 1.

(iii) $P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$

The probability of impossible event is 0

(iv) $0 \le P(E) \le 1$

The probability value always lies from 0 to 1

(v) $P(\overline{E}) = 1 - P(E)$

(vi) $P(E) + P(\overline{E}) = 1$

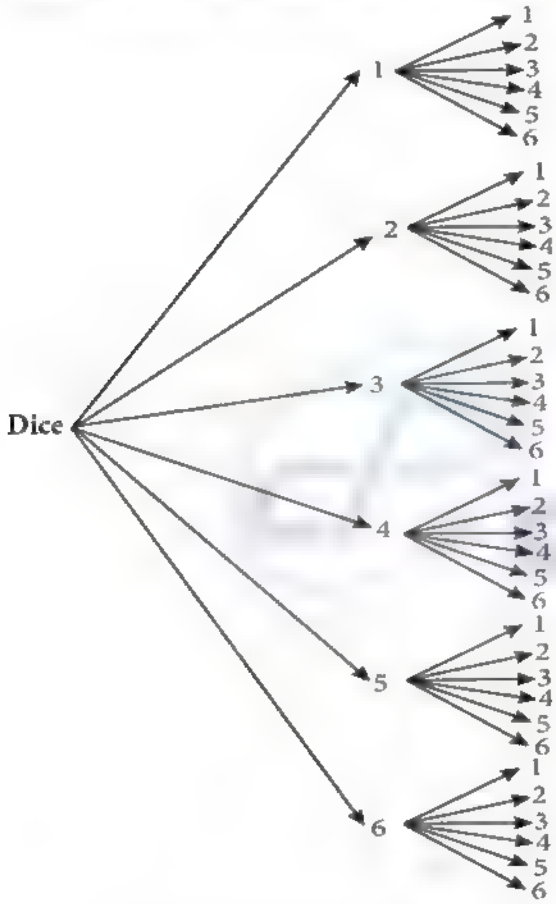
Worked Examples

8.18 **Express the sample space for rolling two dice using tree diagram.**

Sol :

When we roll two dice, since each die contain 6 faces marked with 1, 2, 3, 4, 5, 6 the tree diagram will look like

Hence, the sample space can be written as Figure



$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)$
$(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)$
$(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)$
$(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)$
$(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)$
$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

8.19 **A bag contains 5 blue balls and 4 green balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is (i) blue (ii) not blue.**

Sol : Total number of possible outcomes

$n(S) = 5 + 4 = 9$

(i) Let A be the event of getting a blue ball. Number of favourable outcomes for the event A. Therefore, $n(A) = 5$

Probability that the ball drawn is blue.

Therefore, $P(A) = \frac{n(A)}{n(S)} = \frac{5}{9}$

(ii) $\overline{A}$ will be the event of not getting a blue ball.

So, $P(\overline{A}) = 1 - P(A) = 1 - \frac{5}{9} = \frac{4}{9}$

8.20 **Two dice are rolled. Find the probability that the sum of outcomes is**

(i) equal to 4 **(ii) greater than 10**

(iii) less than 13

Sol :

When we roll two dice, the sample space is given by

$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)$
$(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)$
$(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)$
$(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)$
$(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)$
$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\};$

$n(S) = 36$

(i) Let A be the event of getting the sum of outcome values equal to 4.

Then $A = \{(1, 3), (2, 2), (3, 1)\}$; $n(A) = 3$.

Probability of getting the sum of outcomes values equal to 4 is

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(ii) Let B be the event of getting the sum of outcome values greater than 10.

$B = \{(5, 6), (6, 5), (6, 6)$; $n(B) = 3$

Probability of getting the sum of outcomes greater than 10 is

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

(iii) Let C be the event of getting the sum of outcomes less than 13. Here all the outcomes have the sum value less than 13. Hence $C = S$.

Therefore, $n(C) = n(S) = 36$

Probability of getting the total value less than 13 is

$$P(C) = \frac{n(C)}{n(S)} = \frac{36}{36} = 1.$$

8.21 **Two coins are tossed together. What is the probability of getting different faces on the coins?**

Sol :

When two coins are tossed together, the sample space is S = {HH, HT, TH, TT}; n(S) = 4

Let A be the event of getting different faces on the coins.

A = {HT, TH}; n(A) = 2

Probability of getting different faces on the coins is

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

8.22 **From a well shuffled pack of 52 cards, one card is drawn at random. Find the probability of getting (i) red card (ii) heart card (iii) red king (iv) face card (v) number card**

Sol :

n(S) = 52

(i) Let A be the event of getting a red card.

n(A) = 26

Probability of getting a red card is

$$P(A) = \frac{n(C)}{n(S)} = \frac{26}{52} = \frac{1}{2}$$

(ii) Let B be the event of getting a heart card.

n(B) = 13

Probability of getting a heart card is

$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

(iii) Let C be the event of getting a red king card. A red king card can be either a diamond king or a heart king.

n(C) = 2

Probability of getting a red king card is

$$P(C) = \frac{n(C)}{n(S)} = \frac{2}{52} = \frac{1}{26}$$

(iv) Let D be the event of getting a face card. The face cards are Jack (J), Queen (Q), and King (K).

n(D) = 4 × 3 = 12

Probability of getting a face card is

$$P(D) = \frac{n(D)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

(v) Let E be the event of getting a number card. The number cards are 2, 3, 4, 5, 6, 7, 8, 9 and 10.

n(E) = 4 × 9 = 36

Probability of getting a number card is

$$P(E) = \frac{n(E)}{n(S)} = \frac{36}{52} = \frac{9}{13}$$

8.23 **What is the probability that a leap year selected at random will contain 53 Saturdays. (Hint: 366 = 52 × 7 + 2)**

Sol :

A leap year has 366 days. So it has 52 full weeks and 2 days. 52 Saturdays must be in 52 full weeks.

The possible chances for the remaining two days will be the sample space.

S = {(Sun-Mon, Mon-Tue, Tue-Wed, Wed-Thu, Thu-Fri, Fri-Sat, Sat-Sun)}

n(S) = 7

Let A be the event of getting 53rd Saturday.

Then A = {Fri-Sat, Sat-Sun}

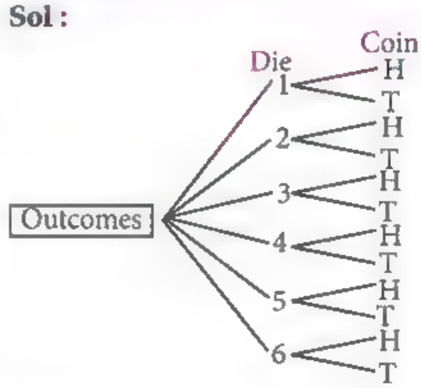
n(A) = 2

Probability of getting 53 Saturdays in a leap year is

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{7}$$

8.24 **A die is rolled and a coin is tossed simultaneously. Find the probability that the die shows an odd number and the coin shows a head.**

Sol :



Sample space

S = {1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T};

n(S) = 12

Let A be the event of getting an odd number and a head.

A = {1H, 3H, 5H}; n(A) = 3

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

8.25 **A bag contains 6 green balls, some black and red balls. Number of black balls is as twice as number of red ball. Probability of getting a green ball is thrice the probability of getting a red ball. Find (i) number of black balls (ii) total number of balls.**

Sol : Number of Green balls is n(G) = 6

Let number of red balls is n(R) = x

Therefore, number of black balls is n(B) = 2x

Total number of balls n(S) = 6 + x + 2x = 6 + 3x

It is given that, P(G) = 3 × P(R)

$$\frac{6}{6+3x} = 3 \times \frac{x}{6+3x}$$

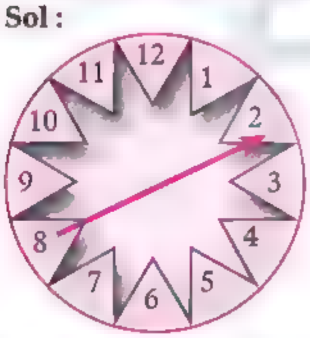
$$3x = 6 \Rightarrow x = 2$$

(i) Number of black balls = $2 \times 2 = 4$

(ii) Total number of balls = $6 + (3 \times 2) = 12$

8.26 **A game of chance consists of spinning an arrow which is equally likely to come to rest pointing to one of the numbers 1, 2, 3, ... 12. What is the probability that it will point to (i) 7 (ii) a prime number (iii) a composite number?**

Sol :



Sample space S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

n (S) = 12

(i) Let A be the event of resting in 7. n(A) = 1

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{12}$$

(ii) Let B be the event that the arrow will come to rest in a prime number.

B = {2, 3, 5, 7, 11}; n(B) = 5

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{12}$$

(iii) Let C be the event that arrow will come to rest in composite number.

C = {4, 6, 8, 9, 10, 12}; n(C) = 6

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{12} = \frac{1}{2}$$

Progress Check

1. An experiment in which a particular outcome cannot be predicted is called _____.

Ans : Random Experiment

2. The set of all possible outcomes is called _____.

Ans : Sample space

3. Which of the following values cannot be a probability of an event? (a) –0.0001 (b) 0.5 (c) 1.001 (d) 1 (e) 20% (f) 0.253 (g) $\frac{1-\sqrt{5}}{2}$ (h) $\frac{\sqrt{3}+1}{4}$

Ans : Probability of an event lies between 0 and 1

$0 \le P(E) \le 1$

∴ (a) -0.0001 (< 0)

(c) 1.001 (> 1)

(g) $\frac{1-\sqrt{5}}{2}$ (< 0)

are the numbers cannot be a probability.

Thinking Corner

1. What will be the probability that a non leap year will have 53 Saturdays?

Ans : In a non leap year will be 52 saturdays and 1 day will be left.

This one day can be any of the 7 days.

∴ Sample space S = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

n (S) = 7

Let A be the event of getting saturday

∴ n (A) = 1

$$P(A) = \frac{n(A)}{n(S)}$$

$$P(A) = \frac{1}{7}$$

∴ Probability that a non leap year will have 53 saturdays is $\frac{1}{7}$.

2. What is the complement event of an impossible event?

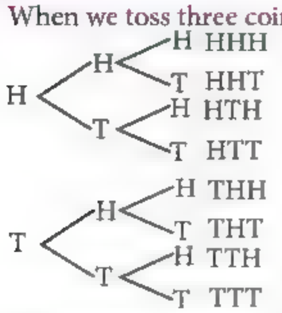
Ans : Complement event of an impossible event is the sample space.

Exercise 8.3

1. Write the sample space for tossing three coins using tree diagram.

Sol :

When we toss three coins the outcome will be

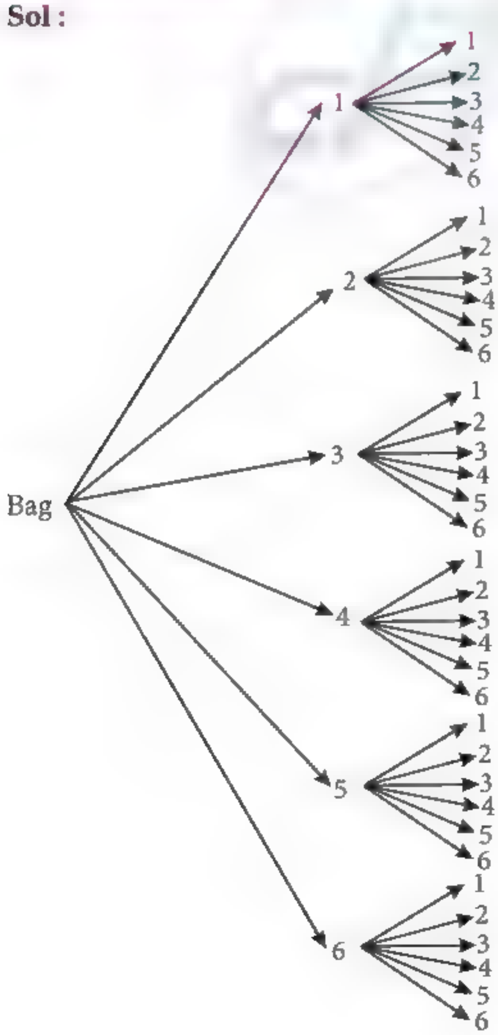


$\therefore$ The sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Total number of outcomes = 8

2. Write the sample space for selecting two balls from a bag containing 6 balls numbered 1 to 6 (using tree diagram).

Sol :



Sample Space

S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

$\therefore$ Total number of outcomes = 36

3. If A is an event of a random experiment such that P(A) : P($\bar{A}$) = 17 : 15 and n(S) = 640 then find (i) P($\bar{A}$) (ii) n(A)

Sol :

Given P(A) : P($\bar{A}$) = 17 : 15

(i) $$\frac{P(A)}{P(\bar{A})} = \frac{17}{15}$$

$$\frac{P(A)}{1-P(A)} = \frac{17}{15} \quad [\because P(\bar{A}) = 1 - P(A)]$$

$$15\,P(A) = 17\,[1 - P(A)]$$

$$15\,P(A) = 17 - 17\,P(A)$$

$$32\,P(A) = 17; \quad P(A) = \frac{17}{32}$$

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{17}{32} = \frac{32-17}{32} = \frac{15}{32}$$

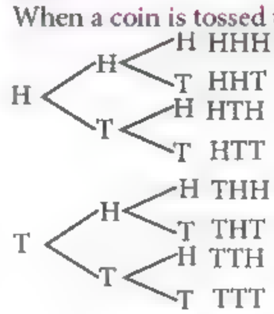
(ii) $$P(A) = \frac{n(A)}{n(S)} = \frac{17}{32}$$

$$n(A) = 17$$

4. A coin is tossed thrice. What is the probability of getting two consecutive tails?

Sol :

When a coin is tossed thrice, the outcome will be



The sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

n(S) = 8

Let A be the event of getting two consecutive tails

A = {HTT, TTH, TTT}

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

Probability of getting two consecutive tails $= \frac{3}{8}$

5. At a fete, cards bearing numbers 1 to 1000, one number on one card are put in a box. Each player selects one card at random and that card is not replaced. If the selected card has a perfect square number greater than 500, the player wins a prize. What is the probability that (i) the first player wins a prize (ii) the second player wins a prize, if the first has won?

Sol :

Sample space $S = \{1, 2, 3, 4, ... 1000\}$

$$n(S) = 1000$$

Let 'A' be the event of selecting a card that is perfect square greater than 500

$A = \{23^2, 24^2, 25^2, 26^2, 27^2, 28^2, 29^2, 30^2, 31^2\}$

$$n(A) = 9$$

(i) Probability of the first player wins a prize

By picking one of the cards from A he may win a prize.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{9}{1000}$$

(ii) Probability of the second player winning a prize if the first has won

Let 'B' be the event of second player wins a prize.

Since the card picked in **(i)** is not replaced

$$n(B) = 8$$

$$n(S) = 1000 - 1 = 999$$

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{8}{999}$$

6. A bag contains 12 blue balls and x red balls. If one ball is drawn at random (i) what is the probability that it will be a red ball? (ii) If 8 more red balls are put in the bag and if the probability of drawing a red ball will be twice that of the probability in (i) then find x.

Sol :

Total number of balls = blue balls + red balls

$$n(S) = 12 + x$$

(i) The probability that it will be a red ball:

Let A be the event of selecting a red ball

$$\therefore \quad n(A) = x$$

$$P(A) = \frac{n(A)}{n(S)}$$

$$P(A) = \frac{x}{12+x}$$

(ii) If 8 more red balls are put in the bag then the number of red balls = x + 8

Total number of balls = 12 + x + 8

Probability of drawing a red ball

$$P(B) = \frac{n(B)}{n(S)}$$

$$= \frac{x+8}{12+x+8} = \frac{x+8}{x+20}$$

If the probability of drawing a red ball will be twice that of the probability **(i)**, then

$$\frac{x+8}{x+20} = 2 \times \left[\frac{x}{12+x}\right]$$

$$(x + 8)(12 + x) = 2x(x + 20)$$

$$12x + 96 + x^2 + 8x = 2x^2 + 40x$$

$$2x^2 - x^2 + 40x - 12x - 8x - 96 = 0$$

$$x^2 + 20x - 96 = 0$$

$$(x - 4)(x + 24) = 0$$

x = 4 and x = – 24 (not possible)

$\therefore$ By applying the value of x in P(A)

$$\text{we get } P(A) = \frac{4}{12+4} = \frac{4}{16}$$

$$P(A) = \frac{1}{4}$$

$$x = 4$$

7. Two unbiased dice are rolled once. Find the probability of getting

(i) a doublet (equal numbers on both dice)

(ii) the product as a prime number

(iii) the sum as a prime number

(iv) the sum as 1

Sol : When two unbiased dice are rolled, the Sample Space

S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (6, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (6, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

$$n(S) = 36$$

(i) Let A be the event of getting a doublet

A = {(1, 1) (2, 2) (3, 3) (4, 4) (5, 5) (6, 6)}

$n(A) = 6$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) Let B be the event of getting the product as a prime number.

B = {(1, 2) (1, 3) (1, 5) (2, 1) (3, 1) (5, 1)}

$n(B) = 6$

$$P(B) = \frac{6}{36} = \frac{1}{6}$$

(iii) Let C be the event of getting the sum as a prime number.

C = {(1, 1) (1, 2) (1, 4) (1, 6) (2, 1) (2, 3) (2, 5) (3, 2) (3, 4) (4, 1) (4, 3) (5, 2) (5, 6) (6, 1) (6, 5)}

$n(C) = 15$

$$P(C) = \frac{n(C)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

(iv) Let D be the event of getting the sum as 1. Since it is an impossible event.

$n(D) = 0$ and $P(D) = 0$

8. Three fair coins are tossed together. Find the probability of getting

(i) all heads **(ii) atleast one tail**

(iii) atmost one head

(iv) atmost two tails

Sol :

When three fair coins are tossed together the sample space

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$n(S) = 8$

(i) Let A be the event of getting all heads

A = {HHH}

$n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$$

(ii) Let B be the event of getting atleast one tail

B = {HHT, HTH, HTT, THH, THT, TTH, TTT}

$n(B) = 7$

$$P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

(iii) Let C be the event of getting atmost one head

C = {HTT, THT, TTH, TTT}

$n(C) = 4$

$$P(C) = \frac{n(C)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

(iv) Let D be the event of getting atmost two tails

D = {HHH, HHT, HTH, HTT, THH, THT, TTH}

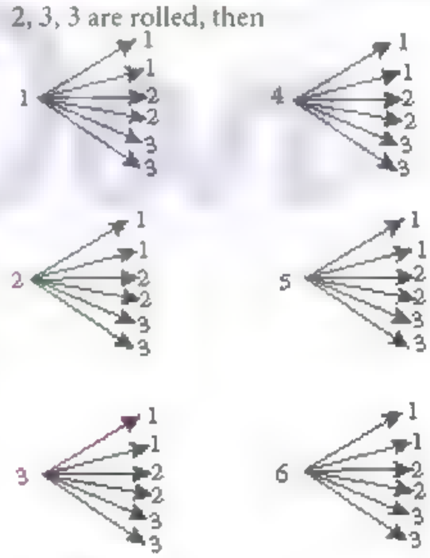
$n(D) = 7$

$$P(D) = \frac{n(D)}{n(S)} = \frac{7}{8}$$

9. Two dice are numbered 1, 2, 3, 4, 5, 6 and 1, 1, 2, 2, 3,3 respectively. They are rolled and the sum of the numbers on them is noted. Find the probability of getting each sum from 2 to 9 separately.

Sol :

When two dice numbered 1, 2, 3, 4, 5, 6 and 1, 1, 2, 2, 3, 3 are rolled, then



Sample Space

S = {(1, 1) (2, 1) (3, 1) (4, 1) (5, 1) (6, 1)
(1, 1) (2, 1) (3, 1) (4, 1) (5, 1) (6, 1)
(1, 2) (2, 2) (3, 2) (4, 2) (5, 2) (6, 2)
(1, 2) (2, 2) (3, 2) (4, 2) (5, 2) (6, 2)
(1, 3) (2, 3) (3, 3) (4, 3) (5, 3) (6, 3)
(1, 3) (2, 3) (3, 3) (4, 3) (5, 3) (6, 3)}

$n(S) = 36$

Let A, B, C, D, E, F, G, H be the event of getting the sum 2, 3, 4, 5, 6, 7, 8 and 9 respectively.

(i) A = {(1, 1) (1, 1)}

$n(A) = 2$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{36}$$

(ii) Sum 3

$$B = \{(2, 1)\ (2, 1)\ (1, 2)\ (1, 2)\}$$
$$n(B) = 4$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{36}$$

(iii) Sum 4

$$C = \{(3, 1)\ (3, 1)\ (2, 2)\ (2, 2)\ (1, 3)\ (1, 3)\}$$
$$n(C) = 6$$
$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{36}$$

(iv) Sum 5

$$D = \{(4, 1)\ (4, 1)\ (3, 2)\ (3, 2)\ (2, 3)\ (2, 3)\}$$
$$n(D) = 6$$
$$P(D) = \frac{n(D)}{n(S)} = \frac{6}{36}$$

(v) Sum 6

$$E = \{(5, 1)\ (5, 1)\ (4, 2)\ (4, 2)\ (3, 3)\ (3, 3)\}$$
$$n(E) = 6$$
$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36}$$

(vi) Sum 7

$$F = \{(6, 1)\ (6, 1)\ (5, 2)\ (5, 2)\ (4, 3)\ (4, 3)\}$$
$$n(F) = 6$$
$$P(F) = \frac{n(F)}{n(S)} = \frac{6}{36}$$

(vii) Sum 8

$$G = \{(6, 2)\ (6, 2)\ (5, 3)\ (5, 2)\ (5, 3)\}$$
$$n(G) = 4$$
$$P(G) = \frac{n(G)}{n(S)} = \frac{4}{36}$$

(viii) Sum 9

$$H = \{(6, 3)\ (6, 3)\}$$
$$n(H) = 2$$
$$P(H) = \frac{n(H)}{n(S)} = \frac{2}{36}$$

10. A bag contains 5 red balls, 6 white balls, 7 green balls, 8 black balls. One ball is drawn at random from the bag. Find the probability that the ball drawn is

(i) white **(ii) black or red**

(iii) not white

(iv) neither white nor black

Sol :

Total number of balls = 5 red + 6 white + 7 green + 8 black

$$n(S) = 26$$

(i) Let A be the event of getting white ball

$$n(A) = 6$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{26} = \frac{3}{13}$$

(ii) Let B be the event of getting black or red

$$n(B) = 5 + 8 = 13$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{26} = \frac{1}{2}$$

(iii) Since A is the event of getting white ball $\overline{A}$ is the event of not getting white ball

$$P(\overline{A}) = 1 - P(A)$$
$$= 1 - \left(\frac{3}{13}\right) = \frac{13-3}{13}$$
$$P(\overline{A}) = \frac{10}{13}$$

(iv) Let 'C' be the event of getting neither white nor black.

$$C = 5 \text{ red} + 7 \text{ green}$$
$$n(C) = 12$$
$$P(C) = \frac{n(C)}{n(S)} = \frac{12}{26} = \frac{6}{13}$$

11. In a box there are 20 non-defective and some defective bulbs. If the probability that a bulb selected at random from the box found to be defective is $\frac{3}{8}$ then, find the number of defective bulbs.

Sol :

Number of non defective bulbs = 20.

Let x be the number of defective bulbs

Then total number of bulbs n(S) = 20 + x

Let 'A' be the event of getting defective bulbs

$$P(A) = \frac{x}{20+x} = \frac{3}{8}$$

$$8x = 3(20 + x)$$
$$8x = 60 + 3x$$
$$8x - 3x = 60$$
$$5x = 60$$
$$x = \frac{60}{5}$$
$$x = 12$$

$\therefore$ Number of defective bulbs = 12

12. The king and queen of diamonds, queen and jack of hearts, jack and king of spades are removed from a deck of 52 playing cards and then well shuffled. Now one card is drawn at random from the remaining cards. Determine the probability that the card is (i) a clavor (ii) a queen of red card (iii) a king of black card

Sol :

Suits of playing cards	Spade ♠	Heart ♥	Clavor ♣	Diamond ♦
Existing cards	A	A	A	A
	2	2	2	2
	3	3	3	3
	4	4	4	4
	5	5	5	5
	6	6	6	6
	7	7	7	7
	8	8	8	8
	9	9	9	9
	10	10	10	10
			J	J
	Q		Q	
		K	K	
Set of playing cards in each suit	13	13	13	13
Remaining cards after removing some of them	11	11	13	11

Total number of cards remaining

$= 11 + 11 + 13 + 11 = 46$

$$n(S) = 46$$

(i) Let A be the event of getting a clavor card

$$n(A) = 13$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{13}{46}$$

(ii) Let B be the event of getting queen of red card

$$n(B) = 0 \text{ (Removed red queens)}$$

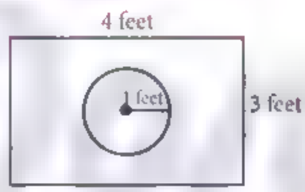
$$P(B) = \frac{n(B)}{n(S)} = 0$$

(iii) Let C be the event of getting a king of black card

$$n(C) = 1$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{1}{46}$$

13. Some boys are playing a game, in which the stone thrown by them landing in a circular region (given in the figure) is considered as win and landing other than the circular region is considered as loss. What is the probability to win the game?



Sol :

Area of the rectangle = (length × breadth) square units

$= 4 \times 3$ square feet

$n(S) = 12$ square feet

Area of the circle $= \pi r^2$ square units

$= \pi (1)^2$ square feet

$= \pi$ square feet

Let 'A' be the event of winning the game,

then $n(A) = \pi$

Probability of winning the game $= \frac{n(A)}{n(S)} = \frac{\pi}{12}$

$$= \frac{3.14}{12} = \frac{1.57}{6} = \frac{157}{600}$$

14. Two customers Priya and Amuthan are visiting a particular shop in the same week (Monday to Saturday). Each is equally likely to visit the shop on any one day as on another day. What is the probability that both will visit the shop on

(i) the same day (ii) different days

(iii) consecutive days?

Sol :

If Priya and Vidhya are visiting the shop in the same week, the sample space

S = {(Mon, Mon) (Mon, Tue) (Mon, Wed)
(Mon, Thur) (Mon, Fri) (Mon, Sat)
(Tue, Mon) (Tue, Tue) (Tue, Wed)
(Tue, Thur) (Tue, Fri) (Tue, Sat)
(Wed, Mon) (Wed, Tue) (Wed, Wed)
(Wed, Thur) (Wed, Fri) (Wed, Sat)
(Thur, Mon) (Thur, Tue) (Thur, Wed)
(Thur, Thur) (Thur, Fri) (Thur, Sat)
(Fri, Mon) (Fri, Tue) (Fri, Wed) (Fri, Thur)
(Fri, Fri) (Fri, Sat) (Sat, Mon) (Sat, Tue)
(Sat, Wed) (Sat, Thur) (Sat, Fri) (Sat, Sat)}

$n(S) = 36$

(i) Let 'A' be the event that both will visit the shop on the same day.

A = {(Mon, Mon) (Tue, Tue) (Wed, Wed) (Thur, Thur) (Fri, Fri) (Sat, Sat)}

$n(A) = 6$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) Both will visit the shop on different days

$$P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{6}$$

$$P(\bar{A}) = \frac{6-1}{6} = \frac{5}{6}$$

i.e., probability that both will visit the shop on different days $= \frac{5}{6}$

(iii) Let 'B' be the event that both will visit the shop on consecutive days

B = {(Mon, Tue) (Tue, Wed) (Wed, Thur) (Thur, Fri) (Fri, Sat)}

$n(B) = 5$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}$$

15. In a game, the entry fee is ₹ 150. The game consists of tossing a coin 3 times. Dhana bought a ticket for entry. If one or two heads show, she gets her entry fee back. If she throws 3 heads, she receives double the entry fees. Otherwise she will lose. Find the probability that she (i) gets double entry fee (ii) just gets her entry fee (iii) loses the entry fee.

Sol :

In tossing a coin 3 times, the sample space

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$n(S) = 8$

(i) Let 'A' be the event of getting double entry fee. She received double entry fee when she throws 3 heads.

$\therefore$ A = {HHH}

$n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{8}$$

Probability of getting double entry fee $= \frac{1}{8}$

(ii) Let 'B' be the event of getting the entry fee back. She receives her entry fee back when she throws one or two heads.

$\therefore$ B = {HHT, HTH, HTT, THH, THT, TTH}

$n(B) = 6$

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{8} = \frac{3}{4}$$

Probability of getting back entry fee $= \frac{3}{4}$

(iii) Let 'C' be the event of losing the entry fee

C = {TTT}

$n(C) = 1$

$$P(C) = \frac{n(C)}{n(S)} = \frac{1}{8}$$

Probability of losing the entry fee $= \frac{1}{8}$

ALGEBRA OF EVENTS

Key Points

In a random experiment, let S be the sample space. Let $A \subseteq S$ and $B \subseteq S$. Then A and B are events.

- $A \cap B$ is an event that occurs only when both A and B occurs.
- $A \cup B$ is an event that occurs only when atleast one of A or B occurs.
- $\overline{A}$ is an event that occurs only when A doesn't occur.
- $A \cap \overline{A} = \phi$
- $A \cup \overline{A} = S$
- If A and B are mutually exclusive events, the n $P(A \cup B) = P(A) + P(B)$
- P(Union of events) = Σ (Probability of events)
- $P(\overline{A \cup B}) = P(\overline{A} \cap \overline{B})$

Theorem 1

If A and B are two events associated with a random experiment, then

1. $P(A \cap \overline{B}) = P(\text{only } A) = P(A) - P(A \cap B)$
2. $P(\overline{A} \cap B) = P(\text{only } B) = P(B) - P(A \cap B)$

Addition Theorem of Probability

1. If A and B are any two non mutually exclusive events then
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. If A, B and C are any three non mutually exclusive events then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Worked Examples

8.27 If $P(A) = 0.37$, $P(B) = 0.42$, $P(A \cap B) = 0.09$ then find $P(A \cup B)$.

Sol :

$P(A) = 0.37$, $P(B) = 0.42$, $P(A \cap B) = 0.09$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.37 + 0.42 - 0.09 = 0.7$

8.28 What is the probability of drawing either a king or a queen in a single draw from a well shuffled pack of 52 cards?

Sol :

Total number of cards = 52

Number of king cards = 4

Probability of drawing a king card = $\frac{4}{52}$

Number of queen cards = 4

Probability of drawing a queen card = $\frac{4}{52}$

Both the events of drawing a king and queen are mutually exclusive

$\Rightarrow P(A \cup B) = P(A) + P(B)$

Therefore, probability of drawing either a king or a queen is = $\frac{4}{52} + \frac{4}{52} = \frac{2}{13}$

8.29 Two dice are rolled together. Find the probability of getting a doublet or sum of faces as 4.

Sol :

When two dice are rolled together, there will be $6 \times 6 = 36$ outcomes.

Let S be the sample space. Then n(S) = 36

Let A be the event of getting a doublet and B be the event of getting face sum 4.

Then $A = \{(1, 1), (2, 2) (3, 3), (4, 4), (5, 5) (6, 6)\}$

$B = \{(1, 3), (2, 2), (3, 1)\}$

Therefore, $A \cap B = \{(2, 2)\}$

Then, $n(A) = 6, n(B) = 3, n(A \cap B) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$$

Therefore, P (getting a doublet or a total of 4)

$$= P(A \cup B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{36} + \frac{3}{36} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9}$$

Hence, the required probability is $\frac{2}{9}$.

8.30 If A and B are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and P (A and B) $= \frac{1}{8}$, find (i) P(A or B) (ii) P(not A and not B).

Sol :

(i) $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$$P(A \text{ or } B) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$$

(ii) $P(\text{not } A \text{ and not } B) = P(\bar{A} \cap \bar{B})$

$= P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$$P(\text{not } A \text{ and not } B) = 1 - \frac{5}{8} = \frac{3}{8}$$

8.31 A card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

Sol :

Total number of cards = 52; n(S) = 52

Let A be the event of getting a king card,

n(A) = 4

$$P(B) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

Let B be the event of getting a heart card,

n(B) = 13

$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{52}$$

Let C be the event of getting a red card,

n(C) = 26

$$P(C) = \frac{n(C)}{n(S)} = \frac{26}{52}$$

$P(A \cap B)$ = P(getting heart king) $= \frac{1}{52}$

$P(B \cap C)$ = P(getting red and heart) $= \frac{13}{52}$

$P(A \cap C)$ = P(getting red king) $= \frac{2}{52}$

$P(A \cap B \cap C)$ = P(getting heart, king which is red) $= \frac{1}{52}$

Therefore, required probability is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}.$$

8.32 In a class of 50 students, 28 opted for NCC, 30 opted for NSS and 18 opted both NCC and NSS. One of the students is selected at random. Find the probability that

(i) The student opted for NCC but not NSS.

(ii) The student opted for NSS but not NCC.

(iii) The student opted for exactly one of them.

Sol :

Total number of students n(S) = 50

Let A and B be the events of students opted for NCC and NSS respectively.

$n(A) = 28, n(B) = 30, n(A \cap B) = 18$

$$P(A) = \frac{n(A)}{n(S)} = \frac{28}{50}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{30}{50}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{18}{50}$$

(i) Probability of the students opted for NCC but not NSS

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$
$$= \frac{28}{50} - \frac{18}{50} = \frac{1}{5}$$

(ii) Probability of the students opted for NSS but not NCC.

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$
$$= \frac{30}{50} - \frac{18}{50} = \frac{6}{25}$$

(iii) Probability of the students opted for exactly one of them

$$= P[(A \cap \bar{B}) \cup (\bar{A} \cap B)]$$
$$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$$
$$= \frac{1}{5} + \frac{6}{25} = \frac{11}{25}$$

(Note that $(A \cap \bar{B}), (\bar{A} \cap B)$ are exclusive events).

8.33 A and B are two candidates seeking admission to IIT, the probability that A getting selected is 0.5 and the probability that both A and B getting selected is 0.3. Prove that the probability of B being selected is at most 0.8.

Sol :

$P(A) = 0.5$, $P(A \cap B) = 0.3$

We have $P(A \cup B) \leq 1$

$$P(A) + P(B) - P(A \cap B) \leq 1$$
$$0.5 + P(B) - 0.3 \leq 1$$
$$P(B) \leq 1 - 0.2$$
$$P(B) \leq 1 - 0.2$$
$$P(B) \leq 0.8$$

Therefore, probability of B getting selected is atmost 0.8.

Progress Check

1. P(only A) = _____

   Ans : $P(A) - P(A \cap B)$

2. $P(\bar{A} \cap B)$ = _____

   Ans : $P(B) - P(A \cap B)$

3. $A \cap B$ and $\bar{A} \cap B$ are _____ events.

   Ans : Mutually exclusive

4. $P(\bar{A} \cap \bar{B})$ = _____

   Ans : $P(\overline{A \cup B})$

5. If A and B are mutually exclusive events then $P(A \cap B)$ = _____

   Ans : 0

6. If $P(A \cap B) = 0.3$, $P(\bar{A} \cap B) = 0.45$, then P(B) = _____

   Ans : 0.75

Exercise 8. 4

1. If $P(A) = \frac{2}{3}$, $P(B) = \frac{2}{5}$, $P(A \cup B) = \frac{1}{3}$ then find $P(A \cap B)$.

Sol :

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\frac{1}{3} = \frac{2}{3} + \frac{2}{5} - P(A \cap B)$$
$$P(A \cap B) = \frac{2}{3} + \frac{2}{5} - \frac{1}{3}$$
$$P(A \cap B) = \frac{10+6-5}{15} = \frac{16-5}{15} = \frac{11}{15}$$

2. A and B are two events such that, P(A) = 0.42, P(B) = 0.48 and $P(A \cap B) = 0.16$. Find (i) P(not A) (ii) P(not B) (iii) P(A or B)

Sol :

(i) Given $P(A) = 0.42$

$$\therefore P(\text{not } A) = 1 - P(A)$$
$$P(\bar{A}) = 1 - 0.42 = 0.58$$

(ii) Given $P(B) = 0.48$

$P(\text{not } B) = 1 - P(B)$

$P(\bar{B}) = 1 - 0.48 = 0.52$

(iii) $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 0.42 + 0.48 - 0.16$

$= 0.90 - 0.16$

$P(A \text{ or } B) = 0.74$

3. If A and B are two mutually exclusive events of a random experiment and P(not A) = 0.45, $P(A \cup B)$ = 0.65, then find P(B).

Sol :

Since A and B are mutually exclusive events

$P(A \cap B) = 0$

$P(\text{not } A) = 0.45$

$\therefore \quad P(A) = 1 - P(\text{not } A)$

$P(A) = 1 - 0.45 = 0.55$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.65 = 0.55 + P(B) - 0$

$P(B) = 0.65 - 0.55 = 0.1$

4. The probability that atleast one of A and B occur is 0.6. If A and B occur simultaneously with probability 0.2, then find $P(\bar{A}) + P(\bar{B})$.

Sol :

$P(A \text{ or } B) = P(A \cup B) = 0.6$

$P(A \cap B) = 0.2$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.6 = P(A) + P(B) - 0.2$

$P(A) + P(B) = 0.6 + 0.2 = 0.8$

$[\because P(A) = 1 - P(\bar{A});\ P(B) = 1 - P(\bar{B})]$

$[1 - P(\bar{A})] + [1 - P(\bar{B})] = 0.8$

$1 - P(\bar{A}) + 1 - P(\bar{B}) = 0.8$

$2 - [P(\bar{A}) + P(\bar{B})] = 0.8$

$P(\bar{A}) + P(\bar{B}) = 2 - 0.8 = 1.2$

5. The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then find the probability that neither A nor B happen.

Sol : $P(A) = 0.5$

$P(B) = 0.3$

Since A and B are mutually exclusive events

$P(A \cap B) = 0$

$P(\text{either } A \text{ or } B) = P(A \cup B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 0.5 + 0.3 - 0 = 0.8$

$P(\text{neither } A \text{ nor } B) = P(\overline{A \cup B})$

$P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.8 = 0.2$

Probability of neither A nor B happen = 0.2

6. Two dice are rolled once. Find the probability of getting an even number on the first die or a total of face sum 8.

Sol :

When two dice are rolled once, the sample space

S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6)
(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(5, 1) (5, 2) (5, 3) (5, 4) (5,5) (5, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

$n(S) = 36$

Let 'A' be the event of getting an even number on the first die.

A= {(2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6)
(4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6)
(6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

$n(A) = 18$

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36}$$

Let B be the event of getting total face sum 8.

B = {(2, 6) (3, 5) (4, 4) (5, 3) (6, 2)}

$n(B) = 5$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{36}$$

$A \cap B = \{(4, 4) (6, 2) (2, 6)\}$

$n(A \cap B) = 3$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{36}$$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{18}{36} + \frac{5}{36} - \frac{3}{36}$$

$$P(A \cup B) = \frac{18 + 5 - 3}{36} = \frac{23 - 3}{36} = \frac{20}{36} = \frac{5}{9}$$

Probability of getting even number in the first die or a total face sum 8 is $\frac{5}{9}$.

7. From a well-shuffled pack of 52 cards, a card is drawn at random. Find the probability of it being either a red king or a black queen.

Sol :

$$\text{Total number of cards} = 52$$
$$n(S) = 52$$

Let 'A' be the event of getting a red king card

$$n(A) = 2$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{52}$$

Let 'B' be the event of getting black queen card

$$n(B) = 2$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{52}$$

Since A and B are mutually exclusive events

$$A \cap B = 0$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = \frac{2}{52} + \frac{2}{52} - 0$$
$$P(A \cup B) = \frac{4}{52} = \frac{1}{13}$$

Probability of getting either a red king or a black queen = $\frac{1}{13}$.

8. A box contains cards numbered 3, 5, 7, 9, ... 35, 37. A card is drawn at random from the box. Find the probability that the drawn card have either multiples of 7 or a prime number.

Sol :

Cards in the box = $\{3, 5, 7, 9, \ldots\ldots, 37\}$

$$\text{Number of cards} = \frac{l-a}{d} + 1$$
$$= \frac{37-3}{2} + 1 = \frac{34}{2} + 1 = 17 + 1$$

$\therefore$ Number of cards= 18

$$n(S) = 18$$

Let A be the event of selecting a number which is multiple of 7.

$$A = \{7, 21, 35\}$$
$$n(A) = 3$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{18}$$

Let B be the event of selecting a prime number

$B = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$

$n(B) = 11$

$$P(B) = \frac{n(B)}{n(S)} = \frac{11}{18}$$
$$A \cap B = \{7\}$$
$$n(A \cap B) = 1$$
$$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$
$$P(A \cap B) = \frac{1}{18}$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = \frac{3}{18} + \frac{11}{18} - \frac{1}{18}$$
$$= \frac{3+11-1}{18} = \frac{13}{18}$$

Probability of drawing a card either multiple of 7 or a prime number is $\frac{13}{18}$.

9. Three unbiased coins are tossed once. Find the probability of getting atmost 2 tails or at least 2 heads.

Sol :

When three coins are tossed the sample space

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$$n(S) = 8$$

Let A be the event of getting atmost 2 tails (i.e., 0 tail, 1 tail, 2 tails}

A = {HHH, HHT, HTH, HTT, THH, THT, TTH}

$$n(A) = 7$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{7}{8}$$

Let B be the event of getting atleast 2 heads. (i.e., 2 heads, 3 heads)

B = {HHT, HTH, THH, HHH}

$$n(B) = 4$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{8}$$

$A \cap B$ = {HHT, HTH, THH, HHH}

$n(A \cap B) = 4$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{4}{8}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{7}{8} + \frac{4}{8} - \frac{4}{8} = \frac{7+4-4}{8} = \frac{7}{8}$$

Probability of getting atmost two tails or atleast 2 heads $= \frac{7}{8}$.

10. The probability that a person will get an electrification contract is $\frac{3}{5}$ and the probability that he will not get plumbing contract is $\frac{5}{8}$. The probability of getting atleast one contract is $\frac{5}{7}$. What is the probability that he will get both?

Sol :

Let A be the event of getting electrification contract.

$$P(A) = \frac{3}{5}$$

Let B be the event of getting plumbing contract

$$P(\bar{B}) = \frac{5}{8}$$

$$1 - P(B) = \frac{5}{8}$$

$$P(B) = 1 - \frac{5}{8} = \frac{3}{8}$$

Also $P(A \cup B) = \frac{5}{7}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{5}{7} = \frac{3}{5} + \frac{3}{8} - P(A \cap B)$$

$$P(A \cap B) = \frac{3}{5} + \frac{3}{8} - \frac{5}{7}$$

$$= \frac{168 + 105 - 200}{280} = \frac{73}{280}$$

Probability of getting both contracts $= \frac{73}{280}$.

11. In a town of 8000 people, 1300 are over 50 years and 3000 are females. It is known that 30% of the females are over 50 years. What is the probability that a chosen individual from the town is either a female or over 50 years?

Sol :

Total number of people = 8000

$n(S) = 8000$

Let A be the event of selecting a female

Number of females $n(A) = 3000$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3000}{8000}$$

Let 'B' be the event of selecting an individual as over 50 years.

Number of people who are over 50 years = 1300

$n(B) = 1300$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1300}{8000}$$

$n(A \cap B) = 30\%$ of 3000

$$n(A \cap B) = \frac{30}{100} \times 3000 = 900$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{900}{8000}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{3000}{8000} + \frac{1300}{8000} - \frac{900}{8000}$$

$$P(A \cup B) = \frac{3000 + 1300 - 900}{8000}$$

$$= \frac{3400}{8000} = \frac{17}{40}$$

$\therefore$ Probability that a chosen individual is either female or over 50 years is $\frac{17}{40}$.

12. A coin is tossed thrice. Find the probability of getting exactly two heads or atleast one tail or two consecutive heads.

Sol :

A coin is tossed thrice by the sample space

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$n(S) = 8$

Let A be the event of getting exactly two heads

$\therefore$ A = {HHT, HTH, THH}

$n(A) = 3$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

Let B be the event of getting atleast one tail.

B = {HHT, HTH, HTT, THH, THT, TTH, TTT}

$$n(B) = 7$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

Let C be the event of getting consecutively two heads.

$$C = \{HHT, THH, HHH\}$$

$$n(C) = 3$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{3}{8}$$

$$A \cap B = \{HHT, HTH, THH\}$$

$$n(A \cap B) = 3$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{3}{8}$$

$$B \cap C = \{HHT, THH\}$$

$$n(A \cap C) = 2$$

$$P(B \cap C) = \frac{n(B \cap C)}{n(S)} = \frac{2}{8}$$

$$A \cap C = \{HHT, THH\}$$

$$n(A \cap C) = 2$$

$$P(A \cap C) = \frac{n(A \cap C)}{n(S)} = \frac{2}{8}$$

$$A \cap B \cap C = \{HHT, THH\}$$

$$n(A \cap B \cap C) = 2$$

$$P(A \cap B \cap C) = \frac{n(A \cap B \cap C)}{n(S)} = \frac{2}{8}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$$P(A \cup B \cup C) = \frac{3}{8} + \frac{7}{8} + \frac{3}{8} - \frac{3}{8} - \frac{2}{8} - \frac{2}{8} + \frac{2}{8}$$

$$= \frac{3+7+3-3-2-2+2}{8}$$

$$= \frac{8}{8} = 1$$

$\therefore$ The required probability is 1.

13. If A, B, C are any three events such that probability of B is twice as that of probability of A and probability of C is thrice as that of probability of A and if $P(A \cap B) = \frac{1}{6}$, $P(B \cap C) = \frac{1}{4}$, $P(A \cap C) = \frac{1}{8}$, $P(A \cup B \cup C) = \frac{9}{10}$ and $P(A \cap B \cap C) = \frac{1}{15}$, then find P(A), P(B) and P(C)?

Sol :

Given $P(A \cap B) = \frac{1}{6}$

$$P(B \cap C) = \frac{1}{4}$$

$$P(A \cap C) = \frac{1}{8}$$

$$P(A \cup B \cup C) = \frac{9}{10}$$

$$P(A \cap B \cap C) = \frac{1}{15}$$

Also given that $P(B) = 2P(A)$

$$P(C) = 3P(A)$$

Now $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$$\frac{9}{10} = P(A) + 2P(A) + 3(P(A)) - \frac{1}{6} - \frac{1}{4} - \frac{1}{8} + \frac{1}{15}$$

$$\frac{9}{10} = 6P(A) - \frac{1}{6} - \frac{1}{4} - \frac{1}{8} + \frac{1}{15}$$

$$6P(A) = \frac{9}{10} + \frac{1}{6} + \frac{1}{4} + \frac{1}{8} - \frac{1}{15}$$

$$6P(A) = \frac{108+20+30+15-8}{120}$$

$$6P(A) = \frac{165}{120}$$

$$P(A) = \frac{165}{120 \times 6} = \frac{11}{48}$$

$$P(B) = 2 \times \frac{11}{48} = \frac{11}{24}$$

$$P(C) = 3 \times \frac{11}{48} = \frac{11}{16}$$

$$\therefore \quad P(A) = \frac{11}{48};\ P(B) = \frac{11}{24};\ P(C) = \frac{11}{16}.$$

14. In a class of 35, students are numbered from 1 to 35. The ratio of boys and girls in 4 : 3. The roll numbers of students begin with boys and end with girls. Find the probability that a student selected is either a boy with prime roll number or a girl with composite roll number or an even roll number.

Sol :

$$\begin{aligned}
\text{Total students} &= 35\\
n(S) &= 35\\
\text{Boys : girls} &= 4:3\\
\text{Let the number of boys} &= 4x\\
\text{and number of girls} &= 3x\\
4x+3x &= 35\\
7x &= 35\\
x &= \frac{35}{7} = 5\\
\therefore \text{Number of boys} &= 4\times 5 = 20\\
\text{Number of girls} &= 3\times 5 = 15\\
\text{Boys are numbered} &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}\\
\text{Girls are numbered} &= \{21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35\}
\end{aligned}$$

Let A be the event of getting a boy with prime roll number.

$$\begin{aligned}
A &= \{2, 3, 5, 7, 11, 13, 17, 19\}\\
n(A) &= 8\\
P(A) &= \frac{n(A)}{n(S)} = \frac{8}{35}
\end{aligned}$$

Let B be the event of getting a girl with composite roll number.

$$\begin{aligned}
B &= \{21, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 35\}\\
n(B) &= 12\\
P(B) &= \frac{n(B)}{n(S)} = \frac{12}{35}
\end{aligned}$$

Let C be the event of getting an even roll number.

$$\begin{aligned}
C &= \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34\}\\
n(C) &= 17\\
P(C) &= \frac{n(C)}{n(S)} = \frac{17}{35}
\end{aligned}$$

Since A and B are mutually exclusive events

$$\begin{aligned}
P(A\cap B) &= 0\\
B\cap C &= \{22, 24, 26, 28, 30, 32, 34\}\\
n(B\cap C) &= 7\\
P(B\cap C) &= \frac{n(B\cap C)}{n(S)} = \frac{7}{35}\\
A\cap C &= \{2\}\\
n(A\cap C) &= 1\\
P(A\cap C) &= \frac{n(A\cap C)}{n(S)} = \frac{1}{35}\\
P(A\cap B\cap C) &= 0 \quad [\because n(A\cap B) = 0]
\end{aligned}$$

$$\begin{aligned}
P(A\cup B\cup C) &= P(A)+P(B)+P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)\\
&= \frac{8}{35}+\frac{12}{35}+\frac{17}{35}-0-\frac{7}{35}-\frac{1}{35}+0\\
&= \frac{8+12+17-7-1}{35} = \frac{29}{35}
\end{aligned}$$

$\therefore$ Required probability $= \dfrac{29}{35}$

Exercise 8.5

Multiple Choice Questions:

1. Which of the following is not a measure of dispersion?

(1) Range
(2) Standard deviation
(3) Arithmetic mean
(4) Variance [Ans: (3)]

2. The range of the data 8, 8, 8, 8, 8 … 8 is

(1) 0 (2) 1
(3) 8 (4) 3 [Ans: (1)]

Sol :

$$\begin{aligned}
\text{Range} &= L - S\\
&= 8 - 8\\
&= 0
\end{aligned}$$

3. The sum of all deviations of the data from its mean is

(1) Always positive (2) Always negative
(3) Zero (4) non-zero integer
[Ans: (3)]

4. The mean of 100 observations is 40 and their standard deviation is 3. The sum of squares of all deviations is

(1) 40000 (2) 160900
(3) 160000 (4) 30000 [Ans: (2)]

Sol :

$$N = 100$$
$$\bar{x} = 40$$
$$\sigma = 3$$
$$\sigma = \sqrt{\frac{\sum d_i^2}{N} - \left(\frac{\sum d_i}{N}\right)^2}$$
$$3^2 = \frac{\sum d_i^2}{100} - 40^2$$
$$9 = \frac{\sum d_i^2}{100} - 1600$$
$$9 + 1600 = \frac{\sum d_i^2}{100}$$
$$1609 \times 100 = \sum d_i^2$$
$$\sum d_i^2 = 1,60,900$$

5. Variance of first 20 natural numbers is

(1) 32.25 (2) 44.25
(3) 33.25 (4) 30 [Ans: (3)]

Sol :

Sum of first 20 natural numbers

$$= \frac{n(n+1)}{2} = \frac{20 \times 21}{2}$$
$$\sum x_i = 210$$
$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$
$$\sum x_i^2 = 1^2 + 2^2 + 3^2 + \ldots + 20^2$$
$$= \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{20 \times 21 \times 41}{6} = 10 \times 7 \times 41$$
$$= 2870$$
$$\sigma = \sqrt{\frac{2870}{20} - \left(\frac{210}{20}\right)^2}$$
$$\text{Variance } \sigma^2 = \frac{2870}{20} - \left(\frac{210}{20}\right)^2$$
$$= 143.5 - \frac{441}{4}$$
$$= 143.5 - 110.25 = 33.25$$

6. The standard deviation of a data is 3. If each value is multiplied by 5 then the new variance is

(1) 3 (2) 15
(3) 5 (4) 225 [Ans: (4)]

Sol : standard deviation $\sigma = 3$

Each value is multiplied by 5, then standard deviation also gets multiplied by 5.

$\therefore$ New standard deviation $\sigma = 3 \times 5 = 15$

Variance $\sigma^2 = 15^2 = 225$.

7. If the standard deviation of x, y, z is p then the standard deviation of 3x + 5, 3y + 5, 3z + 5 is

(1) 3p + 5 (2) 3p
(3) p + 5 (4) 9p + 15 [Ans: (2)]

Sol : standard deviation $\sigma = p$

If each value is multiplied by 3, the

New standard deviation $\sigma = 3p$

If 5 is added to each value the standard deviation does not change.

$\therefore$ New standard deviation $\sigma = 3p$

8. If the mean and coefficient of variation of a data are 4 and 87.5% then the standard deviation is

(1) 3.5 (2) 3
(3) 4.5 (4) 2.5 [Ans: (1)]

Sol : C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

$$87.5\% = \frac{\sigma}{4} \times 100\%$$
$$87.5 \times 4 = \sigma \times 100$$
$$350.0 = \sigma \times 100$$
$$\sigma = \frac{350}{100} = 3.5$$

9. Which of the following is incorrect?

(1) $P(A) > 1$ (2) $0 \le P(A) \le 1$
(3) $P(\phi) = 0$ (4) $P(A) + P(\bar{A}) = 1$
[Ans: (1)]

10. The probability a red marble selected at random from a jar containing p red, q blue and r green marbles is

(1) $\frac{q}{p+q+r}$ (2) $\frac{p}{p+q+r}$
(3) $\frac{p+q}{p+q+r}$ (4) $\frac{p+r}{p+q+r}$ [Ans: (2)]

Sol : Total marbles = p red + q blue + r green

$$n(S) = p + q + r$$

Let A be the event of selecting a red marbles

$$n(A) = p$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{p}{p+q+r}$$

11. A page is selected at random from a book. The probability that the digit at units place of the page number chosen is less than 7 is

(1) $\frac{3}{10}$ (2) $\frac{7}{10}$

(3) $\frac{3}{9}$ (4) $\frac{7}{9}$ [Ans: (2)]

Sol : The digits that may accur in units place

$$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$n(S) = 10.$$

Let A be the event of selecting page with unit place less than 7.

$$A = \{0, 1, 2, 3, 4. 5, 6\}$$
$$n(A) = 7$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{7}{10}$$

12. The probability of getting a job for a person is $\frac{x}{3}$. If the probability of not getting the job is $\frac{2}{3}$ then the value of x is

(1) 2 (2) 1

(3) 3 (4) 1.5 [Ans: (2)]

Sol : Let A be the event of getting a job.

$$P(A) + P(\bar{A}) = 1$$
$$\frac{x}{3} + \frac{2}{3} = 1$$
$$\frac{x+2}{3} = 1$$
$$x + 2 = 3$$
$$x = 3 - 2$$
$$x = 1.$$

13. Kamalam went to play a lucky draw contest. 135 tickets of the lucky draw were sold. If the probability of Kamalam winning is $\frac{1}{9}$, then the number of tickets bought by Kamalam is

(1) 5 (2) 10

(3) 15 (4) 20 [Ans: (3)]

Sol :

Let A be the event of winning the contest.

$$P(A) = \frac{1}{9}$$
$$\frac{n(A)}{n(S)} = \frac{1}{9}$$
$$\frac{n(A)}{n(S)} = \frac{1 \times 15}{9 \times 15} = \frac{15}{135}$$
$$n(A) = 15.$$

14. If a letter is chosen at random from the English alphabets {a, b, ..., z}, then the probability that the letter chosen precedes x

(1) $\frac{12}{13}$ (2) $\frac{1}{13}$

(3) $\frac{23}{26}$ (4) $\frac{3}{26}$ [Ans: (3)]

Sol : Let S = {a, b, ..., z}

$$n(S) = 26$$

Let A be the event of choosing a letter that precedes x.

$$A = \{a, b, c, d, ..., w\}$$
$$n(A) = 23$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{23}{26}$$

15. A purse contains 10 notes of ₹ 2000, 15 notes of ₹ 500 and 25 notes of ₹ 200. One note is drawn at random. What is the probability that the note is either a ₹ 500 note or ₹ 200 note?

(1) $\frac{1}{5}$ (2) $\frac{3}{10}$

(3) $\frac{2}{3}$ (4) $\frac{4}{5}$ [Ans: (4)]

Sol :

Total notes $n(S) = 10 + 15 + 25$

$$n(S) = 50$$

Let A be the event of drawing ₹ 500 note

$$n(A) = 15$$
$$P(A) = \frac{n(A)}{n(S)}$$
$$P(A) = \frac{15}{50}$$

Let B be the event of drawing ₹ 200 note

$$n(B) = 25$$
$$P(B) = \frac{n(B)}{n(S)} = \frac{25}{50}$$

Since A and B are mutually exclusive events.

$$P(A \cap B) = 0$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = \frac{15}{50} + \frac{25}{50} - 0 = \frac{40}{50} = \frac{4}{5}$$

UNIT EXERCISE - 8

1. The mean of the following frequency distribution is 62.8 and the sum of all frequencies is 50. Compute the missing frequencies f_1 and f_2.

Class Interval	0-20	20-40	40-60	60-80	80-100	100-120
Frequency	5	f_1	10	f_2	7	8

Sol :

Let A = 50 and C = 20.

Class Interval	f_i	x_i	$d_i = x_i - A$, $d_i = x_i - 50$	$d_i = \frac{x_i - 50}{C}$, $d_i = \frac{x_i - 50}{20}$	$f_i d_i$
0-20	5	10	– 40	– 2	– 10
20-40	f_1	30	– 20	– 1	$-f_1$
40-60	10	50	0	0	0
60-80	f_2	70	20	1	f_2
80-100	7	90	40	2	14
100-120	8	110	60	3	24
	$\Sigma f_i = 30 + f_1 + f_2$, $\Sigma f_i = 50$				$\Sigma f_i d_i = 28 + f_2 - f_1$

Given mean $\bar{x} = 62.8$

$$\Sigma f_i = 50$$

$$30 + f_1 + f_2 = 50$$

$$f_1 + f_2 = 50 - 30$$

$$f_1 + f_2 = 20 \quad \ldots (1)$$

$$\text{Mean } \bar{x} = A + C \times \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

$$62.8 = 50 + 20 \times \left[\frac{28 + f_2 - f_1}{50}\right]$$

$$62.8 - 50 = \frac{2}{5}(28 + f_2 - f_1)$$

$$12.8 \times 5 = 56 + 2f_2 - 2f_1$$

$$64.0 - 56 = 2(f_2 - f_1)$$

$$\frac{8}{2} = f_2 - f_1$$

$$f_2 - f_1 = 4$$

$$f_1 - f_2 = -4 \quad \ldots (2)$$

$$(1) + (2) \Rightarrow \quad 2f_1 = 16$$

$$f_1 = \frac{16}{2} = 8$$

$$(2) \Rightarrow \quad f_1 - f_2 = -4$$

$$8 - f_2 = -4$$

$$-f_2 = -4 - 8$$

$$-f_2 = -12$$

$$f_2 = 12$$

$\therefore \quad f_1 = 8$ and $f_2 = 12$.

2. The diameter of circles (in mm) drawn in a design are given below.

Diameters	33-36	37-40	41-44	45-48	49-52
Number of circles	15	17	21	22	25

Calculate the standard deviation.

Sol :

Let A = 42.5 and C = 4

Diameters	Number of circles f_i	x_i	$d_i = \frac{x_i - A}{C}$, $d_i = \frac{x_i - 42.5}{4}$	$f_i d_i$	d_i^2	$f_i d_i^2$
33-36	15	34.5	$\frac{34.5 - 42.5}{4} = -2$	– 30	4	60
37-40	17	38.5	$\frac{38.5 - 42.5}{4} = -1$	– 17	1	17
41-44	21	42.5	$\frac{42.5 - 42.5}{4} = 0$	0	0	0
45-48	22	46.5	$\frac{46.5 - 42.5}{4} = 1$	22	1	22
49-52	25	50.5	$\frac{50.5 - 42.5}{4} = 2$	50	4	100

$$\text{Mean } \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i} \times C$$

$$\bar{x} = 42.5 + \frac{25}{100} \times 4$$

$$\bar{x} = 42.5 + 1 = 43.5$$

Standard Deviation

$$\sigma = C \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$\sigma = 4\times\sqrt{\frac{199}{100}-\left(\frac{25}{100}\right)^2}$$

$$\sigma = 4\times\sqrt{\frac{199}{100}-\frac{625}{10000}}$$

$$\sigma = 4\times\sqrt{\frac{19900-625}{10000}}$$

$$\sigma = 4\sqrt{\frac{19275}{10000}} = 4\sqrt{1.9275}$$

$$= 4\times 1.388 = 5.55$$

$\therefore$ Standard deviation $\sigma = 5.55$.

3. The frequency distribution is given below

x	k	2k	3k	4k	5k	6k
f	2	1	1	1	1	1

In the table k is a positive integer, has a variance of 160. Determine the value of k.

Sol :

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i x_i^2}{N}-\left(\frac{\Sigma f_i x_i}{N}\right)^2}$$

$$\text{Variance } \sigma^2 = \frac{\Sigma f_i x_i^2}{N}-\left(\frac{\Sigma f_i x_i}{N}\right)^2$$

x_i	f_i	x_i^2	$f_i x_i$	$f_i x_i^2$
k	2	k^2	2k	$2k^2$
2k	1	$4k^2$	2k	$4k^2$
3k	1	$9k^2$	3k	$9k^2$
4k	1	$16k^2$	4k	$16k^2$
5k	1	$25k^2$	5k	$25k^2$
6k	1	$36k^2$	6k	$36k^2$
	$N=\Sigma f_i=7$		$\Sigma f_i x_i = 22k$	$\Sigma f_i x_i^2 = 92k^2$

$$\therefore \text{ Variance } \sigma^2 = \frac{92k^2}{7}-\left(\frac{22k}{7}\right)^2$$

$$160 = \frac{92k^2}{7}-\frac{484k^2}{49}$$

$$160 = \frac{644k^2-484k^2}{49}$$

$$160 = \frac{160k^2}{49} \Rightarrow k^2 = \frac{160\times 49}{160}$$

$$k^2 = 49 \Rightarrow k = \pm 7$$

Given that k is a positive integer

$\therefore$ k = 7.

4. The standard deviation of some temperature data in degree Celsius (°C) is 5. If the data were converted into degree Fahrenheit (°F) then what is the variance?

Sol :

Given the standard deviation of some temperature data in degree Celsius is 5.

$$\sigma_C = 5$$

Let the temperature data be

$x_1^o C,\ x_2^o C,\ x_3^o C,\ x_4^o C,\ x_5^o C, \dots$

To convert Celsius to Fahrenheit, we have

$F = \frac{9}{5}\times$ Celsius temperature + 32

If the temperature data are converted into Fahrenheit (°F), then the data will be

$\frac{9}{5}x_1+32,\ \frac{9}{5}x_2+32,\ \frac{9}{5}x_3+32,\ \frac{9}{5}x_4+32,\ \frac{9}{5}x_5+32, \dots$

i.e., Every temperature in Celsius is multiplied by $\frac{9}{5}$ and 32 is added to it.

When we add a constant to each data, the S.D of the given data will not change

Also, when we multiply each data by a constant, the S.D of the new data also get multiplied by the constant.

$\therefore$ Standard deviation of the new data in Fahrenheit will be $\sigma_F = \frac{9}{5}\times\sigma_C$ [$\because$ + 32 will not affect new S.D]

$$= \frac{9}{5}\times 5 = 9^o$$

$\therefore$ New variance $\sigma_F^2 = 9^2 = 81$

$\therefore$ Variance = 81°F.

5. If for a distribution $\Sigma(x-5) = 3$, $\Sigma(x-5)^2 = 43$ and total number of observations is 18. Find the mean and standard deviation.

Sol :

$$\text{Given } \Sigma(x-5) = 3$$

$$N = 18$$

$$\Sigma(x-5) = 3$$

$$\Sigma x - \Sigma 5 = 3$$

$$\Rightarrow \quad \Sigma x - 18\times 5 = 3$$

$$\Sigma x - 90 = 3$$

$$\Sigma x = 3+90 = 93$$

$$\Sigma(x-5)^2 = 43$$

$$\Sigma(x^2+25-10x) = 43$$

$$\Sigma x^2+\Sigma 25-10\Sigma x = 43$$

$$\Sigma x^2+25\times 18-10\times 93 = 43$$

$$\sum x^2 = 43 + 930 - 450 = 523$$

Now standard deviation $\sigma = \sqrt{\frac{\sum x^2}{N} - \left(\frac{\sum x}{N}\right)^2}$

$$= \sqrt{\frac{523}{18} - \left(\frac{93}{18}\right)^2}$$

$$= \sqrt{\frac{523 \times 18 - 93 \times 93}{18 \times 18}}$$

$$= \frac{1}{18}\sqrt{9414 - 8649}$$

$$= \frac{27.66}{18} = 1.53$$

$\therefore$ Standard deviation $= 1.53$

Also mean $= \frac{\sum x}{N}$

$$\bar{x} = \frac{93}{18}$$

$$= 5.166$$

Mean $= 5.17$

6. **Prices of peanut packets in various places of two cities are given below. In which city, prices were more stable?**

Prices in city A	20	22	19	23	16
Prices in city B	10	20	18	12	15

Sol :

Prices of city A

$n = 5$

x_i	20	22	19	23	16	$\sum x_i = 100$
x_i^2	400	484	361	529	256	$\sum x_i^2 = 2030$

Mean $\bar{x} = \frac{\sum x_i}{n}$

$$\bar{x} = \frac{100}{5} = 20$$

Standard deviation $\sigma_A = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$

$$= \sqrt{\frac{2030}{5} - \left(\frac{100}{5}\right)^2}$$

$$= \sqrt{406 - 400} = \sqrt{6} = 2.45$$

Co-efficient of variation C.V. $= \frac{\sigma}{\bar{x}} \times 100\%$

$$= \frac{2.45}{20} \times 100\% = \frac{245}{20}\%$$

$$= \frac{122.5}{10}\% = 12.25\%$$

For prices in city A, Co efficient of variation $= 12.25\%$

For prices in city B.

$n = 5$

x_i	10	20	18	12	15	$\sum x_i = 75$
x_i^2	100	400	324	144	225	$\sum x_i^2 = 1193$

$$\bar{x} = \frac{75}{5} = 15$$

$$\sigma_B = \sqrt{\frac{1193}{5} - \left(\frac{75}{5}\right)^2}$$

$$= \sqrt{238.6 - 15^2}$$

$$= \sqrt{238.6 - 225} = \sqrt{13.6}$$

$$= 3.69$$

For prices in city B, co-efficient of variation

$$\text{C.V.} = \frac{3.69}{15} \times 100\%$$

$$= \frac{369}{15}\% = 24.6\%$$

C.V. of city A < C.V. of city B.

$\therefore$ The prices were more stable in city A.

7. **If the range and coefficient of range of the data are 20 and 0.2 respectively, then find the largest and smallest values of the data.**

Sol :

Given range $= 20$

$L - S = 20$ ... (1)

Co-efficient of range $= 0.2$

$$\frac{L-S}{L+S} = 0.2$$

$$L - S = 0.2\,(L + S)$$

$$20 = 0.2\,(L + S)$$

$$\frac{20}{0.2} = L + S$$

$$L + S = \frac{200}{2}$$

$L + S = 100$ ... (2)

$$L - S = 20$$

$$(1) + (2) \Rightarrow \quad 2L = 120$$

$$L = \frac{120}{2} = 60$$

Put L = 60 in (2)

$$L + S = 100$$

$$60 + S = 100$$

$$S = 100 - 60 = 40$$

$\therefore$ L = 60 and S = 40.

8. If two dice are rolled, then find the probability of getting the product of face value 6 or the difference of face values 5.

Sol : If two dice are rolled, then the sample space

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

n (S) = 36

Let A be the event of getting the product of the face value 6.

A = {(1, 6), (2, 3), (3, 2), (6, 1)}

n (A) = 4

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{36}$$

Let 'B' be the event of getting the difference of the face value 5.

B = {(1, 6), (6, 1)}

n (B) = 2

$$P(B) = \frac{n(B)}{n(S)} = \frac{2}{36}$$

$$A \cap B = \{(1, 6), (6, 1)\}$$

$$n(A \cap B) = 2$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{36}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{4}{36} + \frac{2}{36} - \frac{2}{36} = \frac{4}{36} = \frac{1}{9}$$

$\therefore$ The required probability = $\frac{1}{9}$.

9. In a two children family, find the probability that there is atleast one girl in a family.

Sol : A family has two children

Let girl is denoted by 'g' and boy be denoted by 'b'.

Then the sample space

S = {(g, g), (g, b), (b, g), (b, b)}

n (S) = 4

Let A be the event of having atleast one girl baby.

A = {(g, g), (g, b), (b, g)}

n (A) = 3

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{4}$$

$\therefore$ Probability of getting atleast one girl is $\frac{3}{4}$.

10. A bag contains 5 white and some black balls. If the probability of drawing a black ball from the bag is twice the probability of drawing a white ball then find the number of black balls.

Sol :

Let A be the event of drawing black ball.

Let the number of black balls = x

n (A) = x

Total number of balls = 5 + x

n (S) = 5 + x

$$P(A) = \frac{n(A)}{n(S)} = \frac{x}{5+x}$$

Let 'B' be the event of drawing a white ball

n (B) = 5

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{5+x}$$

Given that $\quad \frac{x}{5+x} = 2\left(\frac{5}{5+x}\right)$

$$\frac{x(5+x)}{5+x} = 2 \times 5$$

$$x = 10$$

$\therefore$ Number of black balls = 10.

11. The probability that a student will pass the final examination in both English and Tamil is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Tamil examination?

Sol :

Let T denote Tamil and E denote English

$$P(T \cap E) = 0.5$$

$$\text{P (Passing neither)} = P(T' \cap E')$$

$$= P(T \cup E)' = 0.1$$

$$P(T \cup E) = 1 - P(T \cup E)'$$

$$= 1 - 0.1 = 0.9$$

Also given $\quad P(E) = 0.75$

$$P(T \cup E) = P(T) + P(E) - P(T \cap E)$$

$$0.9 = P(T) + 0.75 - 0.5$$

$$P(T) = 0.9 - 0.75 + 0.5 = 0.65$$

$\therefore$ Probability of passing Tamil exam = 0.65

$$P(T) = \frac{65}{100} = \frac{13}{20}$$

12. The King, Queen and Jack of the suit spade are removed from a deck of 52 cards. One card is selected from the remaining cards. Find the probability of getting (i) a diamond (ii) a queen (iii) a spade (iv) a heart card bearing the number 5.

Sol :

Suits of playing cards	Spade ♠	Heart ♥	Clavor ♣	Diamond ♦
Existing Cards	A	A	A	A
	2	2	2	2
	3	3	3	3
	4	4	4	4
	5	5	5	5
	6	6	6	6
	7	7	7	7
	8	8	8	8
	9	9	9	9
	10	10	10	10
		J	J	J
		Q	Q	Q
		K	K	K
Total Number of cards in each suit	13	13	13	13
Remaining cards after removing some of them.	10	13	13	13

After removing King, Queen and Jack of spade the remaining number of cards

$$n(S) = 10 + 13 + 13 + 13$$

$$n(S) = 49$$

(i) Let A be the event of selecting a card from diamond.

$$n(A) = 13$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{13}{49}$$

$\therefore$ Probability of getting a diamond card = $\frac{13}{49}$.

(ii) Let 'B' be the event of getting a queen.

$$n(B) = 3$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{49}$$

$\therefore$ Probability of getting a queen = $\frac{3}{49}$.

(iii) Let 'C' be the event of getting a spade

$$n(C) = 10$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{10}{49}$$

$\therefore$ Probability of getting a spade = $\frac{10}{49}$.

(iv) Let D be the event of getting 5 of heart.

$$n(D) = 1$$

$$P(D) = \frac{n(D)}{n(S)} = \frac{1}{49}$$

$\therefore$ Probability of getting 5 of heart = $\frac{1}{49}$.

CREATIVE QUESTIONS

I. Multiple Choice Questions

Measures of Dispersion

1. Statistically, spread or scatterness of observations in a data is called _____
(1) Discriminant (2) Dispersion
(3) Range (4) Standard deviation.
[Ans: (2)]

2. Mean of squared deviations of some observations from their arithmetic mean is called _____
(1) Standard deviation
(2) Variation
(3) Median (4) Mode [Ans: (2)]

3. Positive square root of mean of squared deviations of some observations from the arithmetic mean is called _____
(1) Standard deviation
(2) Variation
(3) Median (4) Mode [Ans: (1)]

4. Sum of deviations of a variable from its mean is always
(1) 0 (2) 1
(3) 2 (4) 5 [Ans: (1)]

5. Standard deviation of first 50 natural numbers is _____
(1) 45.43 (2) 14.43
(3) 20.43 (4) 16.43 [Ans: (2)]

Sol :

Standard deviation of first n natural numbers

$$= \sqrt{\frac{n^2-1}{12}}$$

Standard deviation of first 50 natural numbers

$$= \sqrt{\frac{50^2-1}{12}} = \sqrt{\frac{2500-1}{12}}$$

$$= \sqrt{\frac{2499}{12}} = \sqrt{208.25} = 14.43$$

6. Standard deviation of population is denoted by _____
(1) Ω (2) ω
(3) σ (4) Δ [Ans: (3)]

7. Price of apple per kg for three days are as 98, 97, 100, then the value of standard deviation with assumed mean method is.
(1) 15 (2) 10
(3) 1 (4) 11 [Ans: (3)]

Sol :

x_i	f_i	$d_i = x_i - A = x_i - 98$	d_i^2	f_id_i	$f_id_i^2$
97	1	− 1	1	− 1	1
98	1	0	0	0	0
100	1	2	4	2	4
				$\Sigma f_id_i = 1$	$\Sigma f_id_i^2 = 5$

$$\sigma = \sqrt{\frac{\Sigma f_id_i^2}{N} - \left(\frac{\Sigma f_id_i}{N}\right)^2}$$

$$= \sqrt{\frac{5}{3} - \left(\frac{1}{3}\right)^2}$$

$$= \sqrt{1.666 - 0.333} = \sqrt{1.333}$$

$$= 1.15 \simeq 1$$

8. In statistics, distance or dispersion from central value is classified as
(1) Standard variance
(2) Sample variance
(3) Standard root
(4) Standard deviation [Ans: (4)]

9. Range of the scores 80, 90, 90, 85, 60, 70, 75, 85, 90, 60, 80 is _____
(1) 30 (2) 70
(3) 90 (4) 40 [Ans: (1)]

Sol :

Range $= L - S = 90 - 60 = 30$

10. Coefficient of range of 5, 6, 7, 8, 9, 54 is _____
(1) $\frac{39}{49}$ (2) $\frac{49}{59}$
(3) $\frac{59}{69}$ (4) $\frac{69}{79}$ [Ans: (2)]

Sol : *Co-efficient of range* $= \frac{L-S}{L+S} = \frac{54-5}{54+5} = \frac{49}{59}$

11. If the total sum of squares is 20 and sample variance is 5, then total number of observations are

(1) 15 (2) 25
(3) 4 (4) 35 [Ans: (3)]

Sol : $\sum(x_i - \bar{x})^2 = 20$

$$\text{Variance} = 5$$

$$\sigma = \sqrt{\frac{\sum(x_i-\bar{x})^2}{n}} = \sqrt{\frac{20}{n}}$$

$$\text{Variance } \sigma^2 = \frac{20}{n}$$

$$5 = \frac{20}{n}$$

$$n = \frac{20}{5} = 4$$

Coefficient of Variation

12. If mean is 25 and standard deviation is 5 then co-efficient of variation is

(1) 100% (2) 25%
(3) 20% (4) None of these
[Ans: (3)]

Sol : $C.V. = \frac{\sigma}{\bar{x}} \times 100\% = \frac{5}{25} \times 100\% = 20\%$

13. _____ is used to compare the variation or dispersion in two or more sets of data even though they are measured in different units.

(1) Range
(2) Standard deviation
(3) Co-efficient of variation
(4) Mean deviation. [Ans: (3)]

14. _____ is used to criterion of consistence is for consistence performance.

(1) Range
(2) Standard deviation
(3) Co-efficient of variation
(4) Mean deviation [Ans: (3)]

15. If the co-efficient of variation of marks of Brinda is 25% and that of Buvana is 40%. Who is more stable in scoring?

(1) Brinda (2) Buvana
(3) Both (4) None [Ans: (1)]

Sol :
Less co-efficient of variation means the data are more stable.
∴ Brinda is more stable.

Probability of an Event

16. If a digit is chosen at random from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, then the probability that it is odd is

(1) $\frac{4}{9}$ (2) $\frac{5}{9}$
(3) $\frac{1}{9}$ (4) $\frac{2}{3}$ [Ans: (2)]

Sol : $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$$n(S) = 9$$

A be the event of selecting odd number.

$$A = \{1, 3, 5, 7, 9\}$$

$$n(A) = 5$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{9}$$

17. In a single throw of die, the probability of getting a multiple of 3 is

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$
(3) $\frac{1}{6}$ (4) $\frac{2}{3}$ [Ans: (2)]

Sol :
For a single through of a die S = {1, 2, 3, 4, 5, 6}

$$n(S) = 6.$$

Let E be the event of getting a multiple of 3.

$$E = \{3, 6\}$$

$$n(A) = 2$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

18. The probability of throwing a number greater than 2 with a fair dice is

(1) $\frac{3}{5}$ (2) $\frac{2}{5}$
(3) $\frac{2}{3}$ (4) $\frac{1}{3}$ [Ans: (3)]

Sol : For a die $n(S) = 6$

Let E be the event of throwing a number greater then 2.

$$E = \{3, 4, 5, 6\}$$

$$n(E) = 4$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{6} = \frac{2}{3}$$

19. A card is dropped from a pack of 52 playing cards. The probability that it is an ace is

(1) $\frac{1}{4}$ (2) $\frac{1}{13}$

(3) $\frac{1}{52}$ (4) $\frac{12}{13}$ [Ans: (2)]

Sol : $n(S) = 52.$

Number of ace $n(A) = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

20. The probability of a certain event is

(1) 0 (2) 1

(3) $\frac{1}{2}$ (4) Not exists. [Ans: (2)]

21. The probability of an impossible event is

(1) 0 (2) 1

(3) $\frac{1}{2}$ (4) Not exists. [Ans: (1)]

22. Which of the following is not the probability of accurance of an event

(1) 0.2 (2) 0.4

(3) 0.8 (4) 1.6 [Ans: (4)]

Algebra of Events:

23. If P (E) = 0.05, then P (not E) =

(1) – 0.05 (2) 0.5

(3) 0.9 (4) 0.95 [Ans: (4)]

Sol :

$$P(E') = 1 - P(E) = 1 - 0.05 = 0.95$$

24. Which of the following statement is wrong.

(1) $A \cap B$ is an event that occurs only when both A and B occurs.

(2) $A \cup B$ is an event that occurs only when at least one of A or B occurs.

(3) $\bar{A}$ is an event that occurs only when A does not occur.

(4) $\bar{B}$ is an event that occurs when B occur.

[Ans: (4)]

25. $A \cup \bar{A}$ = _____

(1) 0 (2) 1

(3) ϕ (4) S [Ans: (4)]

26. $A \cap \bar{A}$ = _____

(1) 0 (2) 1

(3) ϕ (4) S [Ans: (3)]

27. $P(\overline{A \cup B})$ = _____

(1) $P(\bar{A} \cup \bar{B})$ (2) $P(\bar{A} \cap \bar{B})$

(3) $P(A \cup B)$ (4) $P(A \cap B)$ [Ans: (2)]

II. Very Short Answer Questions

1. Find the co-efficient of range of the following data 13, 15, 21, 25, 18, 27, 43, 31

Sol :

Largest value $L = 43$

Smallest value $S = 13$

$$\text{Coefficient of range} = \frac{L-S}{L+S} = \frac{43-13}{43+13} = \frac{30}{56}$$

$$\text{Coefficient of range} = \frac{15}{28} = 0.54$$

2. Find the range of the following data 70.3, 43.2, 80.5, 93.4, 100, 13.7

Sol :

Largest value $L = 100$

Smallest value $S = 13.7$

Range $= L - S = 100 - 13.7$

Range $= 86.3$

3. Calculate the range of the following data.

Marks	40-50	50-60	60-70	70-80	80-90	90-100
Students	20	13	27	17	40	8

Sol : Largest value $L = 100$

Smallest value $S = 40$

Range $= L - S = 100 - 40$

Range $= 60$

4. Find the standard deviation of first 50 natural numbers

Sol : Standard deviation of first n natural numbers

$$= \sqrt{\frac{n^2-1}{12}}$$

$$\therefore \text{S.D. of first 50 natural numbers} = \sqrt{\frac{50^2-1}{12}}$$

$$= \sqrt{\frac{2500-1}{12}} = \sqrt{\frac{2499}{12}}$$

$$= \sqrt{208.25} = 14.43$$

Standard deviation of first 50 natural numbers $= 14.43$

5. If the standard deviation of a data is 20.5 and each value of the data is increased by 12, then find the new standard deviation:

Sol :

Given the standard deviation = 20.5.

If we add a fixed constant to each data, the standard deviation never change.

$\therefore$ If we add 12 to each data the new standard deviation will not change.

$\therefore$ New standard deviation = 20.5

6. The standard deviation and mean of a data is given by 60 and 120. Find the coefficient of variation.

Sol :

Standard deviation $\sigma = 60$

Mean $\bar{x} = 120$

Co-efficient of variation

$$\text{C. V} = \frac{\sigma}{\bar{x}} \times 100\%$$

$$= \frac{60}{120} \times 100\% = \frac{6000}{120}$$

$$= 50\%$$

Co-efficient of variation is 50%.

7. The standard deviation and coefficient of variation of some data are 9 and 72. Find the mean.

Sol :

Given the co-efficient of variation C. V = 72

Standard deviation $\sigma = 9$

$$\text{C. V} = \frac{\sigma}{\bar{x}} \times 100\%$$

$$72 = \frac{9}{\bar{x}} \times 100\%$$

$$\bar{x} = \frac{9}{72} \times 100 = \frac{100}{8}$$

$\therefore$ Mean $\bar{x} = 12.5$

8. If the mean and coefficient of variation of a data are 30 and 96 respectively, then find the value of standard deviation.

Sol :

Mean $\bar{x} = 30$

Co-efficient of variation C. V = $\frac{\sigma}{\bar{x}} \times 100\%$

$$96 = \frac{\sigma}{30} \times 100\%$$

$$\sigma = \frac{96 \times 30}{100} = \frac{288}{10}$$

$\therefore$ Standard deviation $\sigma = 28.8$

9. The probability that it will rain tomorrow is 0.85. What is the probability that it will not rain tomorrow?

Sol :

P (will rain) + P (will not rain) = 1

$\therefore$ P (will not rain) = 1 – P (will rain)

= 1 – 0.85 = 0.15

Probability that will not rain tomorrow = 0.15.

10. A die is thrown. Find the probability of getting an even prime number?

Sol :

When a die is thrown, the sample space

$S = \{1, 2, 3, 4, 5, 6\}$

$n(S) = 6$

Let A be the event of getting an even prime

$A = \{2\}$

$n(A) = 1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}$$

Probability of getting an even prime is $\frac{1}{6}$

11. An urn contains 10 red and 8 white balls. One ball is drawn at random. Find the probability that the ball drawn is white.

Sol :

Total number of balls = 10 red + 8 white

$n(S) = 18$

Let A be the event of drawing white ball

$n(A) = 8$

$$P(A) = \frac{8}{18} = \frac{4}{9}$$

12. A bag contains 3 red balls, 5 black balls and 4 white balls. A ball is drawn at random from the bag. What is the probability that the ball drawn is not red?

Sol :

Total number of balls = 3 red + 5 black + 4 white

$n(S) = 12$

Let A be the event of drawing red ball

$n(A) = 3$

$$P(A) = \frac{n(A)}{n(S)}$$

$$P(A) = \frac{3}{12} = \frac{1}{4}$$

$$\therefore \quad P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{4}$$

$$P(\overline{A}) = \frac{4-1}{4} = \frac{3}{4}$$

$\therefore$ Probability of the ball drawn is not red is $\frac{3}{4}$

13. What is the probability that a number selected from the numbers 1, 2, 3, ...15 is a multiple of 4?

Sol : Let $S = \{1, 2, 3, ..., 15\}$

$n(S) = 15$

Let A be the event of selecting a multiple of 4

$A = \{4, 8, 12\}$

$n(A) = 3$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{15} = \frac{1}{15}$$

$\therefore$ Probability of selecting a multiple of 4 is $\frac{1}{15}$.

14. In a lottery there are 10 prizes and 25 blanks. What is the probability of getting prize?

Sol :

Total results in the lottery = 10 prizes + 25 blanks

$n(S) = 35$

Let A be the event of getting prize

$n(A) = 10$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{35} = \frac{2}{7}$$

Probability of getting price is $\frac{2}{7}$

15. The probability of winning a game is 0.3. What is the probability of loosing it?

Sol :

P(winning) + P(loosing) = 1

0.3 + P(loosing) = 1

P(loosing) = 1 – 0.3 = 0.7

$\therefore$ Probability of loosing the game is 0.7

16. A bag contains cards which are numbered from 2 to 90. A card is drawn at random from the bag. Find the probability that it bears a number which is a perfect square?

Sol :

$S = \{2, 3, 4, 5, ..., 90\}$

$n(S) = 89$

Let A be the event of drawing a perfect square

$A = \{4, 9, 16, 25, 36, 49, 64, 81\}$

$n(A) = 8$

$$P(A) = \frac{n(A)}{n(S)} = \frac{8}{89}$$

$\therefore$ Probability of drawing a perfect square = $\frac{8}{89}$

17. If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{4}$, then find $P(A' \cap B')$.

Sol : Given $P(A) = \frac{1}{3}$

$$P(B) = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{4}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{4+6-3}{12} = \frac{7}{12}$$

$$P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$P(\overline{A \cup B}) = 1 - \frac{7}{12} = \frac{12-7}{12} = \frac{5}{12}$$

$$\therefore \quad P(A' \cap B') = \frac{5}{12} \left[\because P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})\right]$$

18. A and B are events such that P (not A or not B) = 0.25. Whether A and B are mutually exclusive or not.

Sol :

We have $P(A \cap B)' = P(A' \cup B')$

Given $P(A' \cup B') = 0.25$

$P(A \cap B)' = 0.25$

$P(A \cap B) = 1 - P(A \cap B)'$

$P(A \cap B) = 1 - 0.25 = 0.75$

$P(A \cap B) \neq 0.$

$\therefore$ A and B are not mutually exclusive.

19. Given $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find P(A or B) if A and B are mutually exclusive events.

Sol : Given A and B are mutually exclusive events.

$\therefore \quad P(A \cap B) = 0$

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{3}{5} + \frac{1}{5}$$

$$P(\text{'A' or 'B'}) = \frac{4}{5}$$

20. If A and B are two events associated with a random experiment such that P(A) = 0.3, P(B) = 0.2 and $P(A \cap B)$ = 0.1. Then find the value of $P(A \cap \overline{B})$.

Sol :

$$P(A \cap \overline{B}) = P\ (A \text{ only})$$
$$= P(A) - P(A \cap B) = 0.3 - 0.1$$
$$\therefore\ P\ (A \cap \overline{B}) = 0.2$$

III. Short Answer Questions:

1. The standard deviation of 50 data is 12.6 and each value of the data is divided by 6, then find the new variance and standard deviation.

Sol :

Standard deviation = 12.6

When each value of the data is divided by a fixed constant, the new standard deviation also get divided by the fixed constant.

$\therefore$ New standard deviation = $\frac{12.6}{6} = 2.1$

New variance $= \sigma^2 = (2.1)^2 = 4.41$

New Standard deviation = 2.1

New variance = 4.41

2. If n = 10, $\overline{x}$ = 1.8, Σx^2 = 1500. Calculate the coefficient of variation.

Sol :

First we calculate the standard deviation.

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$\sigma = \sqrt{\frac{1500}{10} - (1.8)^2} = \sqrt{150 - 3.24} = \sqrt{146.76}$$

$$\sigma = 12.11$$

Now, co-efficient of variation = $\frac{\sigma}{\overline{x}} \times 100\%$

$$= \frac{12.11}{1.8} \times 100\%$$

C. V = 672.78%

3. Coefficient of variation of two distributions are 60 and 70 and their standard deviation are 21 and 16 respectively. What are their arithmetic means?

Sol : Given:

For the I distribution	For the II distribution.
C. V_1 = 60	C. V_2 = 70
σ_1 = 21	σ_2 = 16

$$C.V = \frac{\sigma}{\overline{x}} \times 100\%$$

$$C.V_1 = \frac{\sigma_1}{\overline{x}_1} \times 100\%$$

$$60 = \frac{21}{\overline{x}_1} \times 100$$

$$\overline{x}_1 = \frac{21}{60} \times 100 = 35$$

$$CV_2 = \frac{\sigma_2}{\overline{x}_2} \times 100\%$$

$$70 = \frac{16}{\overline{x}_2} \times 100$$

$$\overline{x}_2 = \frac{16}{70} \times 100 = 22.85$$

$\overline{x}_1 = 35$; $\overline{x}_2 = 22.85$

4. In a lottery of 50 tickets numbered 1 to 50, one ticket is drawn. Find the probability that the drawn ticket bears a prime number.

Sol :

Let S = {1, 2, 3, ...,50}

n(S) = 50

Let A be the event of drawing a ticket bearing prime number.

A = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}

n(A) = 15

$$P(A) = \frac{n(A)}{n(S)} = \frac{15}{50} = \frac{3}{10}$$

$\therefore$ Probability of drawing a ticket bearing prime number is $\frac{3}{10}$

5. A bag contains 6 red, 8 black and 4 white balls. A ball is drawn at random. What is the probability that ball drawn is not black?

Sol :

Total number of balls = 6 red + 8 black + 4 white

n(S) = 18

Let A be the event of drawing a black ball

n(A) = 8

$$P(A) = \frac{n(A)}{n(S)} = \frac{8}{18} = \frac{4}{9}$$

P(not black) = 1 – P(A)

$$P(\overline{A}) = 1 - \frac{4}{9}$$

$$P(\overline{A}) = \frac{9-4}{9} = \frac{5}{9}$$

$\therefore$ Probability that the drawn ball is not black is $\frac{5}{9}$.

6. Two customers are visiting a particular shop in the same week (Monday to Saturday). Each is equally likely to visit the shop on any one day as on another. What is the probability that both will visit the shop on different days?

Sol :

Two customers can visit the shop on any of the 6 days.

S = {(Mon, Mon) (Mon, Tue) (Mon, Wed)
(Mon, Thur) (Mon, Fri) (Mon, Sat)
(Tue, Mon) (Tue, Tue) (Tue, Wed)
(Tue, Thur) (Tue, Fri) (Tue, Sat)
(Wed, Mon) (Wed, Tue) (Wed, Wed)
(Wed, Thur) (Wed, Fri) (Wed, Sat)
(Thur, Mon) (Thur, Tue) (Thur, Wed)
(Thur, Thur) (Thur, Fri) (Thur, Sat)
(Fri, Mon) (Fri, Tue) (Fri, Wed)
(Fri, Thur) (Fri, Fri) (Fri, Sat) (Sat, Mon)
(Sat, Tue) (Sat, Wed) (Sat, Thur) (Sat, Fri)
(Sat, Sat)}

n(S) = 36

Let A be the event of the customers visiting the shop on the same day.

A = {(Mon, Mon) (Tue, Tue) (Wed, Wed)
(Thur, Thur) (Fri, Fri) (Sat, Sat)}

n(A) = 6

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$$P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{6}$$

$$P(\overline{A}) = \frac{6-1}{6} = \frac{5}{6}$$

$\therefore$ Probability that they visit on different days $= \frac{5}{6}$.

7. Find the probability that a number selected from the numbers 1, 2, 3, ...35 is a multiple of 7.

Sol :

S = {1, 2, 3, 4, ...35}

n(S) = 35

Let A be the event of selecting multiple of 7

A = {7, 14, 21, 28, 35}

n(A) = 5

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{35} = \frac{1}{7}$$

$\therefore$ Probability of selecting multiple of 7 is $\frac{1}{7}$.

8. A lot consists of 144 ball pens of which 20 are defective and others good. Leena will buy a pen if it is good, but will not buy if it is defective. The shopkeeper draws one pen at random and gives it to her. What is the probability that she will buy it?

Sol : Total pens = 144

n(S) = 144

Number of good pens = 144 – defective pens
= 144 – 20 = 124

Let A be the event of getting good pen

$\therefore$ n(A) = 124

$$P(A) = \frac{n(A)}{n(S)} = \frac{124}{144} = \frac{31}{36}$$

If she gets good pen, she will buy it.

$\therefore$ Probability of buying the pen $= \frac{31}{36}$.

9. A bag contains 6 red balls and some blue balls. If the probability of drawing a blue ball from the bag is twice that of a red ball, find the number of blue balls in the bag.

Sol : Let the number of blue balls = x

$\therefore$ Total balls n(S) = 6 + x

Let A be the event of drawing a blue ball.

n(A) = x

$$P(A) = \frac{x}{6+x}$$

Let B be the event of drawing red ball

n(B) = 6

$$P(B) = \frac{6}{6+x}$$

given P(A) = 2 [P(B)]

$$P(A) = 2 \times \frac{6}{6+x}$$

$$\frac{x}{6+x} = 2 \times \frac{6}{6+x}$$

$$\frac{x(6+x)}{6+x} = 2 \times 6$$

x = 12

$\therefore$ Number of blue balls = 12.

10. A number x is selected from the number 1, 2, 3 and then a second number y is randomly selected from the numbers 1, 4, 9. What is the probability that the product xy of two number will be less than 9?

Sol :

First number is selected from 1, 2, 3 and second from 1, 4, 9.

S = {(1, 1) (1, 4) (1, 9) (2, 1) (2, 4) (2, 9) (3, 1) (3, 4) (3, 9)}

n(S) = 9

Let A be the event of getting the product of two numbers less than 9.

A = {(1, 1) (1, 4) (2, 1) (2, 4) (3, 1)}

n (A) = 5

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{9}$$

$\therefore$ Required probability = $\frac{5}{9}$.

11. The probability that atleast one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3. Evaluate $P(\bar{A}) + P(\bar{B})$

Sol :

Given $P(A \cup B) = 0.6$

$P(A \cap B) = 0.3$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.6 = P(A) + P(B) - 0.3$

$P(A) + P(B) = 0.6 + 0.3 = 0.9$

$[1 - P(\bar{A})] + [1 - P(\bar{B})] = 0.9$

$1 - P(\bar{A}) + 1 - P(\bar{B}) = 0.9$

$2 - [P(\bar{A}) + P(\bar{B})] = 0.9$

$P(\bar{A}) + P(\bar{B}) = 2 - 0.9 = 1.1$

12. If A and B are mutually exclusive events P(A) = 0.35 and P(B) = 0.45. Find $P(A' \cap B')$.

Sol : $P(A \cup B) = P(A) + P(B)$

$= 0.35 + 0.45 = 0.80$

$P(A' \cap B') = P(A \cup B)'$

$= 1 - P(A \cup B) = 1 - 0.80$

$\therefore P(A' \cap B') = 0.20.$

13. If A and B are two events such that $P(A) = \frac{1}{5}$, $P(B) = \frac{1}{3}$ and P (A and B) = $\frac{1}{7}$ find (i) P (A or B), (ii) P (not A and not B).

Sol :

(i) $P(A \text{ or } B) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$$P(A \text{ or } B) = \frac{1}{5} + \frac{1}{3} - \frac{1}{7} = \frac{21 + 35 - 15}{105}$$

$$P(A \text{ or } B) = \frac{41}{105}$$

(ii) $P(\text{not } A \text{ and not } B) = P(\bar{A} \cap \bar{B})$

$= P(\overline{A \cup B}) = 1 - P(A \cup B)$

$$= 1 - \frac{41}{105} = \frac{105 - 41}{105}$$

$$P(\text{not } A \text{ and not } B) = \frac{64}{105}.$$

IV. Long Answer Questions

1. The mean and standard deviation is the marks of a group of 50 students were 60 and 15 respectively. Later if was found to be the scores 40 and 70 were wrongly entered as 30 and 60. Find the correct mean and standard deviation.

Sol :

Mean $\bar{x} = 60$

Standard deviation $\sigma = 15$.

Wrong scores = 30 and 60.

Correct scores = 40 and 70

$$\text{Old mean} = \frac{\Sigma x_i}{n} = 60$$

$$\frac{\Sigma x_i}{50} = 60$$

Old $\Sigma x_i = 60 \times 50 = 3000$

Correct Σx_i = 3000 – (wrong scores) + (correct scores)

= 3000 – [(30 + 60)] + [(40 + 70)]

= 3000 – [90] + [110]

= 2910 + 110 = 3020

$$\text{Correct mean} = \frac{\text{correct } \Sigma x_i}{n} = \frac{3020}{50} = 60.4$$

Old standard deviation $\sigma = 15$

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$15 = \sqrt{\frac{\Sigma x_i^2}{n} - 60^2}$$

Squaring on both sides

$$225 = \frac{\Sigma x_i^2}{n} - 3600$$

$$225 + 3600 = \frac{\Sigma x_i^2}{50}$$

$$\text{Old } \frac{\Sigma x_i^2}{50} = 3825$$

$$\text{Old } \Sigma x_i^2 = 3825 \times 50 = 191250$$

Correct $\sum x_i^2$ = 191250 – (wrong scores)^2 + (correct scores)^2

$= 191250 - 30^2 - 60^2 + 40^2 + 70^2$

$= 191250 - 900 - 3600 + 1600 + 4900$

$= 191250 - 4500 + 6500$

$= 193250$

$$\text{Correct } \sigma = \sqrt{\frac{193250}{50} - (60.4)^2}$$

$$= \sqrt{3865 - 3648.16} = \sqrt{216.84}$$

Correct $\sigma = 14.7$

2. Calculate standard deviation by direct method. 25, 27, 31, 32, 35

Sol :

x_i	x_i^2
25	625
27	729
31	961
32	1024
35	1225
$\Sigma x_i = 150$	$\Sigma x_i^2 = 4564$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{N}\right)^2}$$

$$= \sqrt{\frac{4564}{5} - \left(\frac{150}{5}\right)^2} = \sqrt{912.8 - 900} = \sqrt{12.8}$$

$\therefore \quad \sigma = 3.578$

3. If $N = 10$, $\Sigma x = 120$, $\Sigma x^2 = 1530$, find the variance.

Sol : $\bar{x} = \frac{\Sigma x}{n} = \frac{120}{10} = 12$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{N}\right)^2}$$

$$\sigma = \sqrt{\frac{1530}{10} - \left(\frac{120}{10}\right)^2}$$

$$\sigma = \sqrt{153 - 144} = \sqrt{9} = 3$$

Variance $\sigma^2 = 3^2 = 9$

4. Calculate standard deviation by assumed mean method 40, 44, 54, 60, 62, 64, 70, 80, 90, 96.

Sol : Let A = 60

x_i	$d_i = x_i - A$ $d_i = x_i - 60$	d_i^2
40	– 20	400
44	– 16	256
54	– 6	36
60	0	0
62	2	4
64	4	16
70	10	100
80	20	400
90	30	900
96	36	1296
N = 10	$\Sigma d_i = 60$	$\Sigma d_i^2 = 3408$

$$\text{Standard deviation } \sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\frac{3408}{10} - \left(\frac{60}{10}\right)^2}$$

$$= \sqrt{340.8 - 36} = \sqrt{304.8}$$

$\therefore$ Standard deviation $\sigma = 17.46$

5. Calculate the standard deviation.

x	10	12	14	16	18	20	22
f	3	5	9	16	8	7	2

Sol :

Let A = 14

x_i	f_i	x_if_i	$d_i = x_i - A$ $d_i = x_i - 14$	f_id_i	d_i^2	$f_id_i^2$
10	3	30	– 4	– 12	16	48
12	5	60	– 2	– 10	4	20
14	9	126	0	0	0	0
16	16	256	2	32	4	64
18	8	144	4	32	16	128
20	7	140	6	42	36	252
22	2	44	8	16	64	128
	$\Sigma f_i = 50$	$\Sigma x_if_i = 800$		$\Sigma f_id_i = 100$	$\Sigma d_i^2 = 140$	$\Sigma f_id_i^2 = 640$

Standard deviation $\sigma = \sqrt{\frac{\Sigma f_id_i^2}{N} - \left(\frac{\Sigma f_id_i}{N}\right)^2}$

$= \sqrt{\frac{640}{50} - \left(\frac{100}{50}\right)^2} = \sqrt{\frac{64}{5} - 4}$

$= \sqrt{\frac{64-20}{5}} = \sqrt{\frac{44}{5}} = \sqrt{8.8}$

$\therefore$ Standard deviation $= 2.97$

6. Compute the standard deviation for the following data.

x	0-10	10-20	20-30	30-40	40-50	50-60	60-70
f	1	4	17	45	26	5	2

Sol :

Let A = 35 and C = 10

x_i	Mid value x_i	f_i	$d_i = x_i - A$ $d_i = x_i - 35$	$d_i = \frac{x_i - A}{C}$	f_id_i	$f_id_i^2$
0-10	5	1	– 30	– 3	– 3	9
10-20	15	4	– 20	– 2	– 8	16
20-30	25	17	– 10	– 1	– 17	17
30-40	35	45	0	0	0	0
40-50	45	26	10	1	26	26
50-60	55	5	20	2	10	20
60 - 70	65	2	30	3	6	18
		$\Sigma f_i = 100$			$\Sigma f_id_i = 14$	$\Sigma f_id_i^2 = 106$

Standard deviation $\sigma = \sqrt{\frac{\Sigma f_id_i^2}{N} - \left(\frac{\Sigma f_id_i}{N}\right)^2} \times C$

$= \sqrt{\frac{106}{100} - \left(\frac{14}{100}\right)^2} \times 10$

$= \sqrt{1.06 - 0.0196} \times 10$

$= \sqrt{1.0404} \times 10$

$= 1.02 \times 10$

Standard deviation $\sigma = 10.2$

7. The following values are calculated in respect of heights and weights of the students of class X.

	Height	Weight
Mean	162.6 cm	52.36 kg
Variance	127.69 cm²	23.1361 kg²

Can we say that the weight show greater variation than the heights

Sol :

To compare the variability, we have to calculate their co-efficient of variation.

Given variance of Height = 127.69 cm²

$\therefore$ Standard deviation of Height

$= \sqrt{127.69\ cm} = 11.3$ cm

Variance of weight = 23.1361 kg

$\therefore$ Standard deviation of weight

$= \sqrt{23.1361\ kg} = 4.81$ kg.

$\therefore$ Co-efficient of variation C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

C. V of Height $= \frac{11.3}{162.6} \times 100 = 6.95\%$

C. V of weights $= \frac{4.89}{52.36} \times 100\% = 9.18\%$

C. V of weight > C. V of Height.

$\therefore$ We can say that weights show more variability than heights.

8. The mean and standard deviation of marks obtained by 50 students of a class in three subjects mathematics, physics and chemistry are given below. Which of these three subjects shows the highest variability in marks and shows the lowest?

Subject	Mathematics	Physics	Chemistry
Mean	42	32	40.9
Standard deviation	12	15	20

Sol :

(i) For mathematics

Mean $\bar{x} = 42$; Standard deviation $\sigma = 12$

Co-efficient of variation C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{12}{42} \times 100\% = \frac{1200}{42}\% = 28.57\%$

(ii) For Physics

Mean $\bar{x} = 32$; SD = 15

C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{15}{32} \times 100\% = \frac{1500}{32}\% = 46.87\%$

(iii) For Chemistry:

Mean $\bar{x} = 40.9$; SD $\sigma = 20$

C. V $= \frac{\sigma}{\bar{x}} \times 100\%$

$= \frac{20}{40.9} \times 100\% = \frac{2000}{40.9} = 48.89\%$

As higher the co-efficient of variation, higher in the variability.

$\therefore$ C. V of chemistry is highest and C. V of mathematics is the least.

$\therefore$ Chemistry shows the highest variability and mathematics shows the least variability.

9. From the prices of shares A and B given below. Find which is more stable value.

A	35	54	52	53	56	58	52	50	51	49
B	108	107	105	105	106	107	104	103	104	101

Given that $\sigma_A = 5.92$ and $\sigma_B = 2$ also mean (A) = 51 and mean (B) = 105.

Sol :

Co-efficient of variation C .V $= \frac{\sigma}{\bar{x}} \times 100\%$

$$\text{C. V}_A = \frac{5.92}{51} \times 100\% = \frac{592}{51}$$

$$\text{C. V}_A = 11.60$$

$$\text{C. V of B} = \frac{2}{105} \times 100 = \frac{200}{105}$$

$$\text{C. V}_B = 1.90$$

Shares, whose co-efficient of variation is lesser is considered to be more stable.

$\therefore$ CV of share B is lesser as compared to C. V of share A.

$\therefore$ Share B is more stable.

10. A jar contains 24 marbles some are green and the others are blue. If a marble is drawn at random from the jar, the probability that it is green is $\frac{2}{3}$. Find the number of blue marbles in the jar.

Sol : Let the number of green marbles be x and the number of blue marbles be y.

Total marbles n(S) = x + y = 24 ... (1)

Let A be the event of getting a green marble

$$P(A) = \frac{n(A)}{n(S)} = \frac{x}{24}$$

but given $P(A) = \frac{2}{3}$

$$\therefore \frac{x}{24} = \frac{2}{3}$$

$$x = \frac{2 \times 24}{3} = 16$$

Put x = 16 in (1)

$$x + y = 24$$
$$16 + y = 24$$
$$y = 24 - 16 = 8$$

$\therefore$ Number of green marbles = 16

Number of blue marbles = 8

11. A jar contains 54 marbles each of which is blue, green or white. The probability of selecting a blue marble at random from the jar is $\frac{1}{3}$ and the probability of selecting a green marble at random is $\frac{4}{9}$. How many white marbles does the jar contain?

Sol :

Let the number of blue balls be b,
number of green balls be g and number of white balls be w.

$$b + g + w = 54 \quad \ldots (1)$$

$$n(S) = 54$$

Let A be the event of selecting blue marble.

$$n(A) = b$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{b}{54}$$

Given $P(A) = \frac{1}{3}$

$$\therefore \quad \frac{1}{3} = \frac{b}{54}$$

$$\Rightarrow \quad b = \frac{54}{3} = 18$$

Let B be the event of selecting a green marble

$$n(B) = g$$

$$P(B) = \frac{g}{54}$$

But $P(B) = \frac{4}{9}$ (given)

$$\therefore \quad \frac{g}{54} = \frac{4}{9}$$

$$g = \frac{4 \times 54}{9} = 24$$

Put $b = 18, g = 24$ in (1)

$$b + g + w = 54$$
$$18 + 24 + w = 54$$
$$42 + w = 54$$
$$w = 54 - 42 = 12$$

$\therefore$ The jar contains 12 white balls.

12. Two dice are thrown simultaneously. Find the probability of getting
(i) an even number as sum
(ii) the total of atleast 10
(iii) a multiple of 3 as the sum.

Sol :

Two dice are thrown, the sample space

S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6) (3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6) (4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6) (5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6) (6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

$$n(S) = 36$$

(i) Let A be the event of getting an even number as sum

A = {(1, 1) (1, 3) (3, 1) (2, 2) (1, 5) (5, 1) (2, 4) (4, 2) (3, 3) (2, 6) (6, 2) (4, 4) (5, 3) (3, 5) (5, 5) (6, 4) (4, 6) (6, 6)}

$$n(A) = 18$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

$\therefore$ Required probability = $\frac{1}{2}$.

(ii) Let B be the event of getting total of atleast 10

B = {(6, 4) (4, 6) (5, 5) (6, 5) (5, 6) (6, 6)}

$$n(B) = 6$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$\therefore$ Probability of getting the sum of atleast 10 is $\frac{1}{6}$.

(iii) Let C be the probability getting a multiple of 3 as sum.

C = { (1, 2) (2, 1) (1, 5) (5, 1) (2, 4) (4, 2) (3, 3) (3, 6) (6, 3) (5, 4) (4, 5) (6, 6)}

$$n(C) = 12$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

$\therefore$ Probability of getting a multiple of 3 as sum = $\frac{1}{3}$.

13. Two dice are thrown. The events A, B and C are as follow: A $\rightarrow$ getting an even number in first die, $B \rightarrow$ getting an odd number on the first die, $C \rightarrow$ getting the sum of the numbers on the dice ≤ 5. Check whether A and C are mutually exclusive.

Sol : Given two dice are thrown

Possible outcomes = $6 \times 6 = 36$

Possible outcomes

S = {(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6) (3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6) (4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6) (5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6) (6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

$$n(S) = 36$$

A is the event of getting even number in the first die

$A = \{(2, 1)\ (2, 2)\ (2, 3)\ (2, 4)\ (2, 5)\ (2, 6)$
$(4, 1)\ (4, 2)\ (4, 3)\ (4, 4)\ (4, 5)\ (4, 6)$
$(6, 1)\ (6, 2)\ (6, 3)\ (6, 4)\ (6, 5)\ (6, 6)\}$

$n(A) = 18$

C is the event of getting the sum of the number less then or equal to 5

$C = \{(1, 1)\ (1, 2)\ (1, 3)\ (1, 4)\ (2, 1)\ (2, 2)\ (3, 1)\ (3, 2)$
$(4, 1)\}$

$n(C) = 10$

$A \cap C = \{(2, 1)\ (2, 2)\ (2, 3)\ (4, 1)\}$

$n(A \cap C) = 4$

$n(A \cap C) \neq 0,$

$\therefore$ A and C are not mutually exclusive.

14. A number is selected from the first 50 natural numbers. What is the probability that it is a multiple of 5 or 11?

Sol : Let S be the sample space

$S = \{1, 2, 3, 4, 5, \ldots 50\}$

$n(S) = 50$

Let A be the set of all multiples of 5.

$A = \{5, 10, 15, 20, 25, 30, 35, 40, 45, 50\}$

$n(A) = 10$

$$P(A) = \frac{n(A)}{n(S)} = \frac{10}{50}$$

Let B be the event of getting multiple of 11

$B = \{11, 22, 33, 44\}$

$n(B) = 4$

$$P(B) = \frac{4}{50}$$

$A \cap B = \{\ \}$

$P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B) = \frac{10}{50} + \frac{4}{50} - 0$$

$$P(A \cup B) = \frac{14}{50} = \frac{7}{25}$$

Probability of getting a multiple of 5 or 11 $= \frac{7}{25}$.

15. A number is selected from the first 25 natural numbers. What is the probability that it would be divisible by 4 or 7?

Sol : Let sample space $S = \{1, 2, 3, 4, \ldots\ldots, 25\}$

$n(S) = 25$

Let A be the event of selecting a number divisible by 4.

$A = \{4, 8, 12, 16, 20, 24\}$

$n(A) = 6$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{25}$$

Let B be the event of selecting a number divisible by 7.

$B = \{7, 14, 21\}$

$n(B) = 3$

$$P(B) = \frac{3}{25}$$

$A \cap B = 0$

$n(A \cap B) = 0$

$P(A \cap B) = 0$

$$P(A \cup B) = P(A) + P(B) = \frac{6}{25} + \frac{3}{25}$$

$$P(A \cup B) = \frac{9}{25}$$

$\therefore$ Probability of selecting a number divisible by 4 or 7 is $\frac{9}{25}$.

16. A number is selected from first 30 natural numbers. What is the probability that if would be divisible by 2 or it is a prime.

Sol :

Sample space $S = \{1, 2, 3, \ldots, 30\}$

$n(S) = 30$

Let A be the event of selecting a number divisible by 2.

$A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}$

$n(A) = 15$

$$P(A) = \frac{n(A)}{n(S)} = \frac{15}{30}$$

Let B be the event of getting a prime

$B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

$n(B) = 10$

$$P(B) = \frac{10}{30}$$

$A \cap B = \{2\}$

$n(A \cap B) = 1$

$$P(A \cap B) = \frac{1}{30}$$

$\therefore$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$P(A \cup B) = \frac{15}{30} + \frac{10}{30} - \frac{1}{30}$$

$$P(A \cup B) = \frac{24}{30} = \frac{4}{5}$$

$\therefore$ Required probability $= \frac{4}{5}$.

17. There are three persons A, B and C having different ages. The probability that A survives another 5 years is 0.80. 'B' survives another 5 years is 0.60 and C survives for another 5 years is 0.50. The probabilities that A and B survives another 5 years is 0.46. B and C another 5 years is 0.32 and A and C survives another 5 years is 0.48. The probability that all these three survive another 5 years is 0.26. Find the probability that at least one of them survive another 5 years.

Sol :

From the information given

$$P(A) = 0.80$$
$$P(B) = 0.60$$
$$P(C) = 0.50$$
$$P(A \cap B) = 0.46$$
$$P(B \cap C) = 0.32$$
$$P(A \cap C) = 0.48$$
$$P(A \cap B \cap C) = 0.26$$

The probability that at least one of them survives another 5 years is $P(A \cup B \cup C)$.

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) \; P(A \cap C) + P(A \cap B \cap C)$$

$$P(A \cup B \cup C) = 0.80 + 0.60 + 0.50 - 0.46 - 0.32 - 0.48 + 0.26$$

$$\therefore \; P(A \cup B \cup C) = 0.90$$

18. The probability that a student will pass his examination is 0.73. The probability of the student getting a compartment is 0.13 and check whether the probability that the student will either pass or get compartment is 0.96.

Sol :

Let A be the event of passing B get compartment.

Given $P(A \cup B) = 0.96$.

$$P(A) = 0.73$$
$$P(B) = 0.13$$
$$P(A \cup B) = 0.96$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.96 = 0.73 + 0.13 - P(A \cap B)$$
$$P(A \cap B) = 0.73 + 0.13 - 0.96$$
$$= 0.86 - 0.96 = -0.10$$
$$P(A \cap B) = -0.10$$

The probability cannot be negative.

$\therefore$ The probability that the student will either pass or get compartment is not 0.96.

X Mathematics

Govt. Model Question Paper 2020

Time : 2.30 hrs **Maximum Marks : 100**

Instructions: i) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

ii) Use Black or Blue ink to write and underline and pencil to draw diagrams.

Note : This question paper contains four parts.

PART - I

(Marks : 14)

(i) Answer all the **15** questions. **14 × 1 =14**

(ii) Choose the correct answer from the given four alternatives and write the option code and the corresponding answer.

1. If n (A×B) = 6 and A = {1, 3} then n (B) is
(1) 1 (2) 2 (3) 3 (4) 6

2. Given $F_1 = 1$, $F_2 = 3$ and $F_n = F_{n-1} + F_{n-2}$ then F_5 is
(1) 3 (2) 5 (3) 8 (4) 11

3. In an A.P, the first term is 1 and the common difference is 4. How many terms of the A.P must be taken for their sum to be equal to 120?
(1) 6 (2) 7 (3) 8 (4) 9

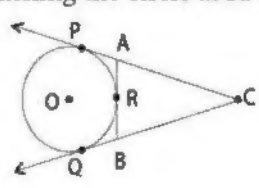
4. f= {(2,a), (3,b), (4,b), (5,c)} is a ______.
(1) identity function (2) one-one function (3) many-one function (4) constant function

5. The number of points of intersection of quadratic polynomial $x^2 + 4x + 4$ with the x axis is
(1) 0 (2) 1 (3) 0 (or) 1 (4) 2

6. The non- diagonal elements in any unit matrix are ______
(1) 0 (2) 1 (3) m (4) n

7. If A is a 2×3 matrix and B is a 3×4 matrix, how many columns does AB have?
(1) 3 (2) 4 (3) 2 (4) 5

8. In figure CP and CQ are tangents to a circle with centre at O. ARB is another tangent touching the circle at R. If CP= 11 cm and BC= 7 cm then the length of BR is
(1) 6 cm (2) 5 cm
(3) 8 cm (4) 4 cm



9. The slope of the line joining (12, 3), (4, a) is $\frac{1}{8}$. The value of 'a' is ______
(1) 1 (2) 4 (3) –5 (4) 2

10. If $x = a\tan\theta$ and $y = b\sec\theta$ then
(1) $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ (2) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

11. A letter is chosen at random from the letter of the word "PROBABILITY". Find the probability that it is not a vowel.

(1) $\frac{1}{5}$ (2) $\frac{2}{3}$ (3) $\frac{1}{3}$ (4) $\frac{3}{5}$

12. The height of a right circular cone whose radius is 5 cm and slant height is 13 cm will be

(1) 12 cm (2) 10 cm (3) 13 cm (4) 5 cm

13. If the mean and co-efficient of variation of a data are 4 and 87.5% then the standard deviation is

(1) 3.5 (2) 3 (3) 4.5 (4) 2.5

14. Variance of first 20 natural numbers is

(1) 32.25 (2) 44.25 (3) 33.25 (4) 30

PART - II

(Marks :20)

Note : Answer 10 questions.

(Question No. 28 is compulsory) **10 × 2 = 20**

15. Define a function.

16. Compute x such that $10^4 \equiv x \pmod{19}$

17. Simplify $\frac{4x^2y}{2z^2} \times \frac{6xz^3}{20y^4}$

18. Pari needs 4 hours to complete the work. His friend Yuvan needs 6 hours to complete the work. How long will it take to complete if they work together?

19. Find the values of x, y and z from the following equation $\begin{pmatrix} 12 & 3 \\ x & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} y & z \\ 3 & 5 \end{pmatrix}$

20. What length of ladder is needed to reach a height of 7 ft along the wall when the base of the ladder is 4 ft from the wall?

21. Prove that , $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \cos ec\theta + \cot\theta$

22. The radius of a sphere increases by 25%. Find the percentage increase in its surface area.

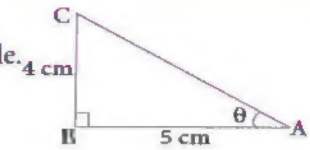
23. The Standard Deviation and Mean of a data are 6.5 and 12.5 respectively. Find the coefficient of variation.

24. If $f(x) = 3 + x$, $g(x) = x - 4$, then check whether $f \circ g = g \circ f$

25. An organization plans to plant saplings in 25 streets in a town in such a way that one sapling for the first street, three for the second, nine for the third and so on. How many saplings are needed to complete the work?

26. Find the 19[th] term of an A.P –11, –15, –19, ...

27. Find the value of $\angle BAC$ in the given triangle.



28. The vertices of a triangle are A(–1,3), B(1,–1) and C(5,1). Find the length of the median through the vertex C.

PART - III

(Marks :50)

Note : Answer 10 questions
(Question No. 42 is compulsory) **10 × 5 = 50**

29. Let f be a function $f : N \to N$ be defined by $f(x) = 3x + 2, x \in N$

    i) Find the images of 1, 2, 3 ii) Find the pre-images of 29, 53 iii) Identify the type of function.

30. Let f: A→B be a function defined by $f(x) = \frac{x}{2} - 1$, where A= {2,4,6,10,12}, B = { 0,1,2,4,5,9}. Represent f by

    i) set of ordered pairs ii) a table iii) an arrow diagram iv) a graph

31. The ratio of 6th and 8th terms of an A.P is 7:9.Find the ratio of 9th term to 13th terms.

32. The sum of first n, 2n and 3n terms of an A.P are S_1, S_2 and S_3 respectively. Prove that S_3, = $3(S_2 - S_1)$

33. Find the values of m and n if the expression $\frac{1}{x^4} - \frac{6}{x^3} + \frac{13}{x^2} + \frac{m}{x} + n$ is a perfect square.

34. If α, β are the roots of the equation $2x^2 - x - 1 = 0$ then form the equation whose roots are $\alpha^2 \beta$, $\beta^2 \alpha$.

35. P and Q are the mid-points of the sides CA and CB respectively of a DABC , right angled at C. Prove that $4(AQ^2 + BP^2) = 5AB^2$.

36. Find the equation of a straight line passing through (1,–4) and has intercepts which are in the ratio 2:5.

37. From the top of the tower 60 m high the angles of depression of the top and bottom of a vertical lamp post are observed to be 38^0 and 60^0 respectively. Find the height of the lamp post ($\tan 38^0 = 0.7813$, $\sqrt{3} = 1.732$).

38. Calculate the weight of a hollow brass sphere if the inner diameter is 14 cm and thickness is 1mm, and whose density is 17.3 g/cm^3.

39. Find the Co-efficient of variation of 24, 26, 33, 37, 29, 31.

40. Two dice, one blue and one grey, are thrown at the same time. Write down all the possible outcomes. What is the probability that the sum of the two numbers appearing on the top of the dice is

    (i) 8 (ii) 13 (iii) less than or equal to 12

41. Find two consecutive positive integers, sum of whose squares is 365.

42. A cylindrical bucket of 32 cm high and with radius of base 18 cm, is filled with sand completely. This bucket is emptied on the ground and a conical heap of sand is formed. If the height of the conical heap is 24 cm, find the radius and slant height of the heap.

PART - IV

(Marks : 16)

Note : Answer both questions. **2 × 8 = 16**

43. (a) PQ is a chord of length 8 cm to a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length of the tangent TP.

(OR)

(b) Draw a triangle ABC of base BC = 8 cm, $\angle A = 60°$ and the bisector of $\angle A$ meets BC at D such that BD = 6 cm.

44. (a) Draw the graph of $y = x^2 + 3x - 4$ and hence use it to solve $x^2 + 3x - 4 = 0$.

(OR)

(b) A motor boat whose speed is 18 km/hr in still water takes 1 hour more to go to 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

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